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from: W. G. Heffron, S. L. Levie, Jr.

subject: Unnormalized Associated Legendre Functions: A Compendium of Graphs, Expansions, Properties, and Integrals -- Case 310

MEMORANDUM FOR FILE

Attached is a useful compendium of graphs, expansions, properties and integrals of the unnormalized associated Legendre functions, denoted $P_\ell^m(x)$, for $|x| \leq 1$. The material which is new (i.e., not available from the compendiums included in the references) is

1. Expressions for $P_\ell^m(x)$ up to P_{10}^{10}.
2. Graphs of $P_\ell^m(x)$ up to P_{10}^{10}.
3. Fourier decompositions of $P_\ell^m(\sin\theta)$ and $P_\ell^m(\cos\theta)$ up to P_{10}^{10}.
4. Least upper bound information.
5. Derivatives of solid spherical harmonics.
6. The integrals $I_a(\ell, i, n, j)$ and $J_p(s, n, m)$.

The term "unnormalized" refers to a particular convention in the definition of $P_\ell^m(x)$. The convention adopted is the one used by Battin (1964), Emde and Jahnke (1945), Kaula (1966), and Korn and Korn (1961), as well as others. It differs by a normalization factor from other conventions in use.

Additional copies of this compendium are available from the authors. Corrections and suggestions are solicited.

Mrs. Sheryl Watson provided the graphs in Figures 1 through 12.

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Attachment

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Unnormalized Associated Legendre Functions: A Compendium of Graphs, Expansions, Properties, and Integrals

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TABLE OF CONTENTS

TABLE OF CONTENTS

(Continued)

UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS

Definitions, Functional Forms, and Graphs

The associated Legendre functions satisfy the differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]u = 0.$$

It will be assumed that ℓ and m are natural numbers, $0 \le m \le \ell$, and $|x| \le 1$. With these restrictions, the unnormalized associated Legendre functions are completely characterized by

$$u \equiv P_\ell^{\,m}(x) = \frac{(1-x^2)^{m/2}}{2^\ell \ell!}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \qquad \text{RODRIGUES' FORMULA}$$

$$= (1-x^2)^{m/2} \sum_{k=0}^{\left[\frac{\ell-m}{2}\right]} T_{\ell m k}\, x^{\ell-m-2k},$$

in which

$[b] \equiv$ greatest integer in b

$$T_{\ell m k} = \frac{(-)^k\,(2\ell-2k)!}{2^\ell k!\,(\ell-k)!\,(\ell-m-2k)!}.$$

When $m=0$ these solutions are called Legendre's polynomials, denoted $P_\ell(x)$.

The definitions above specify the "unnormalized" associated Legendre functions, which are used throughout this compendium. The formulas herein may be converted to other fairly common normalizations with the following relations:

$$P_\ell^m = (-)^m P_{\ell m}^{I} = \sqrt{\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}}\; P_{\ell m}^{II}$$

$$= \sqrt{\frac{(\ell+m)!}{(2-\delta_{m0})(2\ell+1)(\ell-m)!}}\; P_{\ell m}^{III} = \sqrt{\frac{(\ell+m)!}{(2-\delta_{m0})(\ell-m)!}}\; P_{\ell m}^{IV}$$

where

$P_{\ell m}^{I}$ is common in theoretical physics

$P_{\ell m}^{II}$ gives value unity for $\frac{1}{2}\int_{-1}^{1}\left[P_{\ell m}^{II}(x)\right]^2 dx$ (latitude normalization)

$P_{\ell m}^{III}$ gives value unity for $\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\cos^2 m\phi\left[P_{\ell m}^{III}(x)\right]^2 dx\,d\phi$ (sphere normalization)

$P_{\ell m}^{IV}$ gives value $1/(2\ell+1)$ for $\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\cos^2 m\phi\left[P_{\ell m}^{IV}(x)\right]^2 dx\,d\phi$ (another sphere normalization).

The unnormalized associated Legendre functions up to $P_{10}^{10}(x)$ are listed in Table 1 as functions of x. They are plotted in Figures 1 through 12.

The functions $P_\ell^m(\sin\theta)$ may be Fourier-decomposed with the following formula:

$$P_\ell^m(\sin\theta) = \sum_{\substack{q=c \\ \text{by 2's}}}^{\ell} \left\{ \begin{matrix} \cos q\theta \\ \sin q\theta \end{matrix} \right\} \begin{matrix} (\ell-m)\,\text{even} \\ (\ell-m)\,\text{odd} \end{matrix} \sum_{\substack{p=\max(m-\ell,q-m) \\ \text{by 2's}}}^{\min(\ell-m,q+m)} \sum_{\substack{k=|p| \\ \text{by 2's}}}^{\ell-m}$$

$$\left\{ \frac{(-)^{[(3k+\ell-m-p+1)/2]}}{2^{k+\ell+m}} \frac{(k+\ell+m)!}{k!\,\ell!} (2-\delta_{0q}) \times \binom{\ell}{\frac{k+\ell+m}{2}} \binom{k}{\frac{k-p}{2}} \binom{m}{\frac{m+p-q}{2}} \right\},$$

where $c = \begin{cases} 0 \text{ if } \ell = \text{even} \\ 1 \text{ if } \ell = \text{odd} \end{cases}$

and $[b] \equiv$ greatest integer in b.

These decompositions are given explicitly in Table 2 up to $P_{10}^{10}(\sin\theta)$. Table 2 also lists the Fourier decompositions of $P_\ell^m(\cos\theta)$ up to $P_{10}^{10}(\cos\theta)$, which are related to the decompositions of $P_\ell^m(\sin\theta)$ through the replacement of θ by $\frac{\pi}{2}-\theta$.

TABLE 1

EXPRESSIONS FOR UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS UP TO $P_{10}^{10}(x)$

EACH OF THESE FUNCTIONS CAN BE WRITTEN IN THE FORM:

$$P_\ell^m(x) = (1-x^2)^p \sum_{i=0}^{\ell-m} a_i x^i$$

THE TABLE LISTS ℓ, m, p, AND THE a_i FOR EACH OF THE FUNCTIONS UP TO $P_{10}^{10}(x)$.

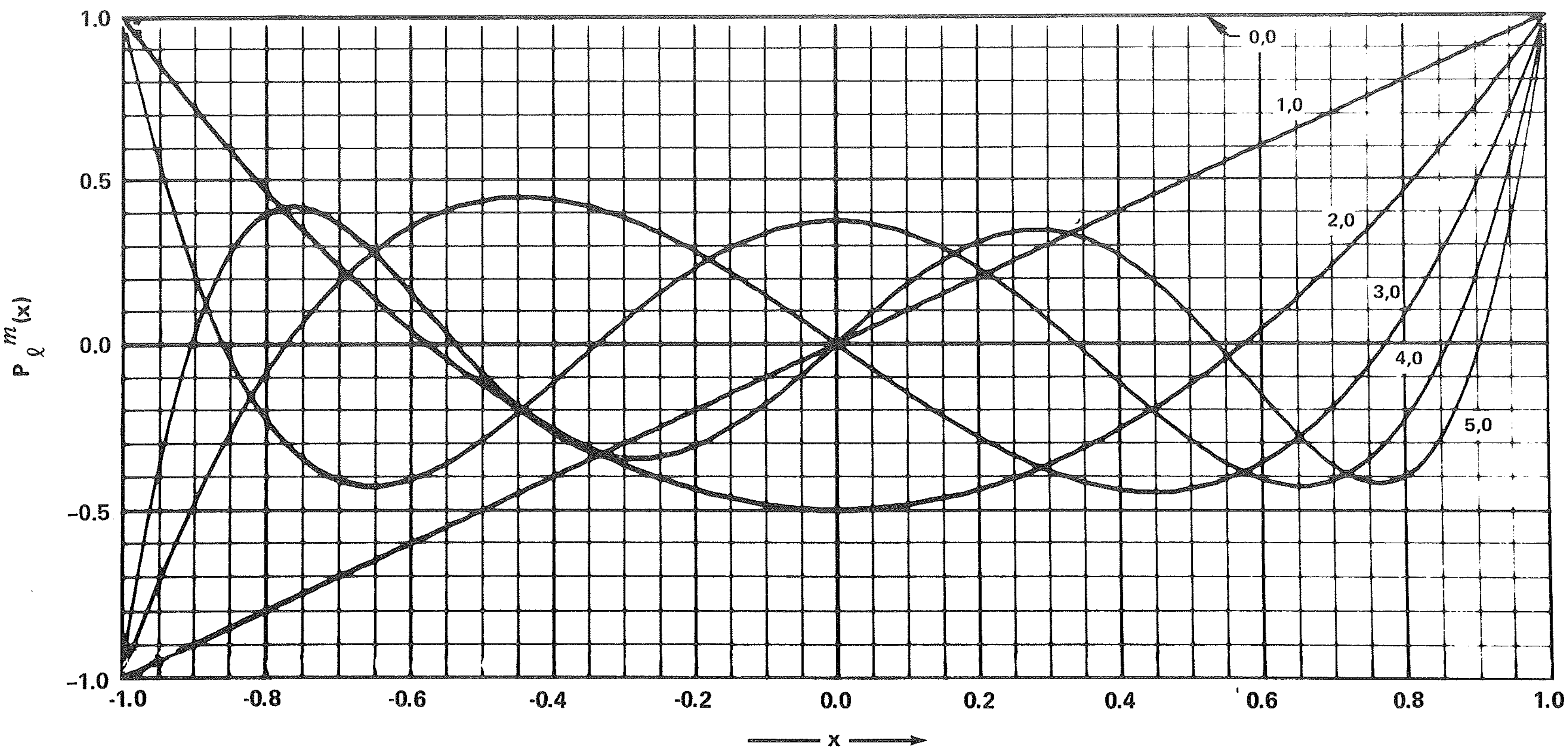
ℓ	m	p	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
0	0	0	1										
1	0	0	0	1									
2	0	0	−1/2	0	3/2								
3	0	0	0	−3/2	0	5/2							
4	0	0	3/8	0	−15/4	0	35/8						
5	0	0	0	15/8	0	−35/4	0	63/8					
6	0	0	−5/16	0	105/16	0	−315/16	0	231/16				
7	0	0	0	−35/16	0	315/16	0	−693/16	0	429/16			
8	0	0	35/128	0	−315/32	0	3465/64	0	−3003/32	0	6435/128		
9	0	0	0	315/128	0	−1155/32	0	9009/64	0	−6435/32	0	12155/128	
10	0	0	−63/256	0	3465/256	0	−15015/128	0	45045/128	0	−109395/256	0	46189/256
1	1	1/2	1										
2	1	1/2	0	3									
3	1	1/2	−3/2	0	15/2								
4	1	1/2	0	−15/2	0	35/2							
5	1	1/2	15/8	0	−105/4	0	315/8						
6	1	1/2	0	105/8	0	−315/4	0	693/8					
7	1	1/2	−35/16	0	945/16	0	−3465/16	0	3003/16				
8	1	1/2	0	−315/16	0	3465/16	0	−9009/16	0	6435/16			
9	1	1/2	35/128	0	−3465/32	0	45045/64	0	−45045/32	0	109395/128		
10	1	1/2	0	3465/128	0	−15015/32	0	135135/64	0	−109395/32	0	230945/128	

TABLE 1 (CONTINUED)

ℓ	m	p	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
2	2	1	3										
3	2	1	0	15									
4	2	1	−15/2	0	105/2								
5	2	1	0	−105/2	0	315/2							
6	2	1	105/8	0	−945/4	0	3465/8						
7	2	1	0	945/8	0	−3465/4	0	9009/8					
8	2	1	−315/16	0	10395/16	0	−45045/16	0	45045/16				
9	2	1	0	−3465/16	0	45045/16	0	−135135/16	0	109395/16			
10	2	1	3465/128	0	−45045/32	0	675675/64	0	−765765/32	0	2078505/128		
3	3	3/2	15										
4	3	3/2	0	105									
5	3	3/2	−105/2	0	945/2								
6	3	3/2	0	−945/2	0	3465/2							
7	3	3/2	945/8	0	−10395/4	0	45045/8						
8	3	3/2	0	10395/8	0	−45045/4	0	135135/8					
9	3	3/2	−3465/16	0	135135/16	0	−675675/16	0	765765/16				
10	3	3/2	0	−45045/16	0	675675/16	0	−2297295/16	0	2078505/16			
4	4	2	105										
5	4	2	0	945									
6	4	2	−945/2	0	10395/2								
7	4	2	0	−10395/2	0	45045/2							
8	4	2	10395/8	0	−135135/4	0	675675/8						
9	4	2	0	135135/8	0	−675675/4	0	2297295/8					
10	4	2	−45045/16	0	2027025/16	0	−11486475/16	0	14549535/16				
5	5	5/2	945										
6	5	5/2	0	10395									
7	5	5/2	−10395/2	0	135135/2								
8	5	5/2	0	−135135/2	0	675675/2							

TABLE 1 (CONCLUDED)

ℓ	m	p	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
9	5	5/2	135135/8	0	−2027025/4	0	11486475/8						
10	5	5/2	0	2027025/8	0	−11486475/4	0	43648605/8					
6	6	3	10395										
7	6	3	0	135135									
8	6	3	−135135/2	0	2027025/2								
9	6	3	0	−2027025/2	0	11486475/2							
10	6	3	2027025/8	0	−34459425/4	0	218243025/8						
7	7	7/2	135135										
8	7	7/2	0	2027025									
9	7	7/2	−2027025/2	0	34459425/2								
10	7	7/2	0	−34459425/2	0	218243025/2							
8	8	4	2027025										
9	8	4	0	34459425									
10	8	4	−34459425/2	0	654729075/2								
9	9	9/2	34459425										
10	9	9/2	0	654729075									
10	10	5	654729075										



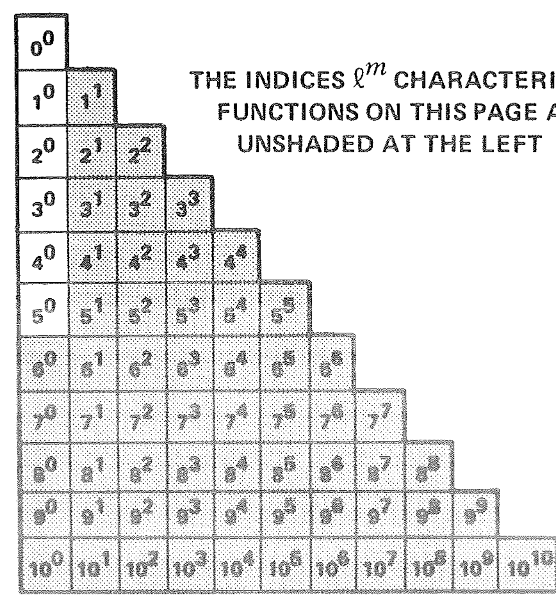


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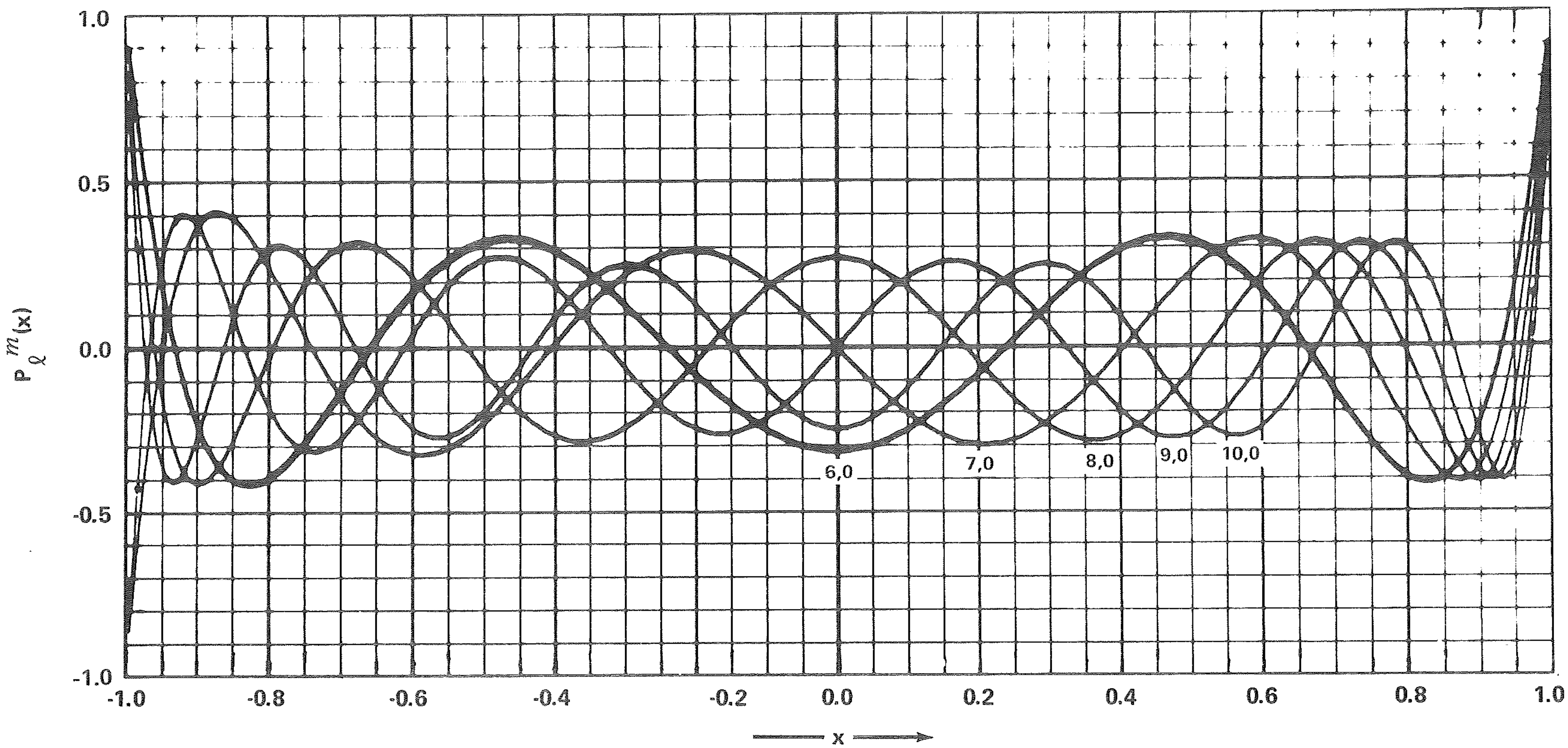
FIGURE 1 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS



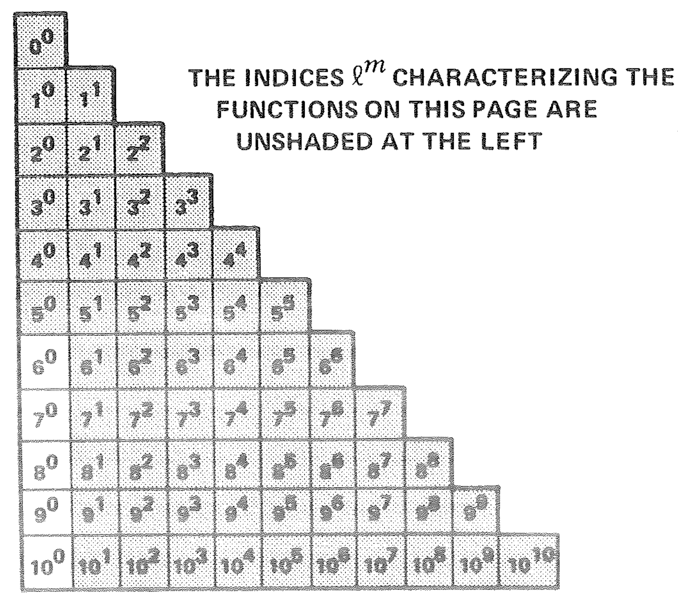
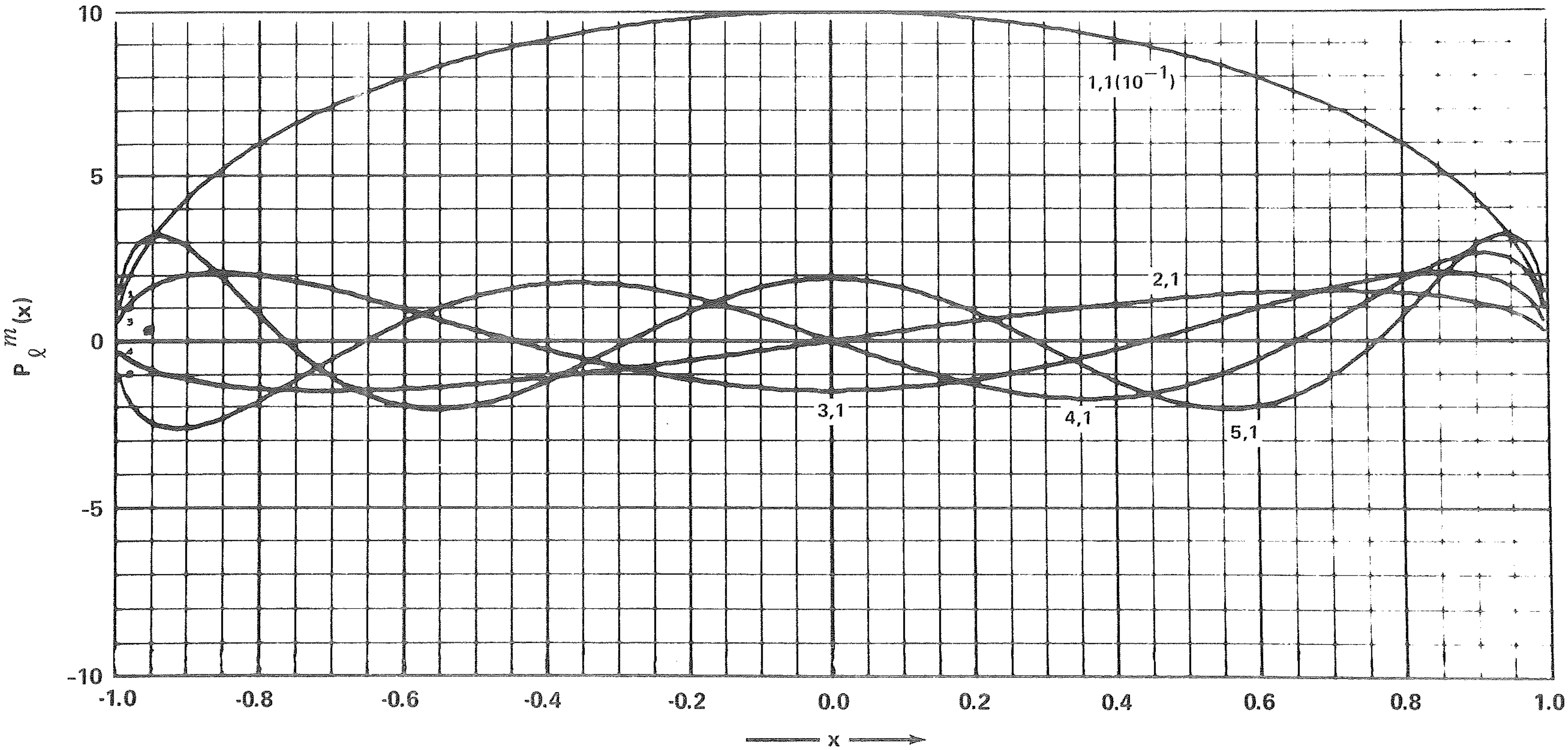


FIGURE 2 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS



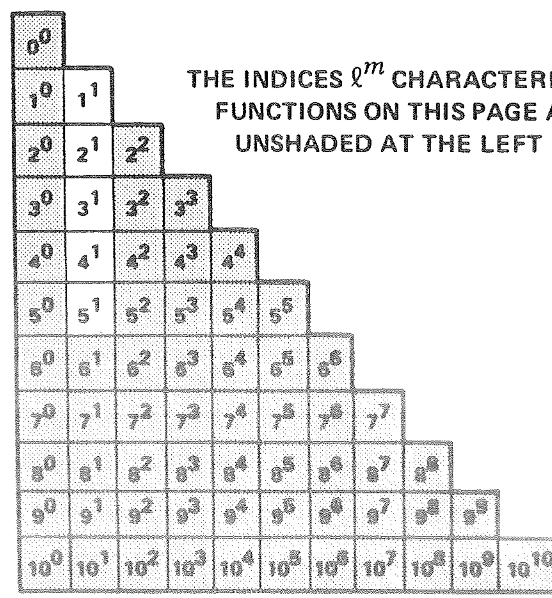


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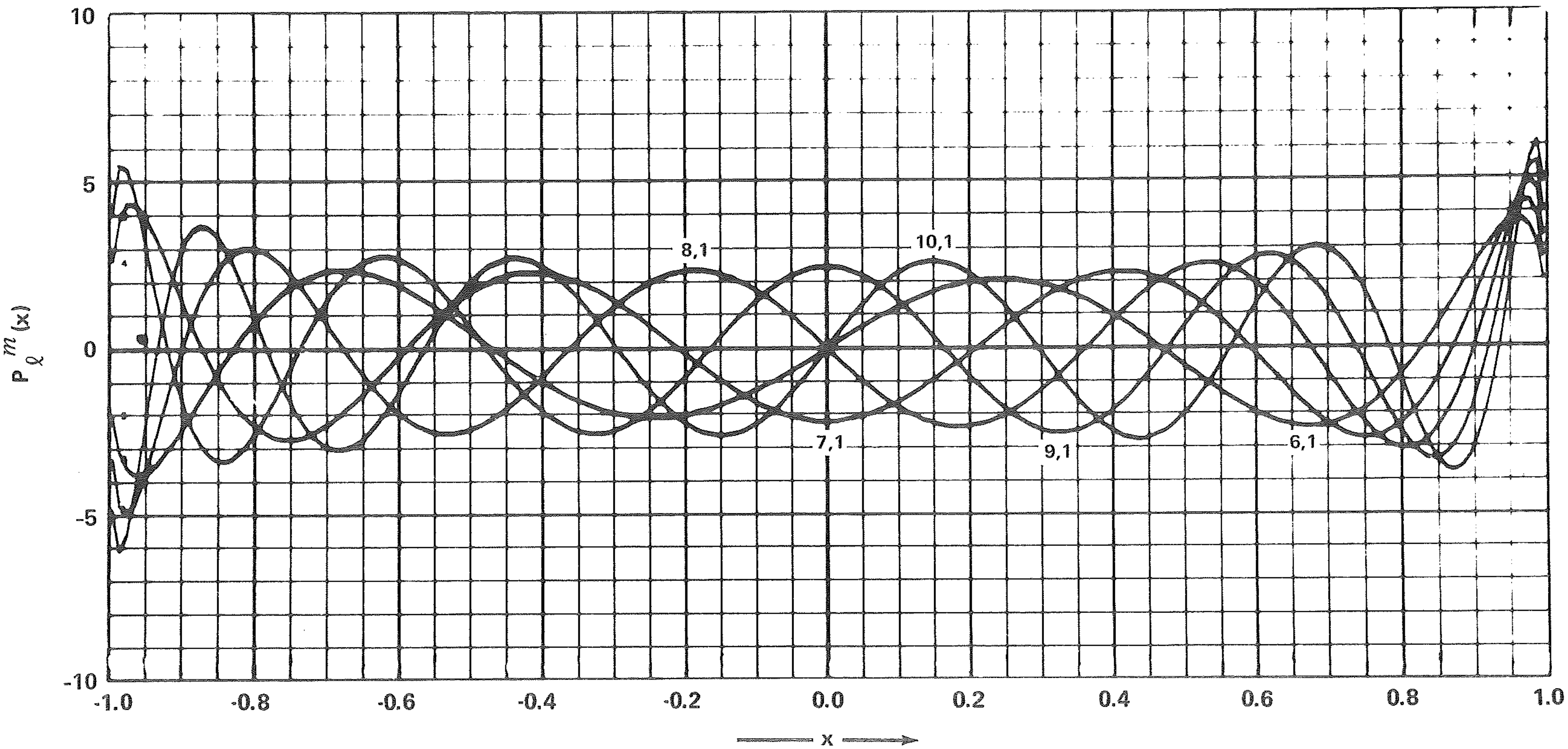
FIGURE 3 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



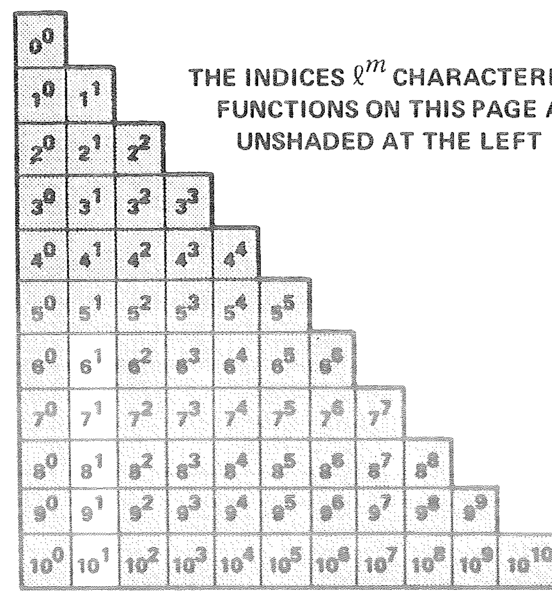


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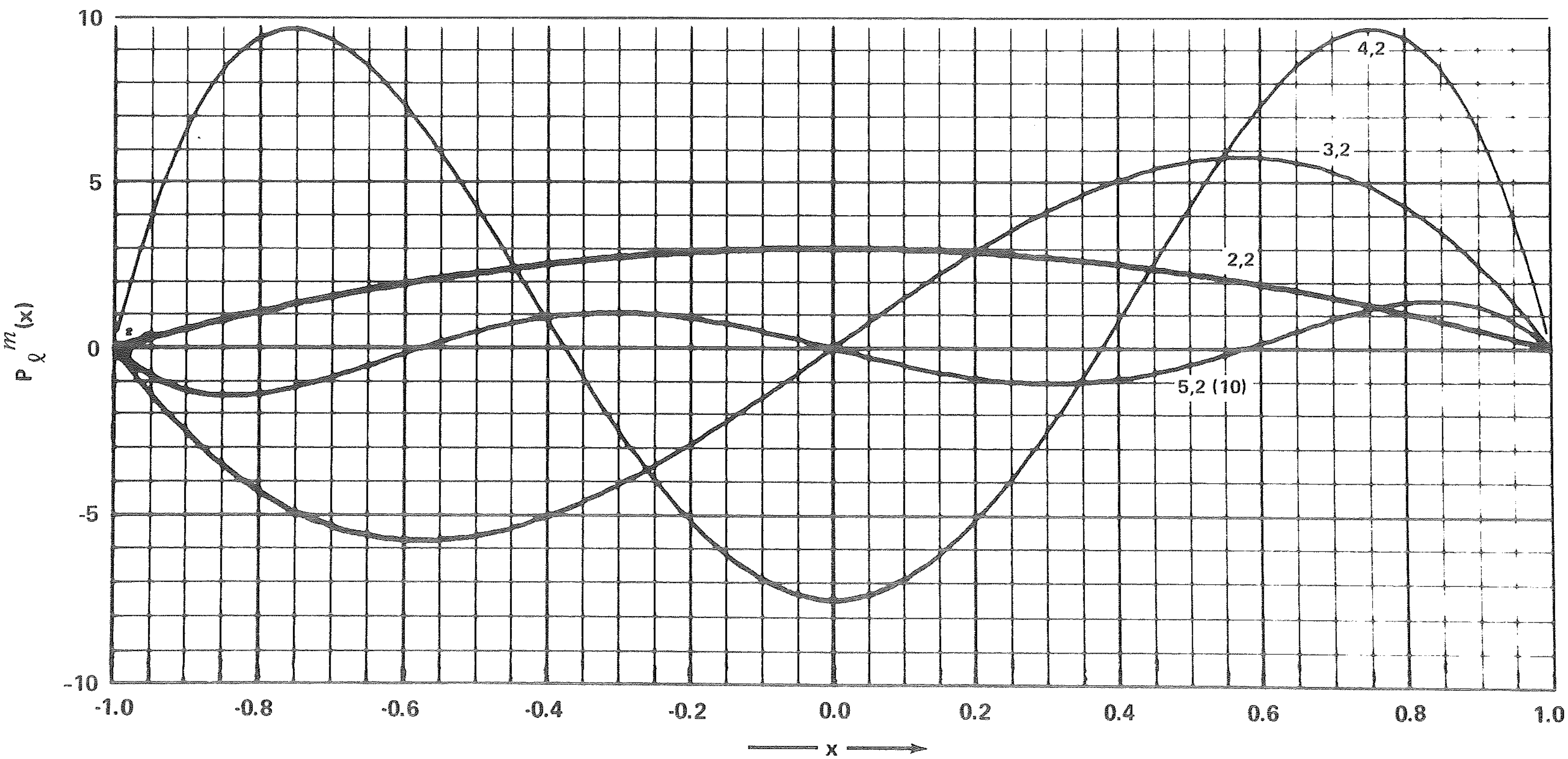
FIGURE 4 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS



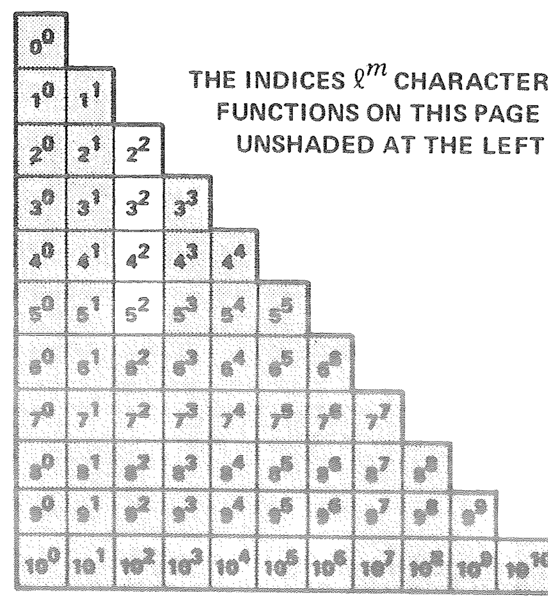


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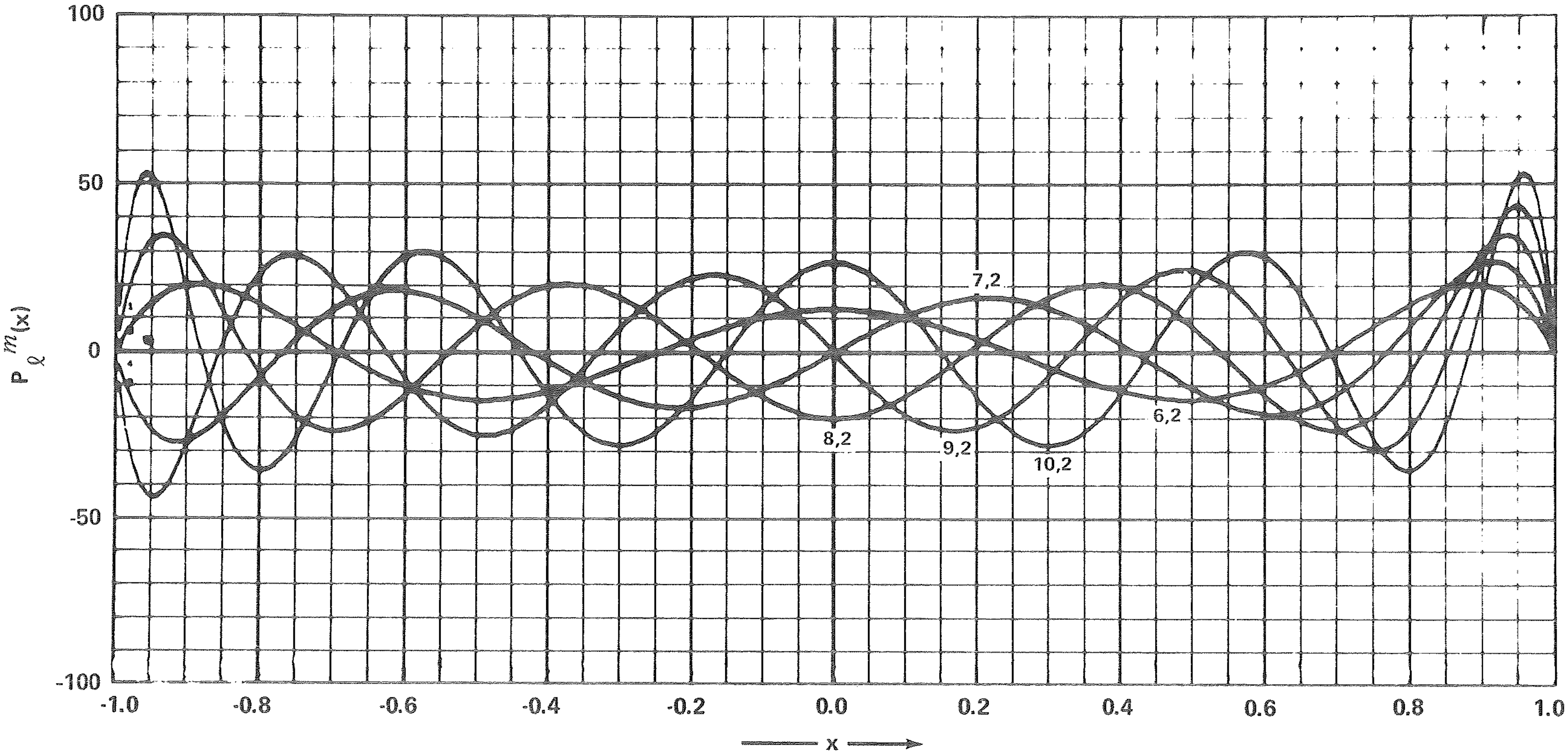
FIGURE 5 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



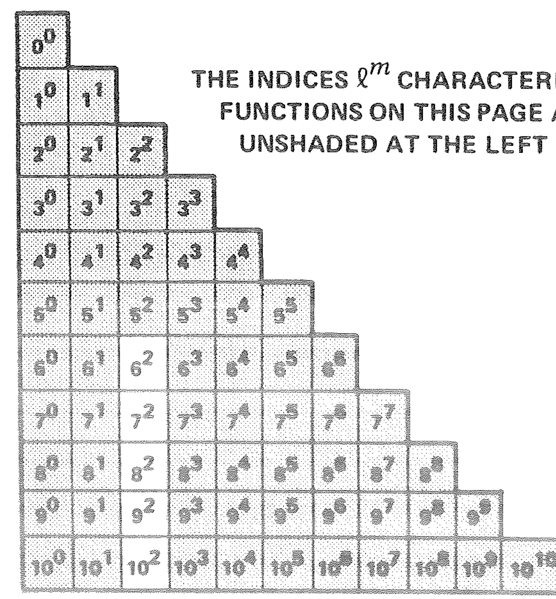


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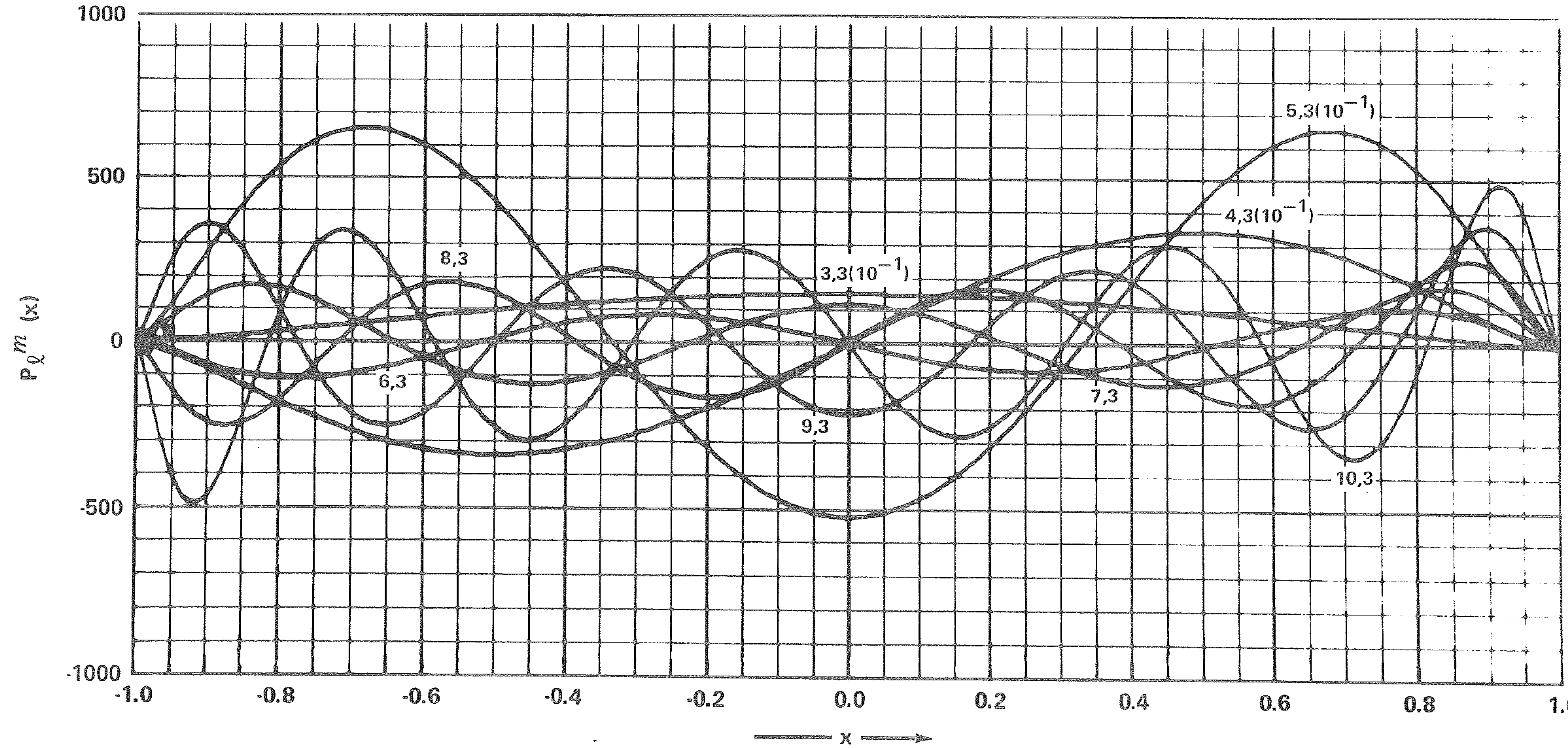
FIGURE 6 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS



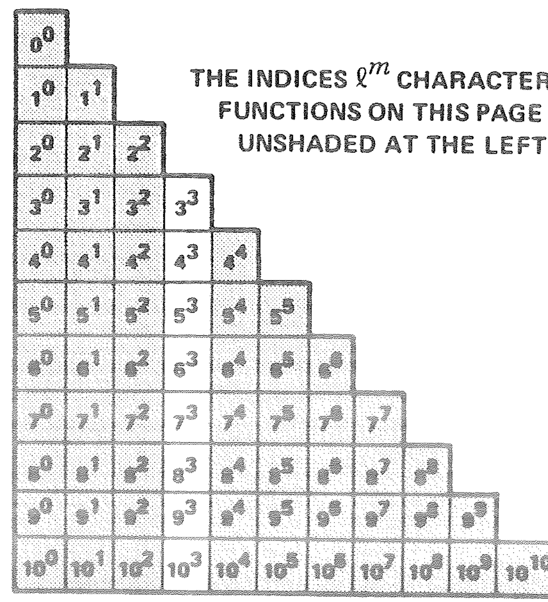


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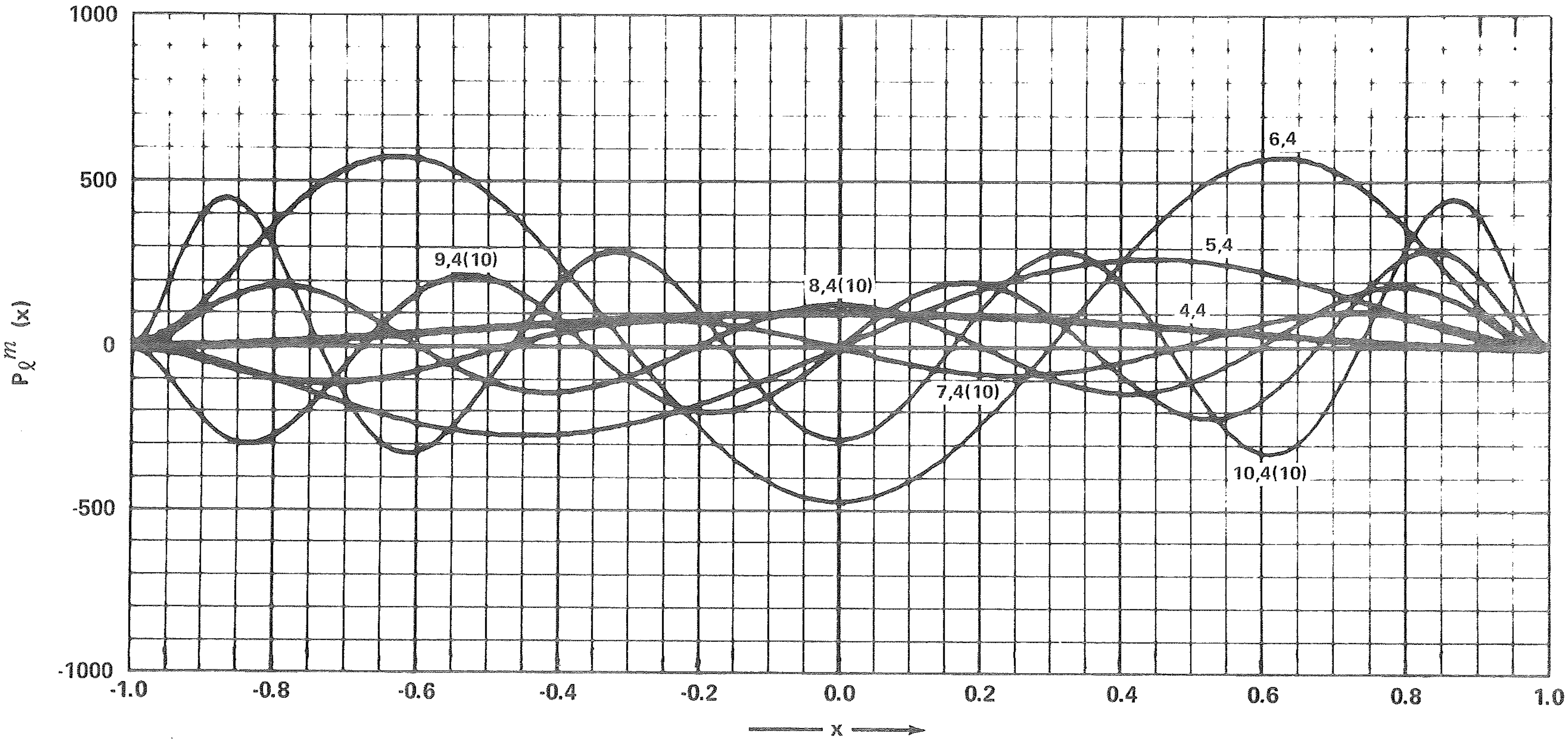
FIGURE 7 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



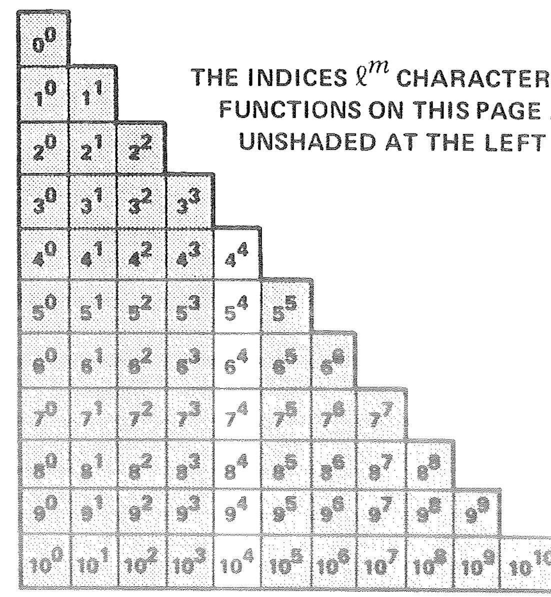


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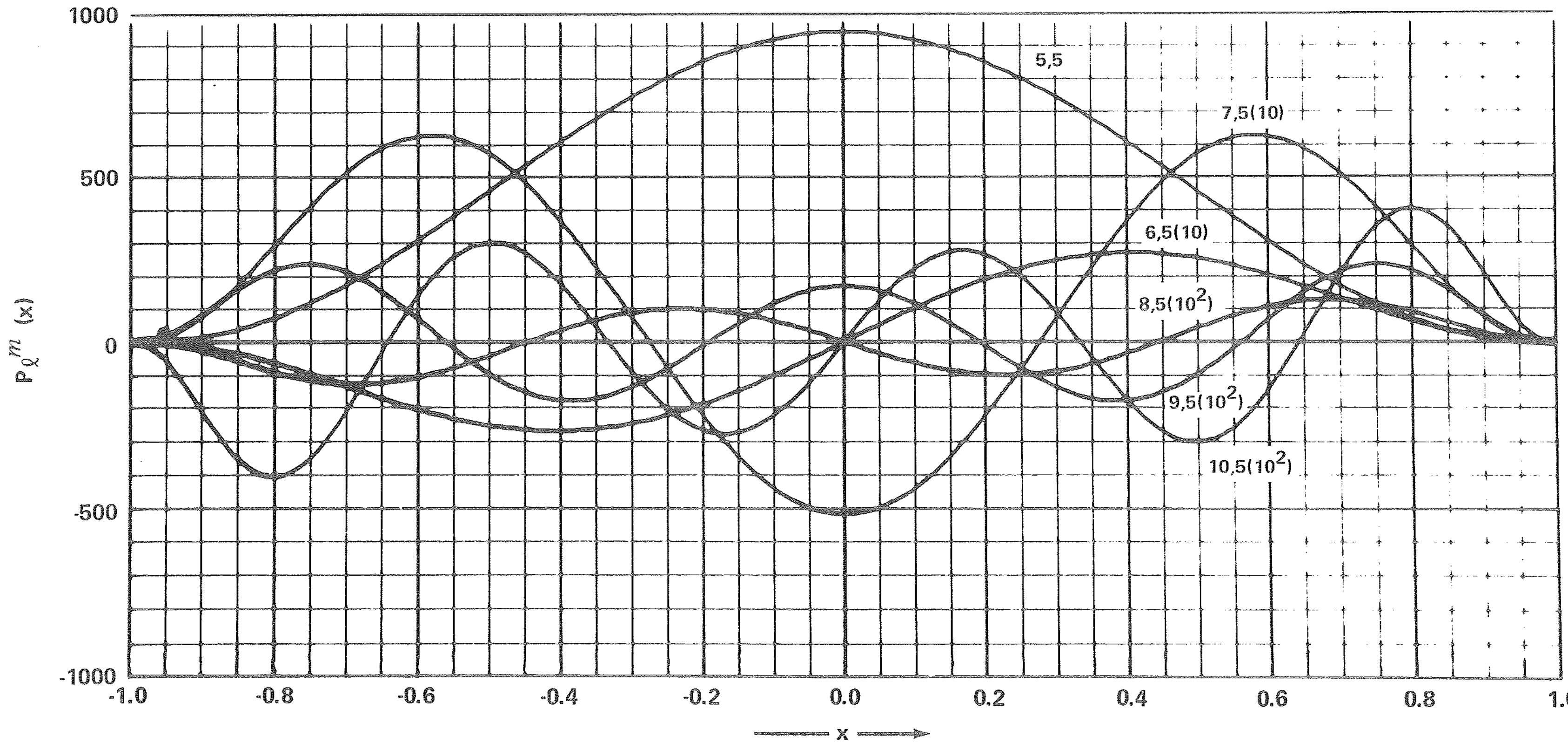
FIGURE 8 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



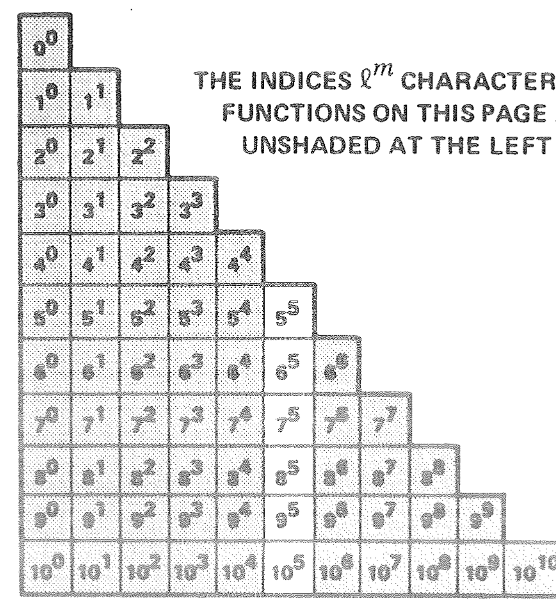


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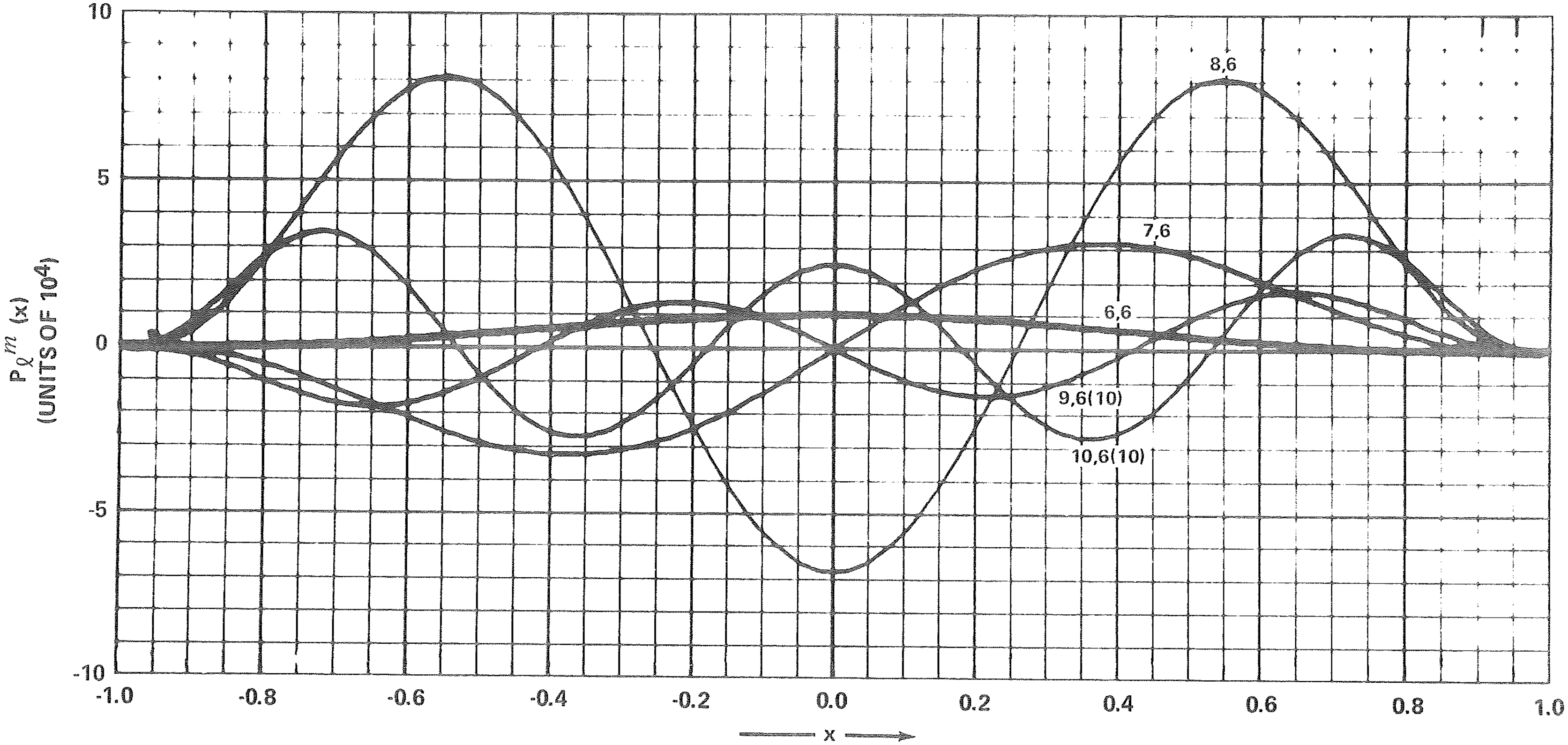
FIGURE 9 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



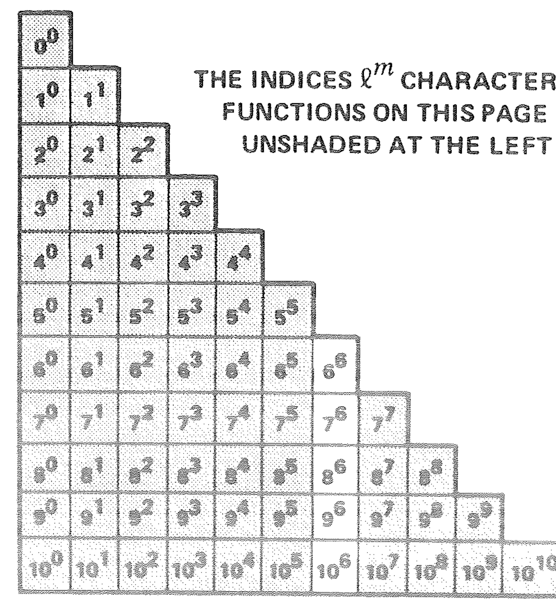


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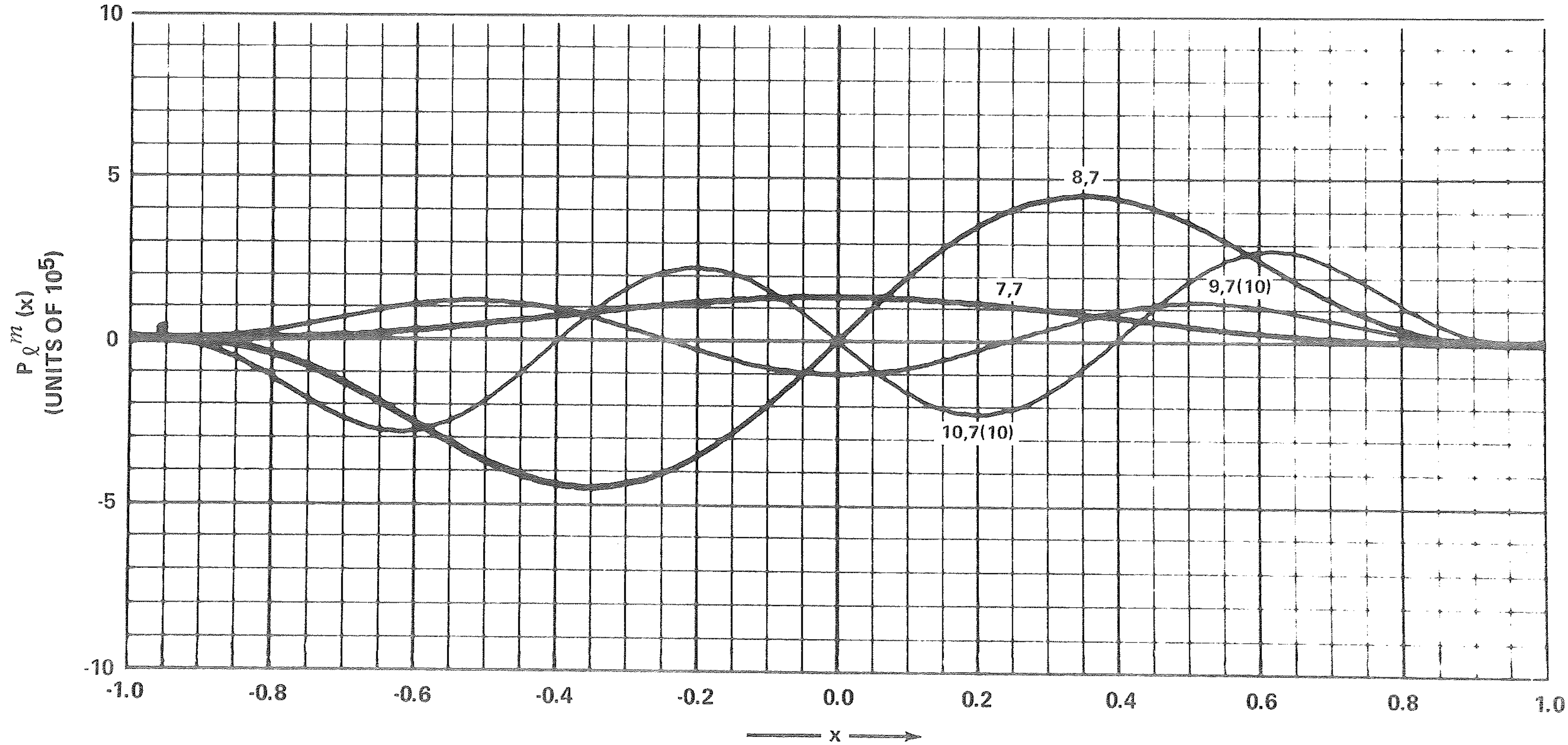
FIGURE 10 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



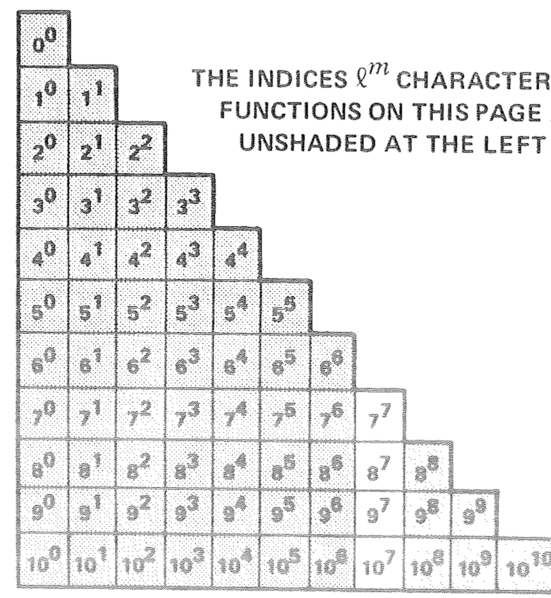


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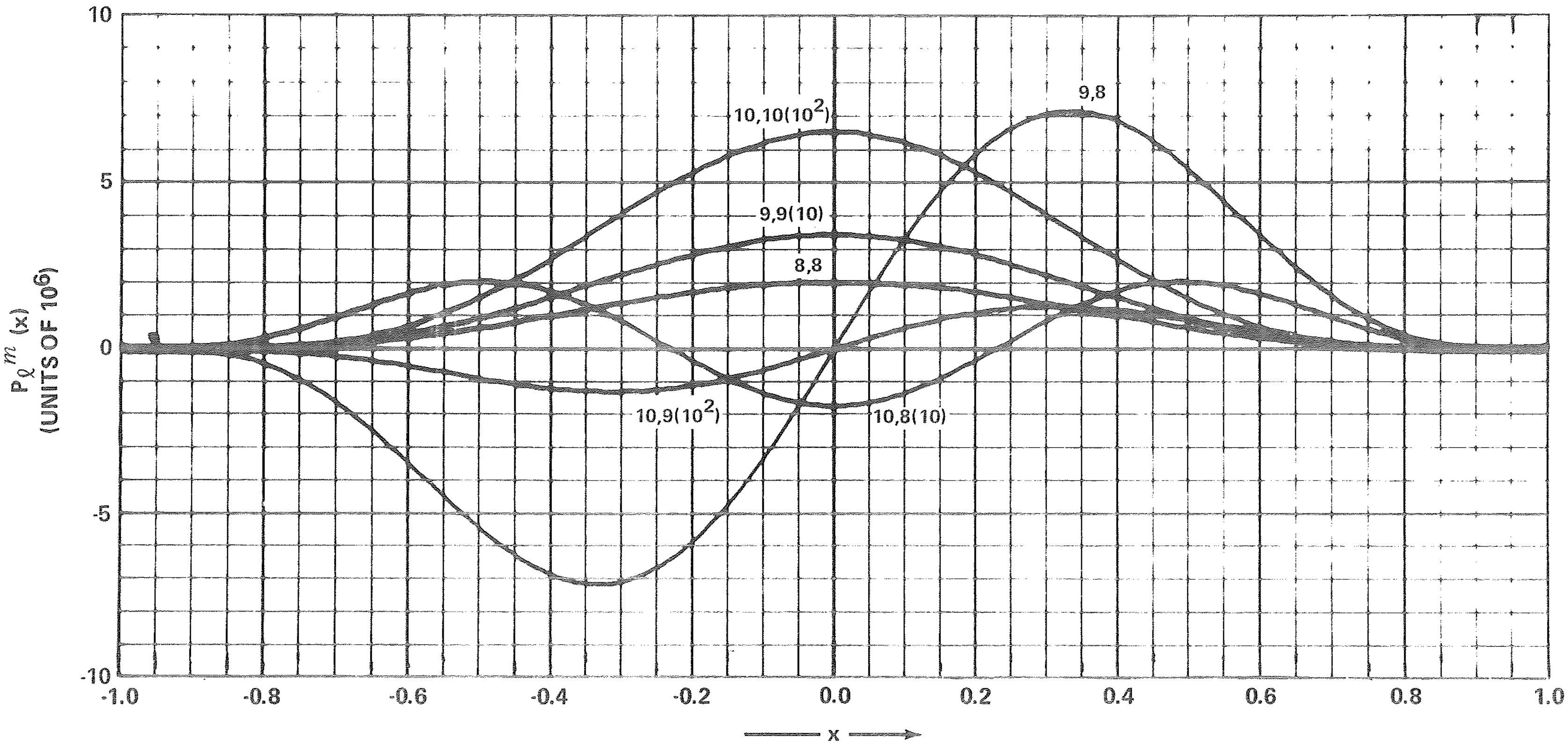
FIGURE 11 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



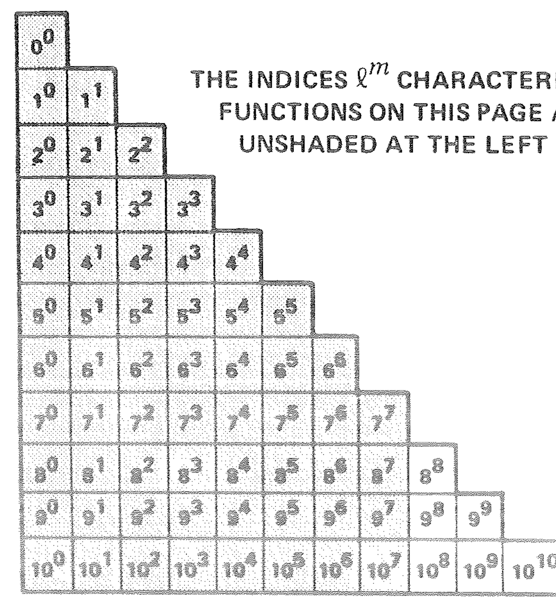


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FIGURE 12 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE

TABLE 2

FOURIER DECOMPOSITIONS OF P_ℓ^m (SINθ) AND P_ℓ^m (COSθ) UP TO $\ell = m = 10$

BOTH P_ℓ^m (SINθ) AND P_ℓ^m (COSθ) DECOMPOSE INTO SUMS OF COSINES AND/OR SINES OF EITHER EVEN OR ODD MULTIPLES OF θ, THE HIGHEST MULTIPLE BEING ALWAYS $\ell\theta$. FOR EACH ℓ AND m, THE TABLE GIVES THE FORM FOR THE EXPANSIONS OF P_ℓ^m (SINθ) AND P_ℓ^m (COSθ) AND SUPPLIES THE COEFFICIENTS FOR EACH TERM.

		1	SINθ	COS2θ	SIN3θ	COS4θ	SIN5θ	COS6θ	SIN7θ	COS8θ	SIN9θ	COS10θ	P_ℓ^m (SINθ)
ℓ	m	1	COSθ	–COS2θ	–COS3θ	COS4θ	COS5θ	–COS6θ	–COS7θ	COS8θ	COS9θ	–COS10θ	P_ℓ^m (COSθ)
0	0	1											
1	0	0	1										
2	0	1/4	0	-3/4									
3	0	0	3/8	0	-5/8								
4	0	9/64	0	-5/16	0	35/64							
5	0	0	15/64	0	-35/128	0	63/128						
6	0	25/256	0	-105/512	0	63/256	0	-231/512					
7	0	0	175/1024	0	-189/1024	0	231/1024	0	-429/1024				
8	0	1225/16384	0	-315/2048	0	693/4096	0	-429/2048	0	6435/16384			
9	0	0	2205/16384	0	-1155/8192	0	1287/8192	0	-6435/32768	0	12155/32768		
10	0	3969/65536	0	-8085/65536	0	2145/16384	0	-19305/131072	0	12155/65636	0	-46189/131072	

TABLE 2 (CONTINUED)

		$\cos\theta$	$\sin 2\theta$	$\cos 3\theta$	$\sin 4\theta$	$\cos 5\theta$	$\sin 6\theta$	$\cos 7\theta$	$\sin 8\theta$	$\cos 9\theta$	$\sin 10\theta$	$P_\ell^m(\sin\theta)$
ℓ	m	$\sin\theta$	$\sin 2\theta$	$-\sin 3\theta$	$-\sin 4\theta$	$\sin 5\theta$	$\sin 6\theta$	$-\sin 7\theta$	$-\sin 8\theta$	$\sin 9\theta$	$\sin 10\theta$	$P_\ell^m(\cos\theta)$
1	1	1										
2	1	0	3/2									
3	1	3/8	0	-15/8								
4	1	0	5/8	0	-35/16							
5	1	15/64	0	-105/128	0	315/128						
6	1	0	105/256	0	-63/64	0	693/256					
7	1	175/1024	0	-567/1024	0	1155/1024	0	-3003/1024				
8	1	0	315/1024	0	-693/1024	0	1287/1024	0	-6435/2048			
9	1	2205/16384	0	-3465/8192	0	6435/8192	0	-45045/32768	0	109395/32768		
10	1	0	8085/32768	0	-2145/4096	0	57915/65536	0	-12155/8192	0	230945/65536	

		1	$\sin\theta$	$\cos 2\theta$	$\sin 3\theta$	$\cos 4\theta$	$\sin 5\theta$	$\cos 6\theta$	$\sin 7\theta$	$\cos 8\theta$	$\sin 9\theta$	$\cos 10\theta$	$P_\ell^m(\sin\theta)$
ℓ	m	1	$\cos\theta$	$-\cos 2\theta$	$-\cos 3\theta$	$\cos 4\theta$	$\cos 5\theta$	$-\cos 6\theta$	$-\cos 7\theta$	$\cos 8\theta$	$\cos 9\theta$	$-\cos 10\theta$	$P_\ell^m(\cos\theta)$
2	2	3/2	0	3/2									
3	2	0	15/4	0	15/4								
4	2	45/16	0	-15/4	0	-105/16							
5	2	0	105/16	0	-105/32	0	-315/32						
6	2	525/128	0	-1785/256	0	315/128	0	3465/256					
7	2	0	4725/512	0	-3591/512	0	693/512	0	9009/512				
8	2	11025/2048	0	-315/32	0	3465/512	0	0	0	-45045/2048			
9	2	0	24255/2048	0	-10395/1024	0	6435/1024	0	6435/4096	0	-109395/4096		
10	2	218295/32768	0	-412335/32768	0	83655/8192	0	-366795/65536	0	-109395/32768	0	2078505/65536	

TABLE 2 (CONTINUED)

		$\cos\theta$	$\sin 2\theta$	$\cos 3\theta$	$\sin 4\theta$	$\cos 5\theta$	$\sin 6\theta$	$\cos 7\theta$	$\sin 8\theta$	$\cos 9\theta$	$\sin 10\theta$	P_ℓ^m (SIN θ)
ℓ	m	$\sin\theta$	$\sin 2\theta$	$-\sin 3\theta$	$-\sin 4\theta$	$\sin 5\theta$	$\sin 6\theta$	$-\sin 7\theta$	$-\sin 8\theta$	$\sin 9\theta$	$\sin 10\theta$	P_ℓ^m (COSθ)
3	3	45/4	0	15/4								
4	3	0	105/4	0	105/8							
5	3	315/16	0	-1365/32	0	-945/32						
6	3	0	2835/64	0	-945/16	0	-3465/64					
7	3	14175/512	0	-36855/512	0	38115/512	0	45045/512				
8	3	0	31185/512	0	-51975/512	0	45045/512	0	135135/1024			
9	3	72765/2048	0	-100485/1024	0	135135/1024	0	-405405/4096	0	-765765/4096		
10	3	0	315315/4096	0	-70785/512	0	1332045/8192	0	-109395/1024	0	-2078505/8192	

		1	$\sin\theta$	$\cos 2\theta$	$\sin 3\theta$	$\cos 4\theta$	$\sin 5\theta$	$\cos 6\theta$	$\sin 7\theta$	$\cos 8\theta$	$\sin 9\theta$	$\cos 10\theta$	P_ℓ^m (SINθ)
ℓ	m	1	$\cos\theta$	$-\cos 2\theta$	$-\cos 3\theta$	$\cos 4\theta$	$\cos 5\theta$	$-\cos 6\theta$	$-\cos 7\theta$	$\cos 8\theta$	$\cos 9\theta$	$-\cos 10\theta$	P_ℓ^m (COSθ)
4	4	315/8	0	105/2	0	105/8							
5	4	0	945/8	0	2835/16	0	945/16						
6	4	4725/32	0	-4725/64	0	-12285/32	0	-10395/64					
7	4	0	51975/128	0	10395/128	0	-86625/128	0	-45045/128				
8	4	363825/1024	0	-51975/128	0	-93555/256	0	135135/128	0	675675/1024			
9	4	0	945945/1024	0	-135135/512	0	-405405/512	0	3108105/2048	0	2297295/2048		
10	4	2837835/4096	0	-4099095/4096	0	-45045/1024	0	11216205/8192	0	-8423415/4096	0	-14549535/8192	

TABLE 2 (CONTINUED)

		COSθ	SIN2θ	COS3θ	SIN4θ	COS5θ	SIN6θ	COS7θ	SIN8θ	COS9θ	SIN10θ	P_ℓ^m (SINθ)
ℓ	m	SINθ	SIN2θ	–SIN3θ	–SIN4θ	SIN5θ	SIN6θ	–SIN7θ	–SIN8θ	SIN9θ	SIN10θ	P_ℓ^m (COSθ)
5	5	4725/8	0	4725/16	0	945/16						
6	5	0	51975/32	0	10395/8	0	10395/32					
7	5	259875/128	0	-343035/128	0	-446985/128	0	-135135/128				
8	5	0	675675/128	0	-405405/128	0	-945945/128	0	-675675/256			
9	5	4729725/1024	0	-4729725/512	0	1216215/512	0	27702675/2048	0	11486475/2048		
10	5	0	23648625/2048	0	-3378375/256	0	-2027025/4096	0	11486475/512	0	43648605/4096	

		1	SINθ	COS2θ	SIN3θ	COS4θ	SIN5θ	COS6θ	SIN7θ	COS8θ	SIN9θ	COS10θ	P_ℓ^m (SINθ)
ℓ	m	1	COSθ	–COS2θ	–COS3θ	COS4θ	COS5θ	–COS6θ	–COS7θ	COS8θ	COS9θ	–COS10θ	P_ℓ^m (COSθ)
6	6	51975/16	0	155925/32	0	31185/16	0	10395/32					
7	6	0	675675/64	0	1216215/64	0	675675/64	0	135135/64				
8	6	4729725/256	0	0	0	-2837835/64	0	-135135/4	0	-2027025/256			
9	6	0	14189175/256	0	4729725/128	0	-10135125/128	0	-42567525/512	0	-11486475/512		
10	6	127702575/2048	0	-89864775/2048	0	-56081025/512	0	480404925/4096	0	356080725/2048	0	218243025/4096	

TABLE 2 (CONTINUED)

		$\cos\theta$	$\sin 2\theta$	$\cos 3\theta$	$\sin 4\theta$	$\cos 5\theta$	$\sin 6\theta$	$\cos 7\theta$	$\sin 8\theta$	$\cos 9\theta$	$\sin 10\theta$	$P_\ell^m(\sin\theta)$
ℓ	m	$\sin\theta$	$\sin 2\theta$	$-\sin 3\theta$	$-\sin 4\theta$	$\sin 5\theta$	$\sin 6\theta$	$-\sin 7\theta$	$-\sin 8\theta$	$\sin 9\theta$	$\sin 10\theta$	$P_\ell^m(\cos\theta)$
7	7	4729725/64	0	2837835/64	0	945945/64	0	135135/64				
8	7	0	14189175/64	0	14189175/64	0	6081075/64	0	2027025/128			
9	7	99324225/256	0	-42567525/128	0	-83108025/128	0	-180405225/512	0	-34459425/512		
10	7	0	562837275/512	0	-11486475/64	0	-1481755275/1024	0	-126351225/128	0	-218243025/1024	

		1	$\sin\theta$	$\cos 2\theta$	$\sin 3\theta$	$\cos 4\theta$	$\sin 5\theta$	$\cos 6\theta$	$\sin 7\theta$	$\cos 8\theta$	$\sin 9\theta$	$\cos 10\theta$	$P_\ell^m(\sin\theta)$
ℓ	m	1	$\cos\theta$	$-\cos 2\theta$	$-\cos 3\theta$	$\cos 4\theta$	$\cos 5\theta$	$-\cos 6\theta$	$-\cos 7\theta$	$\cos 8\theta$	$\cos 9\theta$	$-\cos 10\theta$	$P_\ell^m(\cos\theta)$
8	8	7094578/128	0	14189175/16	0	14189175/32	0	2027025/16	0	2027025/128			
9	8	0	241215975/128	0	241215975/64	0	172297125/64	0	241215975/256	0	34459425/256		
10	8	2170943775/512	0	723647925/512	0	-1137161025/128	0	-9614179575/1024	0	-2033106075/512	0	-654729075/1024	

TABLE 2 (CONCLUDED)

		COSθ	SIN2θ	COS3θ	SIN4θ	COS5θ	SIN6θ	COS7θ	SIN8θ	COS9θ	SIN10θ	P_ℓ^m (SINθ)
ℓ	m	SINθ	SIN2θ	–SIN3θ	–SIN4θ	SIN5θ	SIN6θ	–SIN7θ	–SIN8θ	SIN9θ	SIN10θ	P_ℓ^m (COSθ)
9	9	2170943775/128	0	723647925/64	0	310134825/64	0	310134825/256		34459425/256		
10	9	0	-3435635385/16	0	1964187225/32	0	17677685025/512	0	654729075/64	0	654729075/512	

		1	SINθ	COS2θ	SIN3θ	COS4θ	SIN5θ	COS6θ	SIN7θ	COS8θ	SIN9θ	COS10θ	P_ℓ^m (SINθ)
ℓ	m	1	COSθ	–COS2θ	–COS3θ	COS4θ	COS5θ	–COS6θ	–COS7θ	COS8θ	COS9θ	–COS10θ	P_ℓ^m (COSθ)
10	10	-13735772505/128	0	27076139/256	0	9820936125/64	0	29462808375/512	0	3273645375/256	0	654729075/512	

Zeros, Symmetry, and Special Values

The function $P_\ell^m(x)$ has $(\ell-m)$ zero crossings on the interval (-1, 1). Its properties include the following:

$$P_\ell^m(-x) = (-)^{\ell-m} P_\ell^m(x) \qquad \text{SYMMETRY}$$

$$P_\ell^m(1) = \delta_{m0}$$

$$P_\ell^m(0) = \frac{(-)^{\frac{\ell-m}{2}} (\ell+m)!}{2^\ell \left(\frac{\ell-m}{2}\right)! \left(\frac{\ell+m}{2}\right)!} \times \begin{cases} 0 & [\ell+m = \text{odd}] \\ 1 & [\ell+m = \text{even}] \end{cases}$$

$$P_\ell^\ell(x) = \frac{(2\ell)!}{2^\ell \ell!} (1-x^2)^{\ell/2}$$

Asymptotic Values

Let $\ell >> \frac{1}{\varepsilon} >> \frac{6}{\pi}$ and $0 < \varepsilon \leq \theta \leq \pi - \varepsilon,$

and define $\phi = \left(\ell+\frac{1}{2}\right)\theta + \frac{\pi}{4}.$

Then

$$P_\ell^m(\cos\theta) \approx (-n)^m \sqrt{\frac{2}{\ell\pi \sin\theta}} \sin\left(\phi+\frac{m\pi}{2}\right)$$

and

$$P_\ell (\cos\theta) \approx \sqrt{\frac{2}{\ell\pi \sin\theta}} \left[\left(1-\frac{1}{4\ell}\right) \sin\phi - \frac{1}{8\ell} \operatorname{ctn}\theta \cos\phi\right].$$

Upper Bounds

Table 3 gives the least upper bounds (LUB's) for the unnormalized associated Legendre functions up to $P_{10}^{10}(x)$. The values were taken from Figures 1 through 12. Empirically, up to $P_{10}^{10}(x)$ it is found that

$$\mathrm{LUB}|P_\ell^m(x)| > (1+\delta_{m0})\sqrt{\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}} .$$

Analytical relations are

$$\mathrm{LUB}|P_\ell^\ell(x)| = \frac{(2\ell)!}{2^\ell \ell!}$$

$$\mathrm{LUB}|P_\ell^m(x)| \leq \mathrm{LUB}|P_\ell^\ell(x)| \qquad \text{CONJECTURED}$$

$$|P_\ell^m(x)| < \frac{2}{\sqrt{\pi\ell}} \frac{(\ell+m)!}{\ell!} \frac{1}{(1-x^2)^{\ell+1/2}} \qquad [\ell\neq 0].$$

ℓ \ m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	1.5	3								
3	1	2.2	5.8	15							
4	1	2.7	9.6	34	110						
5	1	3.3	15	66	270	950					
6	1	3.8	20	110	580	2800	1.1E4				
7	1	4.3	27	180	1100	6300	3.2E4	1.4E5			
8	1	4.9	35	260	1900	13,000	8.1E4	4.6E5	2.1E6		
9	1	5.7	44	360	3000	24,000	1.8E5	1.2E6	7.2E6	3.5E7	
10	1	6.3	53	490	4500	41,000	3.5E5	2.9E6	2.1E7	1.3E8	6.6E8

TABLE 3. Least Upper Bounds for $P_\ell^m(x)$.
Values are extracted from Figures 1 through 12.
NOTE: The notation aEb means a x 10^b.

Recurrence Relations

The argument x is understood in the following recurrence relations, which have been taken from [Magnus and Oberhettinger, 1949] and [Emde and Jahnke, 1945]. Restrictions on m may arise due to the convention on page 1.

$$P_\ell^{-m} = \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m$$

$$P_{-\ell-1}^m = P_\ell^m$$

$$x(2\ell+1) P_\ell^m = (\ell-m+1) P_{\ell+1}^m + (\ell+m) P_{\ell-1}^m$$

$$P_\ell^{m+2} - 2(m+1) \frac{x}{\sqrt{1-x^2}} P_\ell^{m+1} + (\ell-m)(\ell+m+1) P_\ell^m = 0$$

$$(\ell-m)(\ell-m+1) P_{\ell+1}^m = (\ell+m)(\ell+m+1) P_{\ell-1}^m - (2\ell+1) \sqrt{1-x^2} P_\ell^{m+1}$$

$$P_{\ell-1}^m - P_{\ell+1}^m + (2\ell+1) \sqrt{1-x^2} P_\ell^{m-1} = 0$$

$$P_{\ell-1}^m - x P_\ell^m + (\ell-m+1) \sqrt{1-x^2} P_\ell^{m-1} = 0$$

$$x P_\ell^m - P_{\ell+1}^m + (\ell+m) \sqrt{1-x^2} P_\ell^{m-1} = 0$$

$$x(\ell-m) P_\ell^m - (\ell+m) P_{\ell-1}^m + \sqrt{1-x^2} P_\ell^{m+1} = 0$$

$$(\ell-m+1)\ P_{\ell+1}{}^{m} - x\ (\ell+m+1)\ P_{\ell}{}^{m} + \sqrt{1-x^2}\ P_{\ell}{}^{m+1} = 0$$

$$(\ell+1)\ P_{\ell+1}{}^{m} + \ell P_{\ell-1}{}^{m} = (2\ell+1)\left[x\ P_{\ell}{}^{m} + m\sqrt{1-x^2}\ P_{\ell}{}^{m-1}\right] - \ell P_{\ell-1}{}^{m}$$

$$(1-x^2)\ \frac{d\ P_{\ell}{}^{m}}{dx} = x\ (\ell+1)\ P_{\ell}{}^{m} - (\ell-m+1)\ P_{\ell+1}{}^{m}$$

$$= -x\ell\ P_{\ell}{}^{m} + (\ell+m)\ P_{\ell-1}{}^{m}$$

$$= \sqrt{1-x^2}\ P_{\ell}{}^{m+1} - xm\ P_{\ell}{}^{m}$$

$$= -(\ell-m+1)(\ell+m)\ \sqrt{1-x^2}\ P_{\ell}{}^{m-1} + xm\ P_{\ell}{}^{m}$$

$$\frac{xdP_{\ell}}{dx} - \frac{dP_{\ell-1}}{dx} = \ell\ P_{\ell}$$

$$\frac{-xdP_{\ell}}{dx} + \frac{dP_{\ell+1}}{dx} = (\ell+1)\ P_{\ell}$$

$$\frac{dP_{\ell+1}}{dx} - \frac{dP_{\ell-1}}{dx} = (2\ell+1)\ P_{\ell}$$

Recurrence Relations for Solid Spherical Harmonics

The infinite series

$$u(h,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(a_{\ell m} U_\ell^m + b_\ell^m V_\ell^m\right),$$

with

$$\begin{pmatrix} U_\ell^m \\ V_\ell^m \end{pmatrix} = \frac{P_\ell^m(\cos\theta)}{r^{\ell+1}} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix},$$

is a solution of Laplace's equation in spherical polar coordinates. The functions U_ℓ^m and V_ℓ^m are called solid spherical harmonics. These have the following useful recurrence properties [DeWitt, 1962]:

$$U_{\ell+1}^m = \frac{1}{r(\ell-m+1)} \left[(2\ell+1) \cos\theta\, U_\ell^m - \frac{\ell+m}{r} U_{\ell-1}^m\right]$$

$$V_{\ell+1}^m = \frac{1}{r(\ell-m+1)} \left[(2\ell+1) \cos\theta\, U_\ell^m - \frac{\ell+m}{r} U_{\ell-1}^m\right]$$

$$U_{\ell+1}^{\ell+1} = \frac{2\ell+1}{r^2} \left[U_\ell^\ell\, C(\phi) - V_\ell^\ell\, S(\phi)\right]$$

$$V_{\ell+1}^{\ell+1} = \frac{2\ell+1}{r^2} \left[V_\ell^\ell\, C(\phi) + V_\ell^\ell\, S(\phi)\right],$$

where

$$\begin{pmatrix} C(\phi) \\ S(\phi) \end{pmatrix} = \sin\theta \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}.$$

Defining $\bar{\nabla}$ as the gradient operator, then

$$2\bar{\nabla}\, U_\ell^m = \begin{bmatrix} C_\ell^m\, U_{\ell+1}^{m-1} - U_{\ell+1}^{m+1} \\ -C_\ell^m\, V_{\ell+1}^{m-1} - V_{\ell+1}^{m+1} \\ -2(\ell-m+1)\; U_{\ell+1}^{m} \end{bmatrix}$$

$$2\bar{\nabla}\, V_\ell^m = \begin{bmatrix} C_\ell^m\, V_{\ell+1}^{m-1} - V_{\ell+1}^{m-1} \\ C_\ell^m\, U_{\ell+1}^{m-1} + U_{\ell+1}^{m+1} \\ -2(\ell-m+1)\; V_{\ell+1}^{m} \end{bmatrix},$$

where $$C_\ell^m = (\ell-m+1)(\ell-m+2).$$

Recurrence relations on $u(r,\theta,\phi)$ may be found in [James, 1969].

<u>Important Expansions</u>

$$\frac{1}{\sqrt{1 - 2hx + h^2}} = \left\{ \begin{matrix} \sum_{\ell=0}^{\infty} h^\ell P_\ell(x) & [h<1] \\ \sum_{\ell=0}^{\infty} \frac{1}{h^{\ell+1}} P_\ell(x) & [h>1] \end{matrix} \right\} \text{GENERATING FUNCTION}$$

Let α be the angle between two points with spherical polar coordinates (θ, ϕ) and (θ', ϕ').

$$P_\ell(\cos\alpha) = \sum_{m=0}^{\ell} (2-\delta_{m0}) \frac{(\ell-m)!}{(\ell+m)!}$$

$$\times P_\ell^{m}(\cos\theta)\, P_\ell^{m}(\cos\theta') \cos m(\phi-\theta')$$

ADDITION THEOREM

<u>Short Table of Integrals</u>

$$\int_{-1}^{1} P_\ell^{m}(x)\, P_{\ell'}^{m}(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}$$

ORTHOGONALITY INTEGRAL MO

$$\int_{-1}^{1} \frac{P_\ell^{m}(x)\, P_\ell^{m'}(x)}{1-x^2}\, dx = \frac{1}{m} \frac{(\ell+m)!}{(\ell-m)!} \delta_{mm'} \qquad [0<m\leq\ell]$$

MO

$$I_a(\ell, i, n, j) \equiv \int_{-a}^{a} P_\ell^{i}(x)\, P_n^{j}(x)\, dx$$

Let $\quad 2p \equiv 2q - 2(t+\nu) + (\ell+n) - (i+j)$

and $\quad T_{\alpha\beta\gamma} \equiv \dfrac{(-)^{\gamma}\,(2\alpha-2\gamma)!}{2^{\alpha}\gamma!\,(\alpha-\gamma)!\,(\alpha-\beta-2\gamma)!}$

and $\quad [b] \equiv$ greatest integer in b

$$I_a(\ell, i, n, j) = 0 \qquad [\text{if } (\ell+n) - (i+j) = \text{odd}]$$

SL

$$I_a(\ell, i, n, j) = 2 \sum_{t=0}^{\left[\frac{\ell-i}{2}\right]} \sum_{v=0}^{\left[\frac{n-j}{2}\right]} \sum_{q=0}^{\left[\frac{i+j}{2}\right]}$$

SL

$$(-)^q \, T_{\ell i t} \, T_{n j v} \binom{\frac{i+j}{2}}{q} \frac{a^{1+2p}}{1+2p}$$

[if $(i+j)$ = even and $(\ell+n)$ = even]

$$I_a(\ell, i, n, j) = 2 \sum_{t=0}^{\left[\frac{\ell-i}{2}\right]} \sum_{v=0}^{\left[\frac{n-j}{2}\right]} \sum_{q=0}^{\frac{i+j-1}{2}} (-)^q \, T_{\ell i t} \, T_{n j v} \binom{\frac{j+\ell-1}{2}}{q}$$

SL

$$\times \int_0^{\arcsin(a)} \sin^{2p}(\theta) \cos^2(\theta) \, d\theta$$ [if $(i+j)$ = odd and $(\ell+n)$ = odd]

$$I_1(\ell, i, n, j) = \pi \sum_{t=0}^{\left[\frac{\ell-i}{2}\right]} \sum_{v=0}^{\left[\frac{n-j}{2}\right]} \sum_{q=0}^{\frac{i+j-1}{2}} (-)^q \, T_{\ell i t} \, T_{n j v} \binom{\frac{j+\ell-1}{2}}{q}$$

SL

$$\times \frac{1}{(2p+1) \; 2^{2p+2}} \binom{2p+2}{p+1}$$

[if $(i+j)$ = odd and $(\ell+n)$ = odd]

$$J_p(s,\ n,\ m) \equiv \int_0^{\pi} (\sin\theta)^{2s-3}\ (\cos\theta)^p\ P_n^{\,m}(\cos\theta)\ P_{n-p}^{\,m}(\cos\theta)\ d\theta$$

$$J_0(0,\ n,\ m) = \frac{(n+m)!}{2m(m^2-1)(n-m)!}\ (n^2+n+m^2-1) \qquad [n-p\geq m\geq 2] \qquad \text{HK}$$

$$J_0(1,\ n,\ m) = \frac{(n+m)!}{m(n-m)!} \qquad [n-p\geq m\geq 1] \qquad \text{HK}$$

$$J_0(2,\ n,\ m) = \frac{2(n+m)!}{(2n+1)(n-m)!} \qquad [n-p\geq m\geq 0] \qquad \text{HK}$$

$$J_1(0,\ n,\ m) = \frac{n(n+m)!}{2m(m^2-1)(n-m-1)!} \qquad [n-p\geq m\geq 2] \qquad \text{HK}$$

$$J_1(1,\ n,\ m) = \frac{(n+m-1)!}{n(n-m-1)!} \qquad [n-p\geq m\geq 1] \qquad \text{HK}$$

$$J_1(2,\ n,\ m) = \frac{2(n+m)!}{(4n^2-1)(n-m-1)!} \qquad [n-p\geq m\geq 0] \qquad \text{HK}$$

$$\int_0^{\pi} \left[\frac{dP_n^{\,m}(\cos\theta)}{d\theta}\right]^2 \sin\theta\ d\theta = \left(\frac{2n(n+1)}{2n+1} - m\right) \frac{(n+m)!}{(n-m)!} \qquad \text{HK}$$

$$\int_0^{\pi} \left[\frac{dP_n^m(\cos\theta)}{d\theta}\right]^2 \frac{d\theta}{\sin\theta} =$$

$$\begin{cases} n(n+1) & [m=0] \\ \frac{m(n^2+n+1-m^2)}{2m^2-1} \quad \frac{(n+m)!}{(n-m)!} & [m\neq 0] \end{cases}$$ HK

$$\int_0^{\pi} \left[\frac{d^2P_n^m(\cos\theta)}{d^2\theta}\right]^2 \sin\theta \; d\theta =$$

$$\begin{cases} \frac{n(n+1)(2n^2-1)}{2n+1} & [m=0] \\ 2n^2(1-m) + \frac{1}{2} m(n^2-3n+m^2+1) + \frac{2n^4}{2n+1} \frac{(n+m)!}{(n-m)!} & [m\neq 0] \end{cases}$$ HK

Other integrals involving the associated Legendre functions may be found in [Bateman, 1953], in [Gradshteyn and Ryzhik, 1965], and in [Magnus and Oberhettinger, 1949].

The references abbreviated beside the formulas listed above are:

MO: [Magnus and Oberhettinger, 1949]
SL: [Levie, 1971]
HK: [Higgens and Kopal, 1968]

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