LESSON 9: MODELLING SWITCHING SYSTEMS AND INCOMING TRAFFIC AND SERVICE TIME CHARACTERIZATION

Objective

The objective here is to learn about modeling switching systems and incoming traffic and service time characterization.

Introduction

Modelling Switching Systems

Traffic carried by a telecommunication network is generated by a large number of individual subscribers connected to the network.

Subscribers generate calls in a random manner. The call generation by the subscribers and therefore the behaviour of the network or the switching systems in it can be described as a Random Process.

Random Process

A random process is also called a stochastic process. It is one in which one or more quantities vary with time in such a way that the instantaneous values of the quantities are not determinable precisely but are predictable with certain probability. Quantities used in random process are referred to as random variables. Thus, a stochastic process is a time-indexed function of one or more random variables. Generally, it is possible to characterize the behaviour of a random process by some statistical properties.

Thus, we can predict the future performance of random process or stochastic process with a certain probability. The telephone traffic quantifies as a stochastic process. Here the number of simultaneously active subscribers and the number of simultaneously busy servers are random variables. We cannot precisely estimate the number of simultaneously active subscribers at a given instant of time but a prediction can be made with a certain probability. A typical fluctuation in the number of simultaneous calls in a 30 minute period shown in Fig. 9.1. This pattern signifies a typical random process. The values of random variables in a random process may be discrete or continuous. In the case of telephone traffic, the number of simultaneous calls is discrete random variable. But, in the case of temperature variations in an experiment, temperature is a continuous random variable.

Similarly, the time index of the random variables can be discrete or continuous. Accordingly, stochastic processes can be classified into four different types:

1. Continuous – time Continuous – state (CTCS)
2. Continuous – time Discrete – state (CTDS)
3. Discrete – time Continuous – state (DTCS)
4. Discrete – time Discrete – state (DTDS)

A discrete - state stochastic process is often called a chain.

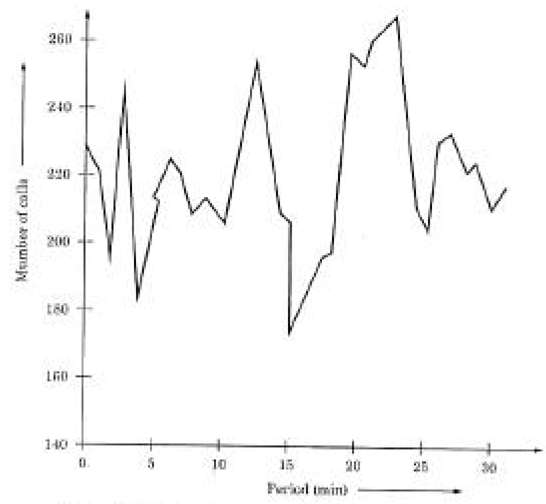
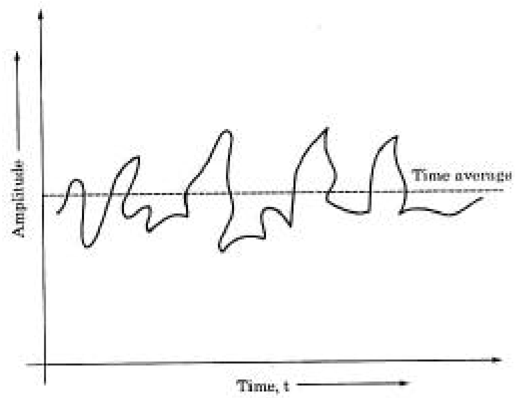


Fig.9.1 Illustration of typical fluctuations in the number of telephone calls We have two ways of obtaining the statistical properties of random process:

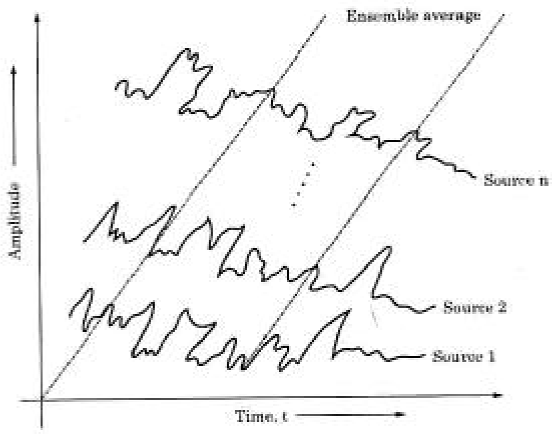
1. By observing the behaviour of a random process over a very long period of time.
2. By observing simultaneously, a very large number of statistically identical random sources at any given instant of time.

The statistical properties obtained using the first method are known as time statistical parameters. The statistical properties obtained using the second method are known as ensemble statistical parameters. The word “ensemble” means the collection sources. Time and ensemble average of a Stochastic process are shown in Figs.9.2 (a) and 9.2(b), respectively.

These methods of obtaining statistical parameters of a random process are analogous to determining the statistical behavior of a die by throwing one die a very large number of times or by throwing a very large number of identical dice all together and observing out comes.



(a) Time average



(b) Ensemble average

Fig. 9.2. Illustration of time and ensemble average of stochastic process. In the first case, a particular outcome would occur one-sixth of the times the experiment is performed. In the second case, one-sixth of the outcomes could correspond to a particular face value of the die. The statistical parameters obtained in both the cases are the same. The ensemble and time statistics in random processes are not same due to two reasons. These reasons are:

The statistical properties of a random process may themselves change with time in which case the ensemble statistics will be different at different instants of time.

Stationary Random Process

Random processes whose statistical parameters do not change with time are called stationary processes.

The statistical properties of the ensemble of sources, though stationary, need not be identical, in which case the time statistics will depend on the sources selected for observation. Thus, the second condition that must be satisfied is that all random processes in the ensemble must have identical statistical properties. If both the conditions are satisfied then the time and the ensemble statistical parameters are identical.

Ergodic Random Processes

The random processes which have identical time and ensemble averages are known as ergodic random processes.

Now, we can note that an ergodic process is a stationary process but a stationary process need not necessarily be ergodic process.

Wide-Sense Stationary Processes

If the mean and the variance alone for a random process are stationary and other higher order moments may vary with time then these random processes are called Wide – Sense (WS) stationary processes.

Teletraffic or telephone traffic is non-stationary. The mean and variance or standard deviation of the telephone traffic is different during different times of the day. In the segment form of telephone traffic during a day (say, 6 – 7 hours, 12 - 13 hours etc.), the traffic may be considered to be stationary.

It is difficult to model and analyse non-stationary random processes.

So, we model and analyze telephone traffic in segments when they can be considered to be stationary.

The behaviour of a telecommunications switching system can also be modelled as a stochastic process. For example, the number of servers busy simultaneously is a discrete random variable. The time at which a server or link or trunk becomes busy or free also exhibits a random behaviour. Hence the entire switching system can be modelled as a random process. Here we have interest in the discrete state random process.

Therefore, in our modelling, we use discrete-state stochastic processes.

Now we discuss following random processes:

1.Markov Processes

2. Birth-Death (B-D) Processes.

1 Markov Processes

Markov processes form an important class random processes that have some special properties. These special properties of random process were first defined and investigated by *A.A.* Markov in 1907. He proposed a simple and highly useful form of dependency among the random variables forming a stochastic process.

Markov processes are of great interest to our modelling of switching systems.

A discrete – time Markov Chain is a discrete-time discrete-state (DTDS) Markov process. It is defined as one that has the following property:

$$P[\{X(t_{n+1}) = x_{n+1}\} / \{X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1\}]$$

$$= P[\{X(t_{n+1}) = x_{n+1}\} / \{X(t_n) = x_n\}] \quad \ldots\ldots\text{Eq.1}$$

where $t_1 < t_2 < t_3 \ldots < t_n < t_{n+1}$ and x_i is the i[th] discrete-state space value.

Equation 1 states the probability that the random variable X takes on the value x_{n+1} at time step n + 1 is entirely found by its state value in the- previous time step *n* und is independent of its state values in earlier time steps; n-1, n-2, etc. Entire past history of the process is also summarised in its current state. Hence the next state is determined only by the current state. If this is the case, the time period for which the process has stayed in the current state should play no role in determining the next state. However, we need some specification of the time that elapses between state transitions. The time specification is available in terms of random variable because the process is random. Therefore, we seek a probability distribution function (pdf) for time which would in no way influence the state transitions of a process.

In other words, the duration for which a process bas stayed in a particular state does not influence the next state transition.

This criterion will be satisfied by only two distribution functions which are:

1. Exponential Distribution Functions:

   It is a Continuous Distribution function.

2. Geometric Distribution Function:

   It is a discrete distribution function.

Thus, the interstate transition time in a Discrete – time Markov process is geometrically distributed. The interstate transition time in a continuous-time Markov process is exponentially distributed. Both distribution functions given above are called memory less.

2 Birth-Death (B-D) Processes

Birth – Death (B-D) process is obtained by applying the restriction that the state transitions of a Markov chain can occur only to the adjacent states. It is customary to talk of population in a B-D process. The number in the population is a random variable and represents the state value of the process.

1. The B-D process moves from its state k to the state k- l if a death occurs
2. The B-D process moves from its state k to state k + 1 if a birth occurs.
3. The B-D process stays in the same state if there is no birth or death during the time period under consideration.

Fig, 9.3 Depects the behaviour of the B-D process.

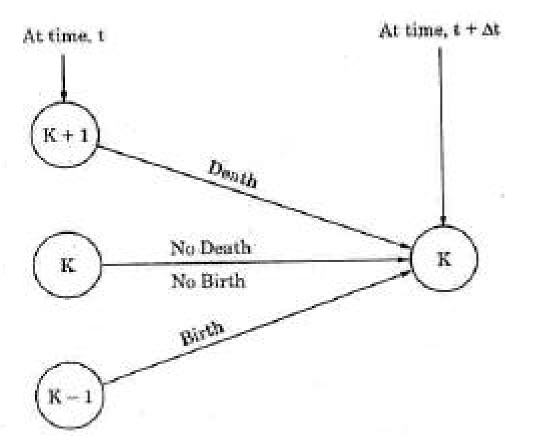


Fig. 9.3 State transitions at a B-D process.

B-D process are very useful in our analysis of telecommunication networks. A telecommunication network can be modelled as a B-D process where the number of busy servers represents the population, a call request means birth and a call termination implies a death. For analysing a B-D process, we shall choose a time interval Dt small enough. The small time interval will satisfy followings:

1. There can almost be only one state transition in that interval
2. There is only one arrival or one termination but not both, and
3. There may be no arrival or termination leaving the state unchanged in the time interval Δt.

We further assume that,

1. The probability of an arrival or termination in a particular interval is independent of what had happened in the earlier time intervals and
2. The probability of an arrival is directly proportional to the time interval Δt.

By considering above-mentioned assumptions, we now proceed to determine the dynamics of a telecommunication switching system modelled as a B-D process.

Now we let following quantities and their symbols used here,

$P_k(t)$ = the probability that the switching system is in state k at time t.

R_k = call arrival rate in state k

r_k = call termination rate in state k

Then, we have the following probabilities in the time interval Δt

P [exactly one arrival] = $R\Delta t$

P [exactly one termination] = $r\,\Delta t$

P [No arrival] = 1 - P [exactly one arrival] = $1 - \Delta R\,t$

P [No termination] = 1 - P [Exactly one termination] = $1 - r\,\Delta t$

Probability of finding the system in state k at time $t + \Delta t$ is given by following equation:

$$P_k(t + \Delta t) = P_{k-1}(t) * R_{k-1} * \Delta t + P_{k+1}(t) * r_{k+1} * \Delta t + (1 - R_k * \Delta t)(1 - r_k * \Delta t) * P_k(t) \quad \text{Eq.2}$$

There are three terms in Right-hand side (RHS) of Eq.2.

The first term on the RHS represents the possibility of' finding the system in state k - 1 at time t and a call request (i.e. birth) occurring during the interval (t, t+Δt). The second term on the RHS represents the possibility of finding the system in state k + 1 at time t and a call termination (i.e. death) occurring during the- interval (t, t+Δt). The last or third term on the RHS represents the no arrival and no termination case.

Now, we are going to expand Eq.2,

$$P_k(t + \Delta t) = P_{k-1}(t) * R_{k-1} * \Delta t + P_{k+1}(t) * r_{k+1} * \Delta t + P_k(t) - P_k(t) * R_k * \Delta t - P_k(t) * r_k * \Delta t + P_k(t) * R_k * r_k * (\Delta t)^2$$

Here $P_k(t) * R_k * r_k * (\Delta t)^2$ is second order Δt term. Ignoring it we get,

$P_k(t + \Delta t) = P_{k-1}(t) * R_{k-1} * \Delta t + P_{k+1}(t) * r_{k+1} * \Delta t + P_k(t) – P_k(t) * R_k * \Delta t - P_k(t) * r_k * \Delta t$

or

$P_k(t + \Delta t) = P_{k-1}(t) * R_{k-1} * \Delta t + P_{k+1}(t) * r_{k+1} * \Delta t + P_k(t) – (R_k + r_k) * P_k(t) * \Delta t$ ……………. Eq.3

Here our interest lies in finding the dynamics of the system. The dynamics of the system is given by the rate of change of the probability P_k with time.

After rearranging Eq.3, we get

$P_k(t + \Delta t) – P_k(t)$

$= P_{k-1}(t) * R_{k-1} + P_{k+1}(t) * r_{k+1} – (R_k + r_k) * P_k(t) \Delta t$

Taking the limit, Δt 0, we get $P_k(t + \Delta t) – P_k(t)$

lim

$= P_{k-1}(t) * R_{k-1} + P_{k+1}(t) * r_{k+1} – (R_k + r_k) * P_k(t) \quad \Delta t \quad \Delta t$ …… Eq.4

or $d P_k(t)$

$= P_{k-1}(t) * R_{k-1} + P_{k+1}(t) * r_{k+1} – (R_k + r_k) * P_k(t)$ …… Eq.4

Δt

This is the differential equation governing the dynamics of a B-D process. This equation can be applied for all values of k > or equal to 1 For k = 0 (i.e. no calls in progress) no termination of calls takes place. In other words, $r_0 = 0$. Further, there can be no state with – 1 as the state value. Therefore, for.k = 0, Eq.4 can be modified as

$d P_0(t)$

$= P_1(t) * r_1 – P_0(t) * R_0$ ……… Eq.5

dt

While the above equations actually give the rate of change of state probabilities, we are often concerned with steady state operation of the networks. Under steady state conditions, the state probabilities reach an equilibrium value and their probabilities do not change with time.

$$P_k(t_1) = P_k(t_2) = P_k(t_i) = P_k$$

For example; when commercial offices start functioning around 10:00 hours, telecommunication switching networks pick up traffic and quickly reach a steady state condition when the state probabilities get stabilized. Under steady state conditions, we have

$$\frac{d P_k(t)}{dt} = 0$$

It means that the B-D process becomes stationary. Therefore, the steady state equations of a B-D process are

$P_{k-1} * R_{k-1} + P_{k+1} * r_{k+1} – (R_k + r_k) * P_k = 0$ for k ³ 1 ……..Eq.6

$P_1 * r_1 – P_0 * R_0 = 0$ for k=0 ………….. Eq.7

Here we can note that the steady state behaviour of a telecommunication switching system is governed by Eqs.6 & 7, when the system is modelled as a Birth-Death (B-D) process.

Incoming Traffic And Service Time Characterization

We know that a Birth-Death (B-D) process is useful in modelling telecommunication-switching networks. Traffic in a telecommunication-switching network is the aggregate of the traffic generated by a large number of individual subscribers connected to the network. Subscribers generate calls in a random manner. Hence the telecommunication traffic (teletraffic) is characterized as n random process. Whenever a subscriber generates or originates a call, he/ she adds one to the number of calls arriving at the network. There is no way by which he/ she can reduce the number of calls that have already arrived. Thus, we need a model for describing an originating process. Therefore, this process can be treated as a special case of B – D process in which the death rate is equal to zero. A B-D process in which no death occurs is known as a **Renewal Process**.

Renewal Process

A renewal process is a pure birth process in the sense that it can only add to the population as the time goes by and cannot deplete the population by itself.

The equations governing the dynamics of a renewal process can be easily obtained from the B-D process equations by setting $r_k = 0$

$$\frac{d P_k(t)}{dt} = P_{k-1}(t) * R_{k-1} – P_k(t) * R_k \quad \text{for } k \geq 0 \text{ …… Eq.7}$$

$$\frac{d P_0(t)}{dt} = – P_0(t) * R_0 \quad \text{…… Eq.8}$$

Steady state equations are not relevant in a renewal process. In renewal process we are counting the number of call origination or birth. Let us assume that we start the observation at the time t. As soon as a birth occurs (say at time $t = t_1$) it is impossible to find the system in state 0. Similarly, after i births, it is not possible for the system to be in state i-1, i-2, etc.

In Eqs.5, 6, 7 and 8, the birth rate is dependent upon the state of the system. If we assume a constant birth rate R which is independent of the state of the system, then we can get a Poisson Process. The governing equations of a Poisson process are given by,

$$\frac{d P_k(t)}{dt} = P_{k-1}(t) * R – P_k(t) * R \quad \text{for } k \geq 0 \text{ …… Eq.9}$$

$$\frac{d P_0(t)}{dt} = P_0(t) * R \quad \text{for } k = 0 \text{ …… Eq.10}$$

For solving Eqs. 9 and 10, we required some boundary conditions. So we are assuming following boundary conditions.

System is in state zero at time $t = 0$, i.e., no births have taken place.

$$P_k(0) = P_k(t) \quad t = 0 = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } k \neq 0 \end{cases} \quad \text{……….. Eq.11}$$

We can get solution of Eq.10 by using Eq.11

$P_0(t) = e^{-Rt}$ …………..Eq.12

From eqs.9 & 12, we obtain, for k = 1

$$\frac{d P_1(t)}{dt} = -R * P_1(t) + R * e^{-Rt} \quad \text{…… Eq.12}$$

After solving eq.12, we get

$P_1(t) = (R\,t)\, e^{-Rt}$ …………………… Eq.13

For k = 2, the solution is

$$P_2(t) = \frac{(R\,t)^2\, e^{-Rt}}{\angle 2} \quad \text{………… Eq.14}$$

We can generalize the solution by using induction method.

$$P_k(t) = \frac{(R\,t)^k\, e^{-Rt}}{\angle k} \quad \text{………… Eq.15}$$

where $\angle k$ is called factorial of k.

Eq.15 is the most celebrated Poisson arrival Process Equation. This equation expresses the probability of finding the system with *k* members in the population at time *t*. In other words, it represents the probability of *k* arrivals in the time interval *t*.

Eq.12 i.e. $P_0(t) = e^{-Rt}$ gives the probability of zero arrival in a given time interval. This probability is nothing but the probability distribution of interarrival times. The interarrival time in a time that elapses between two arrivals. Thus, in a Poisson process, the interarrival time is exponentially distributed. As we know that Poisson process is a Markov process and Markov process demands that the interstate transition times should be exponentially distributed.

Example 9.1: A rural telephone exchange normally experiences 4 call originations per minute. Find the probability of exactly 8 calls occur in an arbitrary chosen interval of 30 seconds.

Solution:

R = 4 calls originating per minute

$$= \frac{4}{60}$$

$$= \frac{1}{15} \text{ calls per second}$$

When t = 30 seconds then

R t = (1/15) * 30

= 2

Therefore, the probability of exactly 8 arrivals is given by,

$$P_k(t) = \frac{(R\,t)^k\, e^{-Rt}}{\angle k}$$

$$P_8(t) = \frac{(R\,t)^8\, e^{-2}}{\angle 8}$$

= 0.00086

Probability of 8 calls is 0.00086

In this example, 8 arrivals in 30 seconds represent a 4-fold increase in the normal traffic. The solution shows that probability of such event occurring is very low. These calculations are useful in sizing a telecommunication switching system.

It is important to recognise that the assumption of an arrival rate independent of the state of the system in a Poisson process. It implies that we are dealing with infinite number of sources or essentially constant number of sources. If a number of arrivals occur immediately before any subinterval, some of sources become busy and cannot generate further requests. The effect of busy sources is to reduce the average arrival rate unless the source population is infinite or large enough so as not to be affected significantly by the busy sources.

Example 9.2: A telecommunication switching system serves 10,000 subscribers with a traffic intensity of 0.1 erlangs per subscriber. If there is a sudden spurt in traffic, it increases the average traffic by 50%. Find the effect on the arrival rate.

Solution:

Number of· active subscribers during

(a) Normal traffic = 1000

(b) Increased traffic = 50% of Normal traffic + Normal traffic

= 500 + 1000

= 1500

Number of available subscribers for generating new traffic during

(a) Normal traffic = 10,000 - 1,000 = 9,000

(b) Increased traffic = 10,000 - 1,500 = 8,500

$$\text{Change in arrival rate} = \frac{\text{Normal traffic} - \text{Increased traffic}}{\text{Normal traffic}}$$

$$= \frac{9000 - 8500}{9000} * 100$$

$$= \frac{500}{9000} * 100$$

= 5.6 %

Relationship between the different Markov processes such as Birth-Death (B-D) process, Renewal process and Poisson process is shown in Fig. 9.4. This relationship is illustrated in the form of Venn diagram.

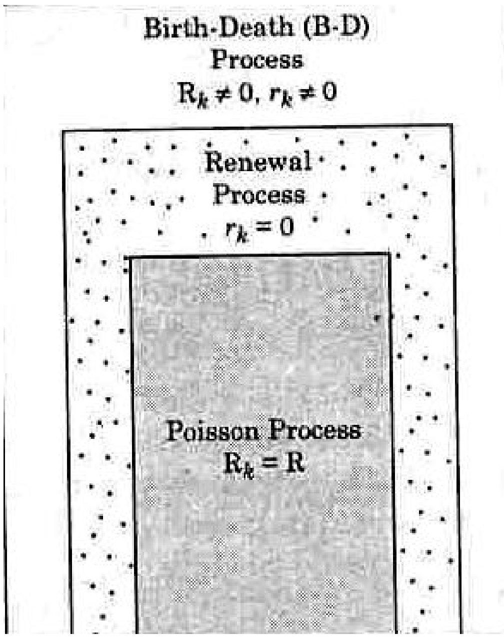


Fig. 9.4 Illustration of relationship among different Markov processes.

By using Venn diagram, we can describe the Poisson process as follows:

1. A pure birth process with constant birth rate.
2. A birth-death (B-D) process with zero death rate and a constant birth rate.
3. A Markov process with state transitions limited to the next higher state or the same state, and having a constant transition rate.

In real life, Poisson process occurs very often. For example:

1. Number of' Telephone calls arriving at a telephone exchange.
2. Number of rainy days in a year.
3. Number of typing errors in a manuscript.
4. Number of bit errors occurring in a data communication system.

Telephone calls arriving at an exchange follow a Poisson process.

- But the process of call generation by the subscribers is not a Poisson process and is a renewal process. Lets see how a renewal process at the subscriber end becomes a Poisson process at the exchange end.

It can be shown that a superposition of a large number of renewal processes results in a Poisson process. This is why we observe-Poisson processes in nature, whenever we study the aggregate behaviour of a large population. All examples of Poisson processes given above represent this phenomenon.

The termination phenomenon in a system modelled as a B-D process can be characterized by pure death process. We obtain a pure death process from a B-D process by setting the birth rate equal to zero.

Thus we can obtain the equations governing the dynamics of a pure death process from Eqs. (4) and (5) as

$$\frac{d P_k(t)}{dt} = r_{k-1} * P_{k+1}(t) - r_k * P_k(t) \text{ for } k \geq 1 \ldots\ldots \text{Eq.16}$$

$$\frac{d P_0(t)}{dt} = P_1(t) * r_1 \ldots\ldots\ldots \text{Eq.17}$$

We can further simplify the behaviour of the pure death process by assuming a constant death rate. We then obtain the governing equations as

$$\frac{d P_k(t)}{dt} = r * P_{k+1}(t) - r * P_k(t) \text{ for } k \geq 1 \ldots\ldots \text{Eq.18}$$

$$\frac{d P_0(t)}{dt} = P_1(t) * r \text{ for } k = 0 \ldots\ldots\ldots \text{Eq.19}$$

We then obtain the equations as

$$\frac{d P_k(t)}{dt} = r * P_{k+1}(t) - r * P_k(t) \text{ for } 0 < k < N \ldots\ldots \text{Eq.20}$$

$$\frac{d P_N(t)}{dt} = r * P_N(t) \text{ for } k = N \ldots\ldots\ldots \text{Eq.21}$$

$$\frac{d P_0(t)}{dt} = r * P_1(t) \text{ for } k = 0 \ldots\ldots\ldots \text{Eq.22}$$

For solving the Eqs.20 & 22, we get

$$P_N(t) = e^{-rt} \quad \text{for } k = N \ldots\ldots\ldots\ldots \text{Eq.23}$$

$$P_k(t) = \frac{(r\,t)^{N-k}\, e^{-rt}}{\angle \underline{N-k}} \quad \text{for } 0 < k < N \ldots\ldots \text{Eq.24}$$

$$P_1(t) = \frac{(r\,t)^{N-1}\, e^{-rt}}{\angle \underline{N-1}} \quad \text{for } k = 1 \ldots\ldots \text{Eq.25}$$

Eq.24 expresses the probability of no termination or death in a given interval as the population remains at the initial level. In other words, it is the probability distribution of remains in a state without a death or termination occurring. The probability of no termination or death is the probability distribution of the service times or the holding times in the case of calls in a telecommunication switching system. Thus a pure death process has a service time distribution, which is exponentially distributed as required in the case of a Markov process.

how long the system

The holding times of conventional voice conversations are well approximated by an exponential distribution. A constant holding time is a better model for call processing and interoffice signalling. For determining the probability distribution of active channels, we straight forward use Poisson equation which is,

$$P_k(t) = \frac{(R\,t)^k\ e^{-R\,t}}{\angle k}$$

Let t_m be the constant holding time. Then, the probability of finding k busy channels at any instant of time is merely the probability that there are k arrivals during the time interval t_m

$$P_k(t_m) = \frac{(R\,t_m)^k\ e^{-R\,t_m}}{\angle k}$$

A constant holding time distribution is useful for modelling telephone exchange functions.

Probability of 8 calls is 0.00086

Change in Arrival rate is 5.6%

Notes