Bellcomm

955 L'Enfant Plaza North, S.W.
Washington, D. C. 20024

B71 05025

date: May 25, 1971

to: Distribution

from: S. L. Levie, Jr.

subject: On Determining the Moon's Density Function
Case 310

ABSTRACT

This paper is a qualitative study of the relations between the moon's density function, figure, external gravitational potential, and physical libration constants. It gives a personal assessment of the consistency of present libration and gravitational information, concluding that the consistency is poor. It is suggested that the moon's density function will be accessible theoretically when the figure and the gravitational potential are known with confidence.

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MEMORANDUM FOR FILE

Introduction

This paper is a qualitative study of the relations between the moon's density function, figure, external gravitational potential, and physical libration constants. It provides a personal assessment of the usefulness of existing lunar data and offers thoughts on what the significance of improved data will be. It concludes that global data of good quality and utility will resolve conflicts between presently available results and permit intercomparison of the various types of results. In particular, the moon's near-homogeneity and near-sphericity will permit the study of the density function if high-quality, high-utility observational data become available.

Relations Between Gravity and Figure

The external gravitational potential of an arbitrary body is the following solution of Laplace's equation:

$$U(r,\theta,\phi) = G\int_0^{2\pi}\int_{-1}^{1}\int_0^{s(\theta',\phi')} \frac{\rho(r',\theta',\phi')}{(r^2+r'^2-2r\,r'\cos\gamma)^{1/2}}\, r'^2\, dr'\, d(\cos\theta')\, d\phi' . \tag{1}$$

In this formula (r, θ, ϕ) are the spherical polar coordinates of an external field point, and (r', θ', ϕ') are the spherical polar coordinates of a differential element of the body's mass; the coordinate origin is near the center of the figure. γ is the angle between the vectors $\bar{r}$ and $\bar{r}'$, G is the universal gravitational constant, ρ is the body's density function, and s is a function specifying the body's surface. It will be assumed that s can be written as an infinite series expansion in surface spherical harmonics:

$$s(\theta',\phi') = s_0\left(1 + \sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell} P_\ell^{\,m}(\cos\theta') \times\left[j_{\ell m}\cos m\phi' + j'_{\ell m}\sin m\phi'\right]\right). \tag{2}$$

In this expression s_0 is the mean radius, and $j_{\ell m}$ and $j'_{\ell m}$ are constants which fully characterize the surface. $P_\ell^{\,m}$ denotes the associated Legendre function with the normalization of Emde and Jahnke (1945).

In order to integrate (1), it will be assumed that $j_{\ell m}$ and $j'_{\ell m}$ are small enough for powers of s to be approximated by keeping only the linear terms in $j_{\ell m}$ and $j'_{\ell m}$ in a binominal expansion of (2). Physically, this assumption restricts the body under consideration to small deviations from sphericity.

Under the additional assumption of homogeneity, the integrals in (1) may be performed*, resulting in

$$U(r,\theta,\phi) = \frac{GM}{r}\left\{1 + \sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell}\left(\frac{s_0}{r}\right)^{\ell} P_\ell^{\,m}(\cos\theta) \times\left[C_{\ell m}\cos m\phi + S_{\ell m}\sin m\phi\right]\right\}, \tag{3}$$

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* The process involves expanding the reciprocal radius factor in solid spherical harmonics, performing the radial integration, keeping only first order terms in the binominal expansion of s, and using the orthogonality of the associated Legendre functions to perform the angular integrations.

where

$$\begin{pmatrix} C_{\ell m} \\ S_{\ell m} \end{pmatrix} = \frac{3}{2\ell+1} \begin{pmatrix} j_{\ell m} \\ j'_{\ell m} \end{pmatrix} \qquad (4)$$

and M is the body's mass. It can be shown that expression (3) is perfectly general, the assumptions of homogeneity and near-sphericity appearing only in (4). With weaker assumptions, generalizations of (4) are possible.

The result (4) shows how the external gravitational potential of a homogeneous, nearly spherical body depends on the body's shape. Since these assumptions are very natural ones to make in the moon's case, the result provides strong motivation for controlled mapping of the entire lunar surface, in order to refine knowledge of the lunar gravitational potential.* This mapping would be practical only if carried out by automatic lunar satellites.**

Controlled mapping may be used to examine the homogeneity hypothesis, by comparing the figure-determined potential against the potential determined from the dynamics of lunar satellites, as described by Lorell (1970) for example. In particular, the structure of the lunar crust, and also independent information about the lunar mass inhomogeneities, may be determined in this manner. In addition, the altitudes and angles available from the mapping satellites constitute new data types usable in the conventional method of determining the lunar gravity coefficients directly by regression techniques [Koch, 1970].

Connection Between the Librations and the Gravitational Potential

Another way of constructing a gravitational potential would be available if all the integrals of inertia of the body were known. This follows from MacMillan's demonstration [MacMillan, 1958] that a gravitational potential function may

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* Mapping of only the nearside, no matter how carefully performed, can be of no use for the kind of studies envisioned.

** Although they were used to map large regions of the moon, the United States' Lunar Orbiter satellites could not perform controlled mapping, since precisely known camera pointing angles were not provided for.

be expanded in an infinite sum over the body's integrals of inertia. In the moon's case, indirect information on the principal moments of inertia is available from astronomical observations of the physical librations.

It appears that the most reliable reductions of the old heliometer data on the librations have been carried out by Koziel [Koziel, 1967], although it is difficult to test reliability or even to compute propagated error limits. Koziel computed the two constants β and f, defined as follows. A principal axis frame for the moon may be defined under the assumption that the x-axis is in the mean direction of earth, the z-axis is in the direction of the north lunar pole, and the y-axis completes the right-handed triad. Letting A, B, and C be the principal moments of inertia about the x-, y-, and z-axes, respectively, β and f are defined as

$$\left.\begin{aligned} \beta &= \frac{C-A}{B} \\ f &= \frac{B}{A}\,\frac{C-B}{C-A} \end{aligned}\right\} . \qquad (5)$$

These two parameters appear along with several others in the Euler equations for the moon's rigid body motion. Assuming that the moon is rigid, that the potential terms above $\ell = 2$ in (3) may be ignored, and that the only external influence on the librations is due to a central body earth [Eckhardt, 1970], Koziel was able to compute all unknown constants by matching the observations to the predictions of the Euler equations, in a least squares sense. The values of β and f from his solution are given in Table 1. The error limits* refer only to goodness of fit; they exclude all observational errors. According to Eckhardt (1970), the error in β is within the error expected from assuming the moon to be rigid. Thus Koziel's results are probably as precise as one can obtain without using a more precise formulation of the dynamical system.

Jeffreys (1967) has given values for β and f based on more recent observations by Yakovkin (unreferenced). These values are also listed in Table 1. That the results are in

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* Koziel did not provide error limits for β; these have been computed by the author under the assumption of independent contributions from all sources.

substantial disagreement with Koziel's was recognized by Jeffreys. More will be said about this in a moment. Still different values from other investigators were collected by Watts (1955).

The values of β and f depend on the moments of inertia in the manner shown above, and the moments in turn depend on the moon's density function and figure. Assuming as in the last section that the moon is homogeneous and nearly spherical, then by employing the same technique used in integrating (1) it can be shown [Goudas, 1964] that

$$\left.\begin{aligned} A &= \frac{2}{5} Ms_0^{\,2} \left(1 + \frac{j_{20}}{2} - 3j_{22} \right) \\ B &= \frac{2}{5} Ms_0^{\,2} \left(1 + \frac{j_{20}}{2} + 3j_{22} \right) \\ C &= \frac{2}{5} Ms_0^{\,2} \left(1 - j_{20} \right) \end{aligned}\right\} . \qquad (6)$$

This says that the moments of inertia depend only on the oblateness and ellipticity of the figure and that these moments are only slightly different from the value for a homogeneous sphere. Using (5) and (6) and keeping only first order terms in β, then

$$\left.\begin{aligned} j_{20} &\doteq -\frac{\beta}{3} (1 + f) \\ j_{22} &= \frac{\beta}{6} (1 - f) \end{aligned}\right\} . \qquad (7)$$

Using (4), it follows that

$$\left.\begin{aligned} C_{20} &= -\frac{\beta}{5} (1 + f) \\ C_{22} &= \frac{\beta}{10} (1 - f) \end{aligned}\right\} . \qquad (8)$$

This was obtained by Lorell (1970), who used instead of (6) theoretical relations involving the principal moments of inertia C_{20} and C_{22}.

Assuming the pairs of β and f given by Koziel and Jeffreys, Table 1 gives the corresponding low-order figure and gravitational coefficients, computed from (7) and (8). The uncertainties in the coefficients are dominated by the uncertainties in f. For comparison, Table 2 gives the low-order gravitational coefficients from several independent determinations using dynamical data from lunar satellites. The results may be grouped into four sets. One set includes the Koziel values from Table 1, Michael's values [Michael et al., 1969], Sjogren's values [Sjogren, 1971], and Boeing Aircraft's Apollo lunar potential model values [Risdal, 1968; and Wollenhaupt, 1970]. Each of the other sets is extracted from individual references. They are Jeffreys (Table 1), Liu and Laing (1971), and Akim (1971).

Unfortunately, about the only thing that can be concluded from Table 2 is that the entries in the first set probably cluster about the true values, since several independent sources produced the clustering. A tempting conclusion is that Koziel's results for β and f are fair values for these constants, since they lead to favorable comparisons with other results. But the error of this conclusion is understood when it is realized that the Boeing coefficients, which compare well with Koziel's, are based on early reductions of Yakovkin's data by Jeffreys,* the libration results of which are grossly inconsistent with Koziel's. Furthermore, Jeffreys' new values for β and f lead to different values of the second degree coefficients. Clearly, at present there is no way of choosing between the astronomical solutions for β and f on the basis of comparisons with lunar satellite results.

This confusion about the libration constants, which has persisted throughout this century, probably can be overcome through the use of modern observational instruments and techniques [Moutsoulas, 1970a]. The data for all of the existing libration calculations suffer from at least one of the following defects, enumerated by Meyer and Ruffin (1965): 1) marginal instrument resolving power, 2) image distortion from atmospheric turbulence, and 3) improperly chosen phase conditions. The first one has been cited many times by Kopal (cf. [Kopal, 1964]) as being a certain corruptor of the

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* A sequence of references connecting Boeing's Apollo coefficients to Yakovkin is [Wollenhaupt, 1970], [Melbourne et al., 1968], [Clark, 1964], [Makemson et al., 1961], [Alexandrov, 1960[, [Jeffreys, 1957], and [Yakovkin, 1952].

heliometer data. Efforts are now under way at the University of Manchester* to obtain new photographs from which the libration reductions may be performed with confidence. In addition, a new method of obtaining libration data, which uses laser ranging between observatories and corner reflectors placed on the moon, is being employed by a group of American scientists [Alley, 1965; 1969]. Thus there is hope of obtaining reliable values of β and f in this century.

Until this happens, the lunar constants determined with artificial satellites will have the most reliable values, although the spread in Table 2 shows that questions remain concerning the quality of those results, too. We have, after all, only one moon. The spread is even worse for higher order coefficients. The questions arise due to, 1) absence of farside tracking for the satellites, 2) incomplete span of orbital inclinations, 3) tracking coverage which was often insufficient for geometrical resolution of the data, and 4) presence of uncoupled thrusting during times of best coverage (cf. [Lorell, 1970]). It must be hoped that these problems will be overcome in the future by a new series of lunar satellites specifically designed for selenodesy.

Conclusions

This paper has arrived at two main conclusions, each suggesting a new system of lunar satellites. The first is that controlled mapping of the lunar surface can be used to make inferences about the lunar gravitational field or, by using the mapping along with good information about the gravitational field, to investigate the moon's internal structure in some detail. The required mapping can only be carried out from lunar satellites due to the inaccessibility of the lunar farside.

The second conclusion is that information on the moon's physical librations and low-order gravitational field is of marginal consistency, with the gravitational information appearing to be of somewhat better quality than the libration information. Correction of the libration difficulties seems to have a good start. Correction of the gravitational difficulties, which is not in sight, will require a carefully planned and operated system of selenodetic satellites. It is possible, but not obvious, that the mapping program can be carried out from the selenodetic satellites.

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Attachments
Tables 1 & 2

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* The indicated work is discussed in the following papers: [Mills, 1967 and 1968], [Mills and Sudbury, 1968], [Moutsoulas, 1970b].

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TABLE 1

Lunar libration coefficients from Koziel and Jeffreys, and their resulting second order figure and gravitational coefficients. The figure and gravitational coefficients are computed under the assumptions of homogeneity and near-sphericity.

		Koziel[a]	Jeffreys[b]
Libration Ratios	$10^4\ \beta$	6.294 ± 0.006	6.279 ± 0.015
	f	0.633 ± 0.004	0.674 ± 0.002
Figure	$10^4\ j_{20}$	-3.426 ± 0.007	-3.503 ± 0.009
	$10^4\ j_{22}$	0.385 ± 0.004	0.342 ± 0.002
Gravity	$10^4\ C_{20}$	-2.056 ± 0.004	-2.102 ± 0.006
	$10^4\ C_{22}$	0.231 ± 0.002	0.205 ± 0.001

a) [Koziel, 1967]. The error in Koziel's value of β was computed by the author under the assumption of independence of error sources.

b) [Jeffreys, 1967]. f and its error were computed by the author from numbers given by Jeffreys under the assumption of independence of error sources.

TABLE 2

Comparison of second order lunar gravitational coefficients from several sources.

Sources	Coefficients	
	$C_{20} \times 10^4$	$C_{22} \times 10^4$
Boeing[a]	-2.07	0.21
Koziel[b]	-2.06	0.23
Michael et al[c]	-2.07	0.22
Sjogren[d]	-2.05	0.22
Jeffreys[e]	-2.01	0.21
Liu and Laing[f]	-2.00	0.24
Akim[g]	-2.06	0.14

a) [Risdal, 1968; and Wollenhaupt, 1970].

b) Table 1.

c) [Michael et al, 1969].

d) [Sjogren, 1971].

e) Table 1.

f) [Liu and Laing, 1971].

g) [Akim, 1971].