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SUBJECT: Representation of a Gravitational
Potential with Fixed Mass Points
Case 340

DATE: December 23, 1968

FROM: W. I. McLaughlin

ABSTRACT

A least squares integral criterion is proposed for measuring goodness of fit between a given Newtonian potential and the potential of a mass point configuration. It is proved that if the mass point locations are held fixed there exists a unique set of mass values which best represents the given potential. A linear equation is derived which enables this set of mass values to be calculated. It is also shown that if a sufficient number of mass points are used, the fit can be made as close as is desired. Finally, a three mass point approximation to an oblate earth is developed as an illustration of the theory.



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MEMORANDUM FOR FILE

INTRODUCTION

NASA's recent Lunar Orbiter series of space vehicles has provided both the means and the incentive for a new effort in selenodesy. The processing of this Lunar Orbiter data, resulting in a spherical harmonic representation of the moon's potential, has proved more difficult than was originally anticipated. For the success of the coming Apollo lunar landings a certain orbit prediction capability is required which would be guaranteed by a sufficiently accurate expression for the moon's potential. In addition, earth satellites, ballistic missiles and planetary probes must have their gravitational environments well defined, so that the need for effective techniques to determine and represent the potential of a body of finite dimensions is a continuing one.

Muller and Sjogren have shown [1] that there are raisins (mascons, Icarites) in the lunar pudding. This, of course, suggests supplementing the traditional spherical harmonic approach with a procedure involving mass points. Here a mass point, in cartesian formulation, is taken to mean a set P of four real numbers m, ξ, η, ζ that defines a function v_p by

$$v_p(x,y,z) = \frac{m}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}}$$

where the mass value m is allowed to be positive or negative.

However, such an attack on the potential representation problem need not be limited to the lunar case where mascon "contaminants" have already been detected. It is our claim that mass point capability should be built into every potential determination effort. In brief, the argument rests on the

flexibility of mass modeling as a means of specifying potentials, the use of Riemann-Stieltjes measures for defining mass distributions, the Lebesgue decomposition theorem (Reference [2]) and, finally, the suitability of mass points in approximating the singular portion of the decomposition theorem.

Acceptance of this thesis inevitably leads to problems of attainability and technique. That is, what mass point representation schemes will work (yield unique answers for the relevant parameters) and which among the workable methods are the most efficient? This paper discusses the simplest scheme: a predetermined number of mass point locations are given and held fixed, and the mass values of these mass points are treated as unknowns.

REPRESENTATION OF A GIVEN POTENTIAL

A criterion must be adopted which enables one to calculate a unique set of mass values given the potential in some region R of space. The criterion defining "goodness of fit" which is used in this paper selects that set of mass values which minimizes the function f_v defined by

$$f_v(m_1, \dots, m_n) = \int_R (v - \tilde{v})^2 d\tau$$

where

$$\tilde{v} = \sum_{j=1}^n \frac{m_j}{[(x-\xi_j)^2 + (y-\eta_j)^2 + (z-\zeta_j)^2]^{1/2}}$$

and v is the given potential function to be represented by \tilde{v} . The ξ_j, η_j and ζ_j are fixed constants. In order to avoid singularities in the above integral it is assumed that the mass point grid is contained in the interior of some region B that does not intersect R . It is also required that the minimization be subject to the constraint

$$(1) \quad \sum_{j=1}^n m_j = M$$

where M is a given constant -- the total mass. Such a least squares criterion is used in numerous and diverse investigations and its appropriateness will not be discussed here (see, for example, Reference [3]).

For this criterion to be of any use it must be shown that f_v exists and has a unique minimum. Also, a means should be provided for the calculation of the values m_1, \dots, m_n which minimize f_v . These requirements are now taken up.

When R is a bounded region, the integral, and hence f_v , clearly exists. For the unbounded case

$$v = \frac{M}{r} + O\left(\frac{1}{r^2}\right)$$

and the constraint $\sum_{j=1}^n m_j = M$ implies that

$$\tilde{v} = \frac{M}{r} + O\left(\frac{1}{r^2}\right)$$

hence

$$v - \tilde{v} = O\left(\frac{1}{r^2}\right)$$

and so

$$(v - \tilde{v})^2 = O\left(\frac{1}{r^4}\right)$$

so that the integral in question converges. The "big oh" notation used here is the standard convention, Reference [4], that a function f of x is of the order of a function g of x as $x \rightarrow \ell$ (written $f = O(g)$ as $x \rightarrow \ell$) if there exists a positive constant K such that $|f(x)| < Kg(x)$ for all x sufficiently near to ℓ . In the present case v is considered a function of the radial coordinate r , and $\ell = \infty$.

Next, the existence of a unique minimum of f_v is demonstrated. To simplify the notation define, for (x, y, z) in R ,

$$\rho_j = [(x - \xi_j)^2 + (y - \eta_j)^2 + (z - \zeta_j)^2]^{1/2}.$$

Since R and B are disjoint regions and (ξ_j, η_j, ζ_j) lies in the interior of B ,

$$\rho_j \neq 0, \quad j=1, \dots, n.$$

Then,

$$\begin{aligned} (2) \quad \tilde{v} &= \sum_{j=1}^n \frac{m_j}{\rho_j} \\ &= \sum_{j=1}^{n-1} \frac{m_j}{\rho_j} + \frac{M - \sum_{j=1}^{n-1} m_j}{\rho_n} \\ &= \sum_{j=1}^{n-1} \left(\frac{1}{\rho_j} - \frac{1}{\rho_n} \right) m_j + \frac{M}{\rho_n} \end{aligned}$$

where the constraint (1) has been solved for m_n and introduced into (2). It is often convenient, though not essential, to let $\xi_n = \eta_n = \zeta_n = 0$ and then $\rho_n = r$, the radius in a system of polar coordinates.

From (2), and the definition of f_v (which is now thought of as a function of the $n-1$ variables m_1, \dots, m_{n-1})

$$(3) \quad \frac{\partial f_v}{\partial m_1} = - \int_R 2 (v - \tilde{v}) \frac{\partial \tilde{v}}{\partial m_1} d\tau$$

$$= -2 \int_R (v - \tilde{v}) \left(\frac{1}{\rho_1} - \frac{1}{\rho_n} \right) d\tau$$

$$(4) \quad \frac{\partial^2 f_v}{\partial m_k \partial m_j} = 2 \int_R \left(\frac{1}{\rho_k} - \frac{1}{\rho_n} \right) \left(\frac{1}{\rho_j} - \frac{1}{\rho_n} \right) d\tau$$

for $k, j = 1, \dots, n-1$. Define the $(n-1) \times (n-1)$ symmetric matrix $A = (a_{kj})$ by

$$(5) \quad a_{kj} = \int_R \left(\frac{1}{\rho_k} - \frac{1}{\rho_n} \right) \left(\frac{1}{\rho_j} - \frac{1}{\rho_n} \right) d\tau .$$

It will be shown later that, under certain mild restrictions on R , A is positive definite.

Now, conditions for a critical point of f_v are

$$(6) \quad \frac{\partial f_v}{\partial m_i} = 0, \quad i=1, \dots, n-1.$$

Using Equations (2), (3) and (5), Equation (6) reads

$$(7) \quad \sum_{j=1}^{n-1} a_{ij} m_j = \int_R \left(v - \frac{M}{\rho_n} \right) \left(\frac{1}{\rho_i} - \frac{1}{\rho_n} \right) d\tau, \quad i=1, \dots, n-1.$$

Let m be the $(n-1)$ -vector whose k^{th} component is m_k and let u be the $(n-1)$ -vector whose k^{th} component is

$$(8) \quad \int_R \left(v - \frac{M}{\rho_n} \right) \left(\frac{1}{\rho_k} - \frac{1}{\rho_n} \right) d\tau.$$

Then Equation (7) can be written as a vector equation

$$(9) \quad Am = u.$$

Two things will be shown: first, that A is a positive definite matrix and second, that f_v is a strictly convex function on E^{n-1} (Euclidean $(n-1)$ -space). If A is positive definite, then A is invertible and Equation (9) can be solved

$$(10) \quad m = A^{-1}u.$$

But f_v then has a local minimum at the critical value of m given by Equation (10) because A is positive definite, and this local minimum must also be an absolute (global) minimum if f_v is strictly convex. For the relevant definitions and theorems on convexity see, for example, Reference [5].

Thus, to show that f_v has a unique minimum it only remains to prove that A is positive definite and that f_v is strictly convex. In addition, Equation (10) gives an explicit way of calculating the mass values specified by the previously stated representation criterion.

To demonstrate that A is positive definite, consider the special case

$$v = v^* = \frac{M}{\rho_n}$$

Then,

$$\begin{aligned} f_{v^*} &= \int_R (v^* - \tilde{v})^2 d\tau \\ &= \int_R \left[\sum_{i=1}^{n-1} \left(\frac{1}{\rho_i} - \frac{1}{\rho_n} \right) m_i \right]^2 d\tau \end{aligned}$$

is certainly a positive semi-definite quadratic form in the m_i , since the integrand is never negative. To show that f_{v^*} is in fact positive definite it suffices that

$$\sum_{i=1}^{n-1} \left(\frac{1}{\rho_i} - \frac{1}{\rho_n} \right) m_i = 0$$

everywhere in R , if and only if $m_i = 0$, $i = 1, \dots, n-1$. But,

$$\sum_{i=1}^{n-1} \left(\frac{1}{\rho_i} - \frac{1}{\rho_n} \right) m_i = 0$$

if and only if

$$(11) \quad \sum_{i=1}^n \frac{m_i}{\rho_i} = \frac{M}{\rho_n}$$

everywhere in R .

Condition (11) is obviously precluded (unless $m_i = 0$, $i=1, \dots, n-1$) if R is sufficiently "big", for example if R is a volume of space rather than a finite number of points. So, f_{v*} is positive definite.

Using Equations (4) and (5)

$$\begin{aligned} \frac{\partial^2 f_{v*}}{\partial m_k \partial m_j} &= 2 \int_R \left(\frac{1}{\rho_k} - \frac{1}{\rho_n} \right) \left(\frac{1}{\rho_j} - \frac{1}{\rho_n} \right) d\tau \\ &= 2a_{kj} \end{aligned}$$

Now since f_{v*} is a positive definite quadratic form, the matrix $B = (b_{kj})$ where

$$b_{kj} = \frac{\partial^2 f_{v*}}{\partial m_k \partial m_j} = 2a_{kj}$$

is positive definite, but

$$A = \frac{1}{2}B$$

and so A is also a positive definite matrix. The positive definiteness of A implies that f_v is strictly convex (see Reference [5]) throughout E^{n-1} . So the desired unique minimum of f_v exists and it occurs at the point m of E^{n-1} given by Equation (10). The particular v^* used in the above proof was chosen only because its introduction facilitates the demonstration that A is a positive definite matrix.

IMPROVING THE MASS POINT REPRESENTATION

If a Newtonian potential is expressed as an infinite series (e.g., in spherical harmonics) then the potential at any point can be approximated as closely as desired by taking a sufficient number of terms of the series. Similarly, in the present situation, one intuitively feels that if enough mass points are used, a given potential can be described as accurately as one might require. That is, enough mass points can be employed so that when their values, as determined by Equation (10), are substituted into the expression for f_v , this function will assume a value as close to zero as is required.

It is a straightforward procedure to prove that this intuitive feeling is correct; consequently, the proof will be supplied with the hope that the mechanism of proof may further expose the role of mass points in mass modeling. The following elementary concepts applying to Riemann integration are assumed to be known: "partition," "refinement of a partition," Darboux sums and their limiting relations to integrals, and the fact that a Darboux sum derived from a refinement of a partition is at least as good an approximation to the integral as the Darboux sum derived from the original partition. References [6] and [7] are useful here.

Assume that all of the matter is contained in the interior of a compact set B and suppose that this mass distribution can be described by a continuous density function σ . The argument holds with slight modifications if only piecewise continuity for σ is assumed.

For a given partition P of B the lower Darboux sum $L(P)$ of the potential integral

$$\int_B \frac{\sigma(\xi, \eta, \zeta)}{\rho(x, y, z, \xi, \eta, \zeta)} d\tau = v(x, y, z)$$

is

$$L(P) = \sum_{i=1}^{n(P)} \frac{\sigma(\xi_1, \eta_1, \zeta_1)}{\rho(x, y, z, \xi_1, \eta_1, \zeta_1)} \Delta\tau_1$$

where $\Delta\tau_1$ is the volume of the i th cell C_1 of the partition and (ξ_1, η_1, ζ_1) is a point of C_1 where the integrand of the potential integral assumes its minimum (more precisely, infimum) in C_1 . Of course ξ_1, η_1 and ζ_1 will in general vary as x, y and z vary.

For each positive integer j let S_j be a sphere of radius j centered at the origin. Since B is compact (hence bounded) there is an integer J such that B is contained in S_j whenever $j > J$. For each $j > J$ define A_j to be the set of all points lying in S_j but not in the interior of B . Then A_j is a compact set.

Let A_j be given and let (x_0, y_0, z_0) be any point of A_j . Then given $\epsilon > 0$ there exists a partition P of B such that

$$(12) \quad \left| \int_B \frac{\sigma(\xi, \eta, \zeta)}{\rho(x_0, y_0, z_0, \xi, \eta, \zeta)} d\tau - L(P) \right| < \sqrt{\frac{\epsilon}{2}} \cdot \sqrt{\frac{3}{4\pi j^3}}.$$

And by continuity, since Equation (12) is an inequality, it holds in some neighborhood of (x_0, y_0, z_0) . Then, given any

other point of A_j , we can refine, if necessary, the partition P so that Equation (12) holds at this new point of A_j (and still at (x_0, y_0, z_0)). Continuing in this manner, find a refinement $P(x, y, z)$ of $P(x_0, y_0, z_0)$ for each point (x, y, z) in A_j so that

$$\left| \int_B \frac{\sigma(\xi, \eta, \zeta)}{\rho(x, y, z, \xi, \eta, \zeta)} d\tau - L(P(x, y, z)) \right| < \sqrt{\frac{\epsilon}{2}} \cdot \sqrt{\frac{3}{4\pi j^3}}$$

holds in a neighborhood $N(x, y, z)$ of (x, y, z) . Now, these neighborhoods constitute an open covering of the compact set A_j and hence the Heine-Borel theorem can be applied in order to find a finite subcover of neighborhoods for A_j . Let \bar{P} be the finest refinement of P associated with this finite subcover (such a finest refinement exists because of the finiteness). Then,

$$(13) \quad \left| \int_B \frac{\sigma(\xi, \eta, \zeta)}{\rho(x, y, z, \xi, \eta, \zeta)} d\tau - L(\bar{P}) \right| < \sqrt{\frac{\epsilon}{2}} \cdot \sqrt{\frac{3}{4\pi j^3}}$$

is valid for every point (x, y, z) in A_j .

Associated with \bar{P} is the lower Darboux sum

$$L(\bar{P}) = \sum_{i=1}^{n(\bar{P})} \frac{\sigma(\xi_i, \eta_i, \zeta_i)}{\rho(x, y, z, \xi_i, \eta_i, \zeta_i)} \Delta\tau_i.$$

Define

$$\sigma(\xi_i, \eta_i, \zeta_i) \Delta\tau_i = m_i, \quad i=1, \dots, n(\bar{P})$$

$$L(\bar{P}) = \tilde{v}_{\bar{P}} = \sum_{i=1}^{n(\bar{P})} \frac{m_i}{\rho(x, y, z, \xi_i, \eta_i, \zeta_i)} .$$

Since the integrand in the definition of f_v (for any v) is $O\left(\frac{1}{r^4}\right)$, there exists a sphere $S_k(\bar{P})$ such that

$$\int_{R-S_k(\bar{P})} (v-\tilde{v}_{\bar{P}})^2 d\tau < \frac{\epsilon}{2}$$

Also,

$$(14) \int_R (v-\tilde{v}_{\bar{P}})^2 d\tau = \int_{A_j} (v-\tilde{v}_{\bar{P}})^2 d\tau + \int_{R-S_j} (v-\tilde{v}_{\bar{P}})^2 d\tau .$$

So, if $k(\bar{P}) \leq j$ (then $S_k(\bar{P})$ is contained in S_j) we have

$$(15) \int_{R-S_j} (v-\tilde{v}_{\bar{P}})^2 d\tau \leq \int_{R-S_k(\bar{P})} (v-\tilde{v}_{\bar{P}})^2 d\tau < \frac{\epsilon}{2} .$$

Then putting Equations (13) and (15) into Equation (14) gives

$$\int_R (v-\tilde{v}_{\bar{P}})^2 d\tau < \left(\sqrt{\frac{\epsilon}{2}} \cdot \sqrt{\frac{3}{4\pi j^3}} \right)^2 \frac{4}{3}\pi j^3 + \frac{\epsilon}{2} = \epsilon .$$

If on the contrary $k(\bar{P}) > j$, then refine \bar{P} (in the same way that P was refined) to \bar{P} so that

$$\left| \int_B \frac{\sigma(\xi, \eta, \zeta)}{\rho(x, y, z, \xi, \eta, \zeta)} d\tau - L(\bar{P}) \right| < \sqrt{\frac{\epsilon}{2}} \cdot \sqrt{\frac{3}{4\pi(k(\bar{P}))^3}}$$

holds for all points of $A_k(\bar{P})$. The inequality (15) is still true and so the approximation theorem has been proved using the masses and positions derived from \bar{P} or $\bar{\bar{P}}$.

It is not difficult to strengthen this result and show that the approximation can be "brute forced" without a knowledge of σ . That is, if enough mass point locations are chosen (unknown mass values) and these positions are packed densely enough in the interior of B , then the mass values determined by Equation (10) will result in f_v assuming a value less than any pre-assigned $\epsilon > 0$. This follows in a straightforward way from the preceding approximation result, the continuity of f_v in mass values and mass point coordinates, and the minimality property of Equation (10).

As to how many points are "enough" and how densely they should be packed to insure a called-for accuracy, no answer can be given without a knowledge of σ . The message is: work hard enough and you will be rewarded. Of course "dense packing" is not suggested as necessarily the best procedure. An intelligent choice of mass point locations may greatly reduce the number of unknown mass values which the computer must handle. Such selective placing can be based on geological data or partial knowledge of the potential from previous studies. These more efficient procedures presuppose the existence of information on the behavior of mass points: a catalog of mass point placements which best fit given situations. The following study on oblateness is a first step in this direction.

THE POTENTIAL OF AN OBLATE BODY

The use of Equation (10) will now be illustrated by the presentation of a simple example. In the course of this presentation an integral formula due to W. W. Ennis, Reference [8], will be used. This formula is of general interest in the application of Equation (10).

It is desired to represent (in the sense of our least squares integral approximation) the potential of a homogeneous solid, whose surface is an oblate spheroid, by using three fixed mass points: one located at the center of mass and the other two on the axis of symmetry at distances $+h$ and $-h$ from the central mass point. Such a configuration, which was suggested by D. H. Novak [9], provides a capability for "pulling in the poles" of the sphere defined by the central mass in order to better represent the oblate spheroid. Negative values for the two satellite masses are anticipated.

In rectangular coordinates the equation of an oblate spheroid is

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1, \quad a > c.$$

Let the total mass of the solid be M . Then in spherical coordinates r, θ, ϕ with ϕ the co-latitude, the potential is, Reference [10],

$$(16) \quad v = 3M \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \cdot \frac{(ae)^{2n}}{r^{2n+1}} P_{2n}(\cos \phi).$$

Here, P_m is the m^{th} Legendre polynomial and

$$e = \frac{\sqrt{a^2 - c^2}}{a}.$$

The expansion given by Equation (16) converges for all $r > a$.

In the previously established notation

$$\frac{1}{\rho_1} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{h}{r}\right)^n P_n(\cos\phi)$$

$$\frac{1}{\rho_2} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{-h}{r}\right)^n P_n(\cos\phi)$$

$$\frac{1}{\rho_3} = \frac{1}{r}$$

results which can be found in Reference [10]. In the present case, Equation (9) reads

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

By symmetry,

$$m_1 = m_2$$

$$u_1 = u_2$$

and from Equation (8)

$$u_1 = \int_R (v - \frac{M}{r}) \left(\frac{1}{\rho_1} - \frac{1}{r} \right) d\tau.$$

Let R be all space outside a sphere of radius b with center at the origin and $b > a$. Then,

$$\begin{aligned}
 u_1 &= \int_{r=b}^{\infty} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \left[3M \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \cdot \frac{(ae)^{2n}}{r^{2n+1}} P_{2n}(\cos\phi) \right] \\
 &\quad \cdot \left[\frac{1}{r} \sum_{n=1}^{\infty} \left(\frac{h}{r} \right)^n P_n(\cos\phi) \right] r^2 \sin\phi d\theta d\phi dr \\
 &= 2\pi \int_{r=b}^{\infty} \int_{\phi=0}^{\pi} 3M \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \cdot \frac{(ae)^{2n}}{r^{2n+1}} P_{2n}(\cos\phi) \right] \\
 &\quad \cdot \left[\frac{1}{r} \sum_{n=1}^{\infty} \left(\frac{h}{r} \right)^n P_n(\cos\phi) \right] r^2 \sin\phi d\phi dr \\
 &= 6\pi M \int_{r=b}^{\infty} \sum_{n=1}^{\infty} \frac{2(-1)^n}{(2n+1)(2n+3)(4n+1)} \cdot \frac{(ae)^{2n}}{r^{2n+1}} \cdot \frac{1}{r} \cdot \left(\frac{h}{r} \right)^{2n} r^2 dr \\
 &= 12\pi M \int_{r=b}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)(4n+1)} \cdot \frac{(aeh)^{2n}}{r^{4n}} dr \\
 &= 12 Mb \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)(16n^2-1)} \cdot \left(\frac{a^2 e^2 h^2}{b^4} \right)^n.
 \end{aligned}$$

It is evident that u_1 (and hence u_2) is a negative quantity.

The orthogonality relations

$$\int_{\phi=0}^{\pi} P_n(\cos\phi) P_m(\cos\phi) \sin\phi d\phi = \begin{cases} 0 & , \text{ if } n \neq m \\ \frac{2}{2n+1} & , \text{ if } n=m \end{cases}$$

were used in evaluating the second integral. Now, Ennis' formula is

$$(17) \int_R \left(\frac{1}{\rho_1} - \frac{1}{\rho_n} \right) \left(\frac{1}{\rho_j} - \frac{1}{\rho_n} \right) d\tau \\ = 4\pi \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cdot \frac{(r_1 r_j)^n}{b^{2n-1}} P_n(\cos\phi_j)$$

where the region R is all space outside of the sphere of radius b with center at the origin and r_1 and r_j are the distances of m_1 and m_j from the origin. The co-latitude of m_j is ϕ_j , and m_1 is assumed to lie on the polar axis ($\phi=0$).

Applying Equation (17)

$$a_{11} = \int_R \left(\frac{1}{\rho_1} - \frac{1}{\rho_3} \right)^2 d\tau \\ = 4\pi \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cdot \frac{h^{2n}}{b^{2n-1}}$$

= a_{22} by symmetry.

Also,

$$\begin{aligned}
 a_{12} &= \int_R \left(\frac{1}{\rho_1} - \frac{1}{\rho_3} \right) \left(\frac{1}{\rho_2} - \frac{1}{\rho_3} \right) d\tau \\
 &= 4\pi \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} \cdot \frac{h^{2n}}{b^{2n-1}} = a_{21} .
 \end{aligned}$$

Then,

$$m_1 = m_2 = \frac{\begin{vmatrix} u_1 & a_{12} \\ u_2 & a_{22} \\ a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{a_{11} + a_{12}}$$

$$\begin{aligned}
 &= \frac{12\pi Mb \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)(16n^2-1)} \left(\frac{a^2 e^{2h^2}}{b^4} \right)^n}{4\pi \sum_{n=1}^{\infty} \frac{1+(-1)^n}{4n^2-1} \cdot \frac{h^{2n}}{b^{2n-1}}} \\
 &= 3M \frac{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)(16n^2-1)} \left(\frac{aeh}{b^2} \right)^{2n}}{\sum_{n=1}^{\infty} \frac{2}{16n^2-1} \left(\frac{h}{b} \right)^{4n}} .
 \end{aligned}$$

It is clear that the series in the numerator and denominator converge even when $b = a$. It is required, of course, that $h < a$. It is easily seen that m_1 and m_2 are negative quantities as anticipated.

SUMMARY AND CONCLUSIONS

It has been shown that fixed mass points can be used to represent any potential unambiguously and with as

high a degree of accuracy as is required. To obtain such a representation it is only necessary to solve a system of linear equations.

Further studies seem to fall naturally into two categories: implementation and extension of the fixed grid method, and development of a "smarter" system, i.e., one with more freedom to minimize residuals. However, any relaxation of the fixed grid method must be done with care. The fact that the potential in R does not uniquely determine the mass distribution in B, and the more complicated mathematics which ensues, are major obstacles to granting mass points complete freedom of choice of location. This non-uniqueness could result in convergence problems for the mass point locations and values. A possible alternative is to impose constraints weaker than fixed grid but stronger than absolute choice of location. The mass points could be required to lie on some given surface, or their distance from the origin allowed to vary along the resulting fixed radius. Again, difficulties may arise, such as non-uniqueness resulting from symmetries in the potential. Nevertheless, relaxation of the fixed grid constraint merits further attention both theoretically and in practical application.

Moreover, interest in the potential of a body usually centers on the dynamical situation in which the body participates. Thus, fixed grid and other mass modeling studies of potential representation need to be extended to an examination of how well this representation reproduces the dynamical behavior of the system. For example, it would be useful to compare the orbit of a satellite about a primary with a specified potential function v to the orbit of the satellite using the same initial conditions but replacing v with a mass point representation. The virtual mass technique, Reference [11], is ideal for computing trajectories in the mass point case, while either numerical integration of the equations of motion or exact analysis (for simple v 's) could be used to compute the comparison orbit.

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Attachment
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