Bellcomm

955 L'Enfant Plaza North, S.W.
Washington, D. C. 20024

date: August 30, 1971

to: Distribution

B71 08040

from: W. G. Heffron

subject: Gravity and Gravity Gradient from Spherical Harmonics - Case 310

ABSTRACT

Beginning with the spherical harmonic expansion for the gravitational potential of a planet, explicit and recursive formulas are presented for the acceleration and gradient of gravity. Figures are given showing the effects on the potential and acceleration of several individual coefficients in the expansion, and also of the potential and acceleration resulting from the sets of coefficients in current use for the earth and moon.

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MEMORANDUM FOR FILE

INTRODUCTION

The memorandum documents explicit and recursive formulas for gravity and gravity gradient using spherical harmonics. It is a companion paper to [1] which is a compendium on the properties of the Legendre polynomials used in the spherical harmonics.

THE POTENTIAL

The spherical harmonic expansion for the gravitational potential of a planet is commonly given as

$$U = \frac{\mu}{r} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(\frac{R}{r}\right)^{\ell} P_{\ell}^{m}(\sin\phi)(C_{\ell m}\cos m\lambda + S_{\ell m}\sin m\lambda)$$

where U is the potential, μ equals GM where G is the universal gravitational constant, M is the planet's mass, r is the radius from the origin of the planet-fixed axes to a field point; ϕ and λ are the field point's latitude and longitude, R is a "reference" radius for the planet, P_{ℓ}^{m} is the unnormalized associated Legendre polynomial of degree ℓ and order m, and $C_{\ell m}$ and $S_{\ell m}$ are the constants which describe the planet.

$C_{00} = 1$ describes the central body (spherical) potential, as the planet would appear from a far distant point. All $S_{\ell 0}$'s are taken as zero. C_{10}, C_{11}, and S_{11} are associated with the center of mass location relative to the origin of the axes:

$$C_{10} = \frac{z_{C.M.}}{R}$$

$$C_{11} = 2\,\frac{x_{C.M.}}{R}$$

$$S_{11} = 2\,\frac{y_{C.M.}}{R}$$

Usually these are assumed to be zero.

If the planet's inertia tensor is

A	-D	-E
-D	B	-F
-E	-F	C

where, typically

$$A = \int (y^2 + z^2)\,dM$$

$$D = + \int xy\ dM$$

then

$$C_{20} = \frac{1}{R^2 M}\left(\frac{A+B}{2} - C\right)$$

$$C_{21} = \frac{1}{R^2 M} E$$

$$S_{21} = \frac{1}{R^2 M} F$$

$$C_{22} = \frac{1}{4MR^2} (B-A)$$

$$S_{22} = \frac{1}{2MR^2} D$$

If the planet is symmetric about its rotational (z) axis, then $C_{\ell m} = 0$ for $m>0$ and all $S_{\ell m} = 0$.

GRAVITY

The acceleration of gravity g is the gradient of the potential. Two decompositions of the acceleration vector are of interest. One projects it on planet axes xyz and the other on radial, East (and horizontal), and North (and horizontal) axes at the field point r,ϕ,λ. Explicit forms are given in Table 1.

Figures 1 - 7 show plots simplified forms of U and of g_r, g_E, and g_N for the coefficients ℓ and m from 2,0 to 3,3. Specifically, what is plotted is

$$U(\ell,m) = P_\ell^m(\sin\phi)\cos m\lambda$$

$$g_r(\ell,m) = -(\ell+1)\, P_\ell^m(\sin\phi)\cos m\lambda$$

$$g_N(\ell,m) = -\left(\frac{m\sin\phi}{\cos\phi} P_\ell^m(\sin\phi) - P_\ell^{m+1}(\sin\phi)\right)\cos m\lambda$$

$$g_E(\ell,m) = -m P_\ell^m \sin m\lambda/\cos\phi$$

The results for g_r, g_E, and g_N are thus in a "per-unit" scaling, and apply for $C_{\ell m}$. For $S_{\ell m}$ add 90° East to the longitude. A satellite's orbit could be affected by g_E and g_N both in-plane and normal to the orbit plane, both horizontally, and in varying amounts if the orbit is eccentric.

Table 2 lists currently used coefficients for the earth [4] and moon [5] and Figures 8 and 9 give plots of U, g_r, g_E, and g_N for these fields.

Recursive formulas for the calculation of g_x, g_y, and g_z have been developed by DeWitt [2], superior for such calculations to those developed by James [3]. DeWitt writes

$$U_\ell^m = \frac{\mu R^\ell}{r^{\ell+1}} P_\ell^m(\sin\phi)\cos m\lambda$$

$$V_\ell^m = \frac{\mu R^\ell}{r^{\ell+1}} P_\ell^m(\sin\phi)\sin m\lambda$$

so that

$$U = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}(C_{\ell m}U_\ell^m + S_{\ell m}V_\ell^m)$$

and develops the recursive formulas

*Note that with μ, R, r all equal 1 and λ equal zero, then $U_\ell^m = P_\ell^m$. The recursive formulas can thus be used to calculate P_ℓ^m, if desired.

$$U_{\ell+1}^{m} = \left(\frac{R}{r}\right) \frac{1}{\ell-m+1} \{\cos\phi\,(2\ell+1)\,U_{\ell}^{m} - (\ell+m)\,U_{\ell-1}^{m}\}$$

$$V_{\ell+1}^{m} = \left(\frac{R}{r}\right) \frac{1}{\ell-m+1} \{\cos\phi\,(2\ell+1)\,V_{\ell}^{m} - (\ell+m)\,V_{\ell-1}^{m}\}$$

and

$$U_{\ell+1}^{\ell+1} = \left(\frac{R}{r}\right) (2\ell+1)\ \{U_{\ell}^{\ell}\cos\lambda - V_{\ell}^{\ell}\sin\lambda\}$$

$$V_{\ell+1}^{\ell+1} = \left(\frac{R}{r}\right) (2\ell+1)\ \{V_{\ell}^{\ell}\cos\lambda - V_{\ell}^{\ell}\sin\lambda\}$$

One starts with $U_0^{\,0} = \mu/r$, $V_0^{\,0} = 0$ and applies the last two formulas to the maximum ℓ desired, and then uses the other two to obtain the remaining needed terms.

The gradient of the potential, in xyz axes, is the vector

$$\nabla U = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} (C_{\ell m}\ \nabla U_{\ell}^{m} + S_{\ell m}\ \nabla V_{\ell}^{m})$$

where

$$\nabla U_{\ell}^{m} = \frac{1}{2R} \begin{bmatrix} (\ell-m+1)\ (\ell-m+2)\ U_{\ell+1}^{m-1} - U_{\ell+1}^{m+1} \\ -(\ell-m+1)\ (\ell-m+2)\ V_{\ell+1}^{m-1} - V_{\ell+1}^{m+1} \\ -2(\ell-m+1)U_{\ell+1}^{m} \end{bmatrix}$$

$$\nabla V_{\ell}^{m} = \frac{1}{2R} \begin{bmatrix} (\ell-m+1)\ (\ell-m+2)\ V_{\ell+1}^{m-1} - V_{\ell+1}^{m+1} \\ (\ell-m+1)\ (\ell-m+2)\ U_{\ell-1}^{m-1} + U_{\ell+1}^{m+1} \\ -2(\ell-m+1)V_{\ell+1}^{m} \end{bmatrix}$$

Thus if ℓ, m are the maximum indices for $C_{\ell m}$ and $S_{\ell m}$, then $\ell+1$, $m+1$ are the necessary maximum indices for U and V.

GRAVITY GRADIENT

The gravity gradient is a 3 x 3 matrix here called G.

Explicit formulas for the elements of G are given in Table 3. Recursive formulas for G were also developed by DeWitt and are

$$G = \nabla\nabla U = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} (C_{\ell m}\ \nabla\nabla U_{\ell}^{m} + S_{\ell m}\ \nabla\nabla V_{\ell}^{m})$$

where

$$\nabla\nabla U_{\ell}^{m} = \begin{bmatrix} \frac{(\ell-m+1)\ (\ell-m+2)\ \nabla U_{\ell+1}^{m-1} - \nabla U_{\ell+1}^{m+1}}{} \\ \frac{-(\ell-m+1)\ (\ell-m+2)\ \nabla V_{\ell+1}^{m-1} - \nabla V_{\ell+1}^{m+1}}{} \\ -2(\ell-m+1)\ \nabla U_{\ell+1}^{m} \end{bmatrix}$$

$$\nabla\nabla V_{\ell}^{m} = \begin{bmatrix} \frac{(\ell-m+1)\ (\ell-m+2)\ \nabla V_{\ell+1}^{m-1} - \nabla V_{\ell+1}^{m+1}}{} \\ \frac{(\ell-m+1)\ (\ell-m+2)\ \nabla U_{\ell+1}^{m-1} - \nabla U_{\ell+1}^{m+1}}{} \\ -2(\ell-m+1)\ \nabla V_{\ell+1}^{m} \end{bmatrix}$$

In this ∇V and ∇U become row vectors instead of column vectors so that, for example, $-2(\ell-m+1)\ \nabla U_{\ell+1}^{m}$ is the last row of $\nabla\nabla U_{\ell}^{m}$. Note that since $\nabla U_{\ell+1}^{m+1}$ involves $U_{\ell+2}^{m+2}$, more coefficients must be computed if the gradient is also required.

2014-WGH-jf

W. G. Heffron

TABLE I

EXPLICIT FORMULAS FOR GRAVITY

$$g_x = -\frac{\mu}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R}{r}\right)^{\ell}\frac{1}{2}\{(C_{\ell m}\cos(m+1)\lambda + S_{\ell m}\sin(m+1)\lambda)P_{\ell+1}^{m+1}(\sin\phi) -$$

$$-(\ell-m+1)(\ell-m+2)(C_{\ell m}\cos(m-1)\lambda + S_{\ell m}\sin(m-1)\lambda)P_{\ell+1}^{m-1}(\sin\phi)\}$$

$$g_y = -\frac{\mu}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R}{r}\right)^{\ell}\frac{1}{2}\{(C_{\ell m}\sin(m+1)\lambda - S_{\ell m}\cos(m+1)\lambda)P_{\ell+1}^{m+1}(\sin\phi) +$$

$$+(\ell-m+1)(\ell-m+2)(C_{\ell m}\sin(m-1)\lambda - S_{\ell m}\cos(m-1)\lambda)P_{\ell+1}^{m-1}(\sin\phi)\}$$

$$g_z = -\frac{\mu}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R}{r}\right)^{\ell}(\ell-m+1)(C_{\ell m}\cos m\lambda + S_{\ell m}\sin m\lambda)P_{\ell+1}^{m}(\sin\phi)$$

$$g_r = -\frac{\mu}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R}{r}\right)^{\ell}(\ell+1)(C_{\ell m}\cos m\lambda + S_{\ell m}\sin m\lambda)P_{\ell}^{m}(\sin\phi)$$

$$g_E = -\frac{\mu}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R}{r}\right)^{\ell}\frac{m}{\cos\phi}(C_{\ell m}\sin m\lambda - S_{\ell m}\cos m\lambda)P_{\ell}^{m}(\sin\phi)$$

$$g_N = -\frac{\mu}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R}{r}\right)^{\ell}(C_{\ell m}\cos m\lambda + S_{\ell m}\sin m\lambda)\left(\frac{m\sin\phi}{\cos\phi}P_{\ell}^{m}(\sin\phi) -\right.$$

$$\left.- P_{\ell}^{m+1}(\sin\phi)\right)$$

TABLE II

POTENTIAL MODELS

EARTH [4]

μ = 3.986012E14 m^3/sec^2

R = 6.37816E6 m

C_{20} = -1.0827E-3	C_{22} = 1.57E-6			
C_{30} = 2.56E-6	C_{31} = 2.10E-6	C_{32} = 2.5E-7	C_{33} = 7.7E-8	
C_{40} = 1.58E-6	C_{41} = -5.8E-7	C_{42} = 7.4E-8	C_{43} = 5.3E-8	C_{44} = -6.5E-9
C_{50} = 1.5E-7				
C_{60} = -5.9E-7				
C_{70} = 4.4E-7				

S_{22} = -8.97E-7			
S_{31} = 1.6E-7	S_{32} = -2.7E-7	S_{33} = 1.73E-7	
S_{41} = -4.6E-7	S_{42} = 1.6E-7	S_{43} = 4.0E-9	S_{44} = 2.3E-9

MOON (L-1 Model, [5])

μ = 4.90278E12 m^3/sec^2

R = 1.738E6 m

C_{20} = -2.07108E-4	C_{22} = 2.0716E-5	
C_{30} = 2.1E-5	C_{31} = 3.4E-5	C_{33} = 2.583E-6

TABLE III

GRAVITY GRADIENT FORMULAS

$$G_{xx} = + \frac{\mu}{r^3} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(\frac{R}{r}\right)^{\ell} \frac{1}{4}\{ (C_{\ell m} \cos(m+2)\lambda + S_{\ell m} \sin(m+2)\lambda) P_{\ell+2}^{m+2}(\sin\phi) -$$

$$- 2(\ell-m+1)(\ell-m+2)(C_{\ell m} \cos m\lambda + S_{\ell m} \sin m\lambda) P_{\ell+2}^{m}(\sin\phi) +$$

$$+ (\ell-m+1)(\ell-m+2)(\ell-m+3)(\ell-m+4)(C_{\ell m} \cos(m-2)\lambda +$$

$$+ S_{\ell m} \sin(m-2)\lambda) P_{\ell+2}^{m-2}(\sin\phi)\}$$

$$G_{xy} = G_{yx} = + \frac{\mu}{r^3} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(\frac{R}{r}\right)^{\ell} \frac{1}{4}\{ (C_{\ell m} \sin(m+2)\lambda -$$

$$- S_{\ell m} \cos(m+2)\lambda) P_{\ell+2}^{m+2}(\sin\phi) - (\ell-m+1)(\ell-m+2)(\ell-m+3)(\ell-m+4) \text{ times}$$

$$(C_{\ell m} \sin(m-2)\lambda - S_{\ell m} \cos(m-2)\lambda) P_{\ell+2}^{m-2}(\sin\phi)\}$$

$$G_{xz} = G_{zx} = \frac{\mu}{r^3} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(\frac{R}{r}\right)^{\ell} \frac{1}{2}\{ (\ell-m+1)(C_{\ell m} \cos(m+1) +$$

$$+ S_{\ell m} \sin(m+1)\lambda) P_{\ell+2}^{m+1}(\sin\phi) - (\ell-m+1)(\ell-m+2)(\ell-m+3) \text{ times}$$

$$(C_{\ell m} \cos(m-1)\lambda + S_{\ell m} \sin(m-1)\lambda) P_{\ell+2}^{m-1}(\sin\phi)\}$$

TABLE III (continued)

$$G_{yy} = \frac{\mu}{r^3} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(\frac{R}{r}\right)^{\ell} \frac{1}{4}\{-(C_{\ell m} \cos(m+2)\lambda + S_{\ell m} \sin(m+2)\lambda)P_{\ell+2}^{m+2}(\sin\phi) -$$

$$- 2(\ell-m+1)(\ell-m+2)(C_{\ell m} \cos m\lambda + S_{\ell m} \sin m\lambda)P_{\ell+2}^{m}(\sin\phi) -$$

$$- (\ell-m+1)(\ell-m+2)(\ell-m+3)(\ell-m+4)(C_{\ell m} \cos(m-2)\lambda +$$

$$+ S_{\ell m} \sin(m-2)\lambda)\ P_{\ell+2}^{m-2}(\sin\phi)\}$$

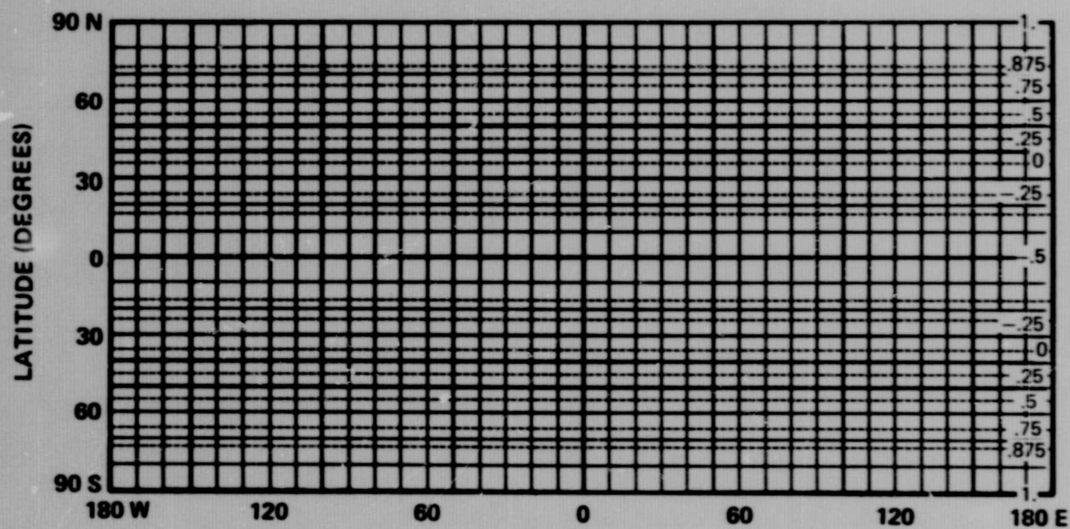
$$G_{yz} = G_{zy} = \frac{\mu}{r^3} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(\frac{R}{r}\right)^{\ell} \frac{1}{2}\{(\ell-m+2)(C_{\ell m} \sin(m+1)\lambda -$$

$$- S_{\ell m} \cos(m+1)\lambda)P_{\ell+2}^{m+1}(\sin\phi) + (\ell-m+1)(\ell-m+2)(\ell-m+3) \text{ times}$$

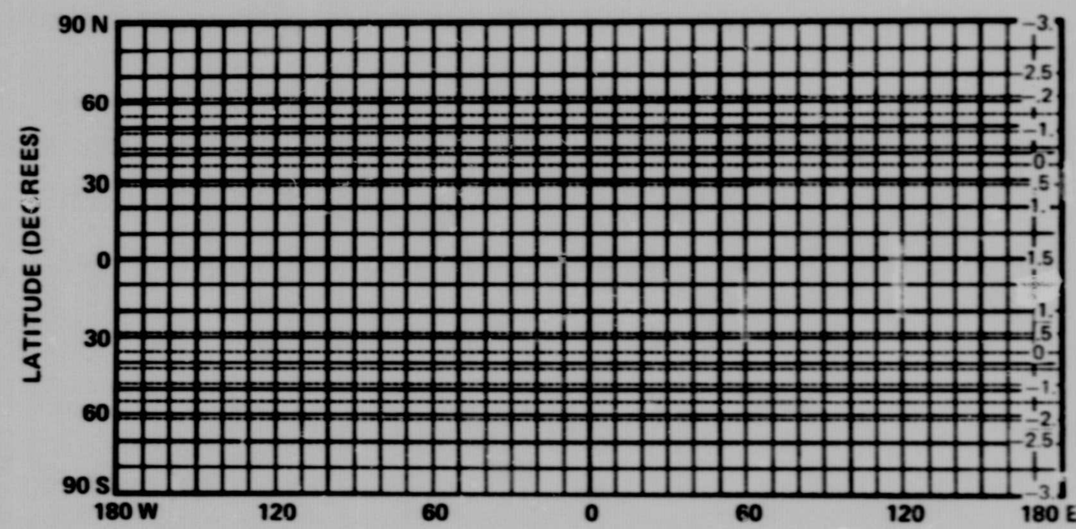
$$(C_{\ell m} \sin(m-1)\lambda - S_{\ell m} \cos(m-1)\lambda)P_{\ell+2}^{m-1}(\sin\phi)\}$$

$$G_{zz} = \frac{\mu}{r^3} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(\frac{R}{r}\right)^{\ell} (\ell-m+1)(\ell-m+2)(C_{\ell m} \cos m\lambda + S_{\ell m} \sin m\lambda)P_{\ell+2}^{m}(\sin\phi)$$

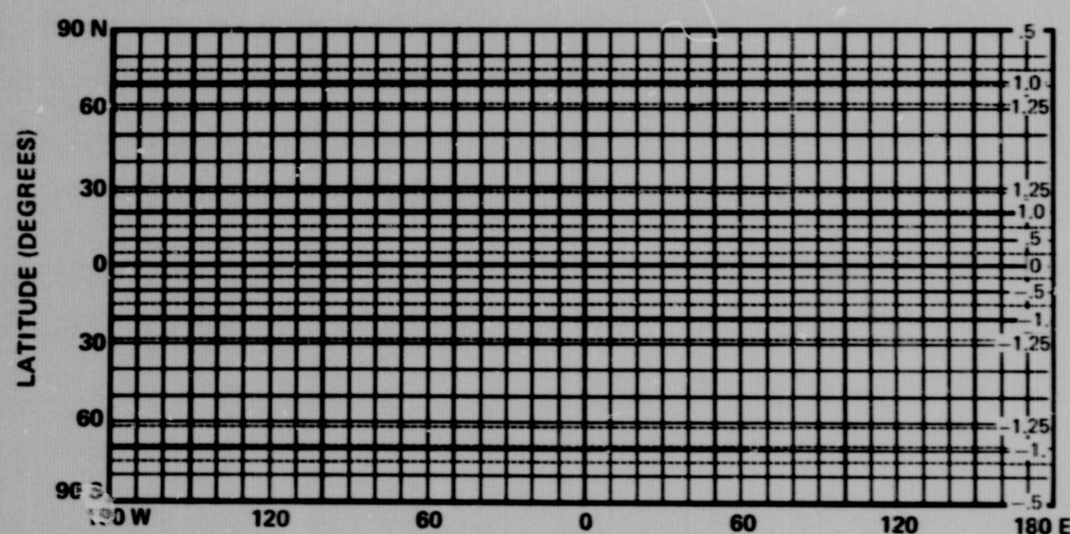
Note: $G_{xx} + G_{yy} + G_{zz} = 0$ (LaPlace's equation)



POTENTIAL EQUAL SURFACES



RADIAL ACCELERATION



NORTH ACCELERATION

ZERO

EAST ACCELERATION

FIGURE 1 - EFFECTS OF C_{20}

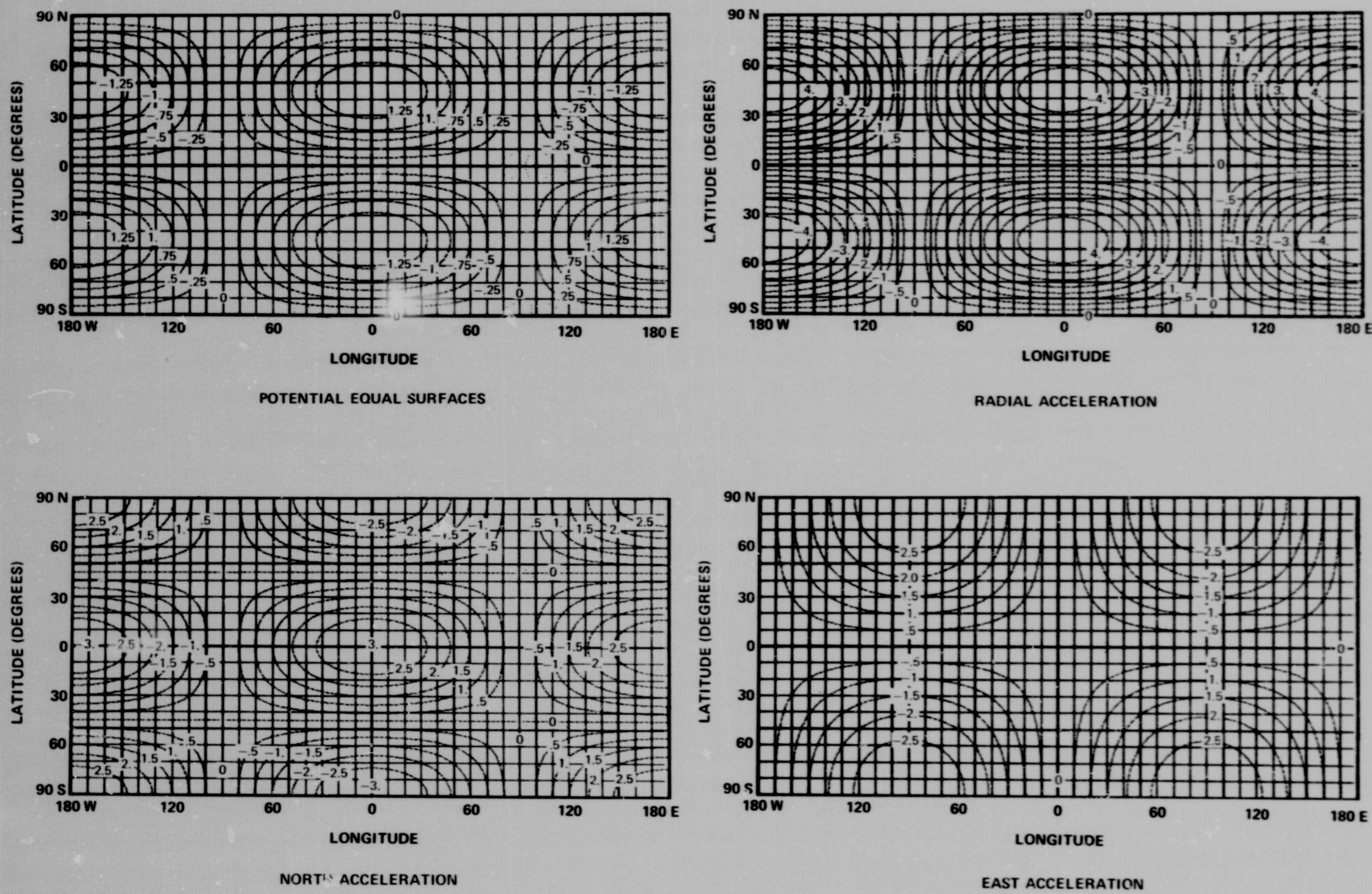
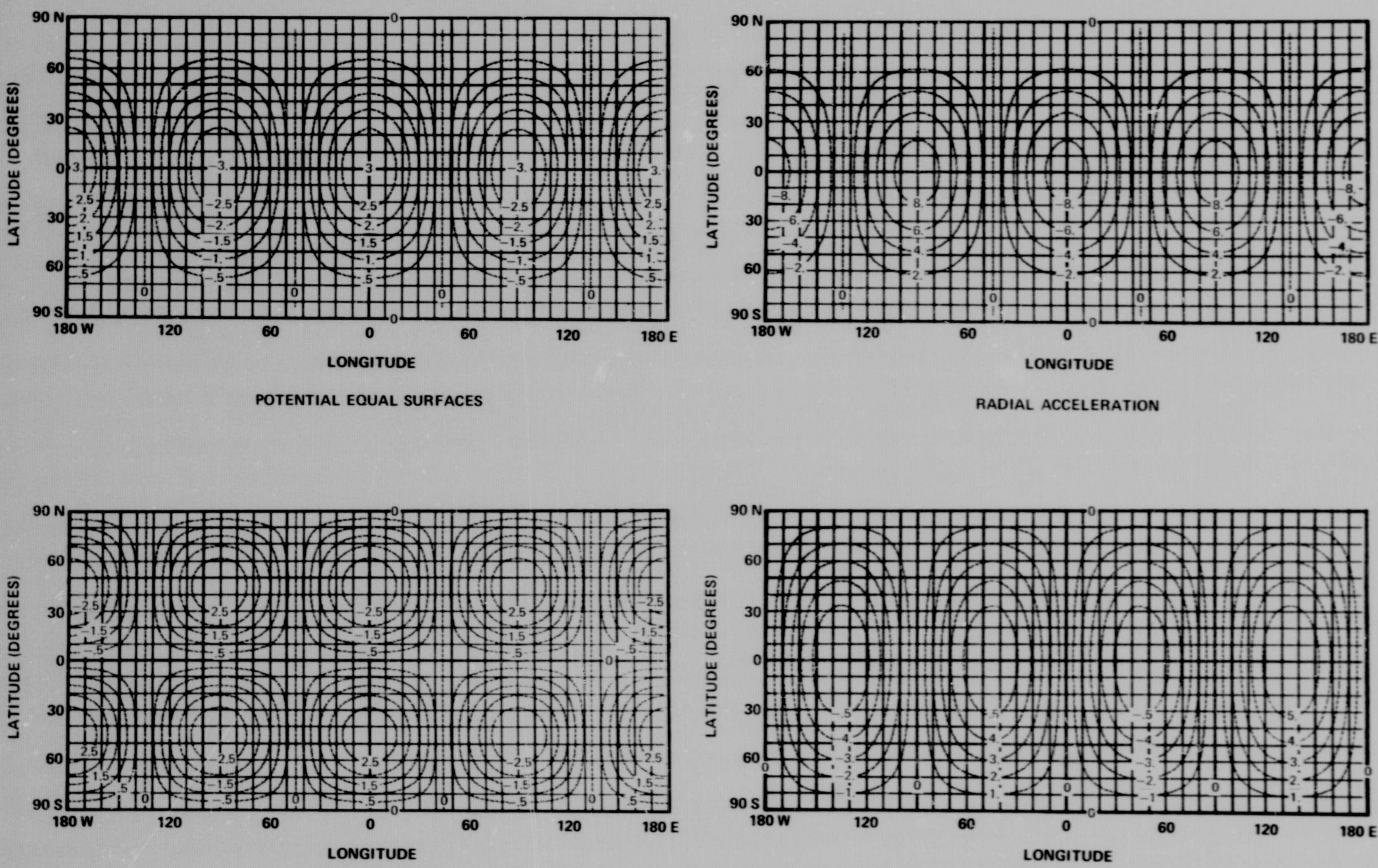


FIGURE 2 - EFFECTS OF C_{21}



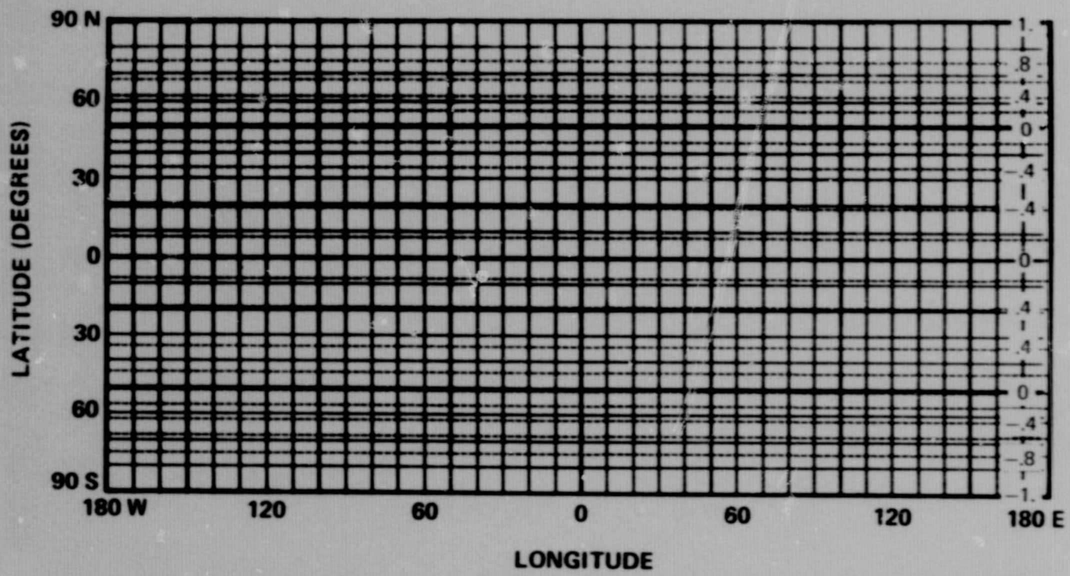
POTENTIAL EQUAL SURFACES

RADIAL ACCELERATION

NORTH ACCELERATION

EAST ACCELERATION

FIGURE 3 - EFFECTS OF C_{22}



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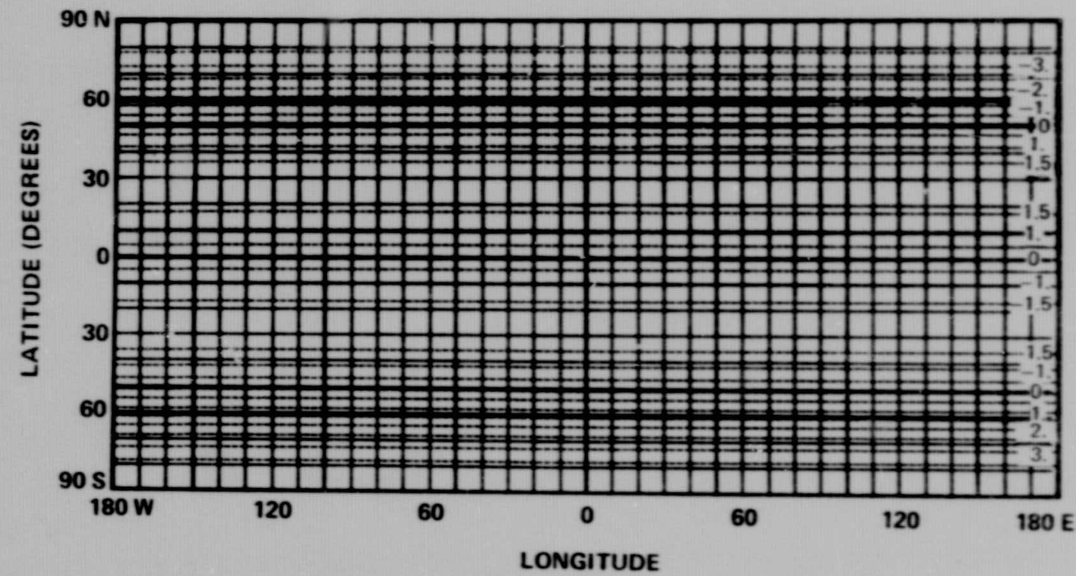

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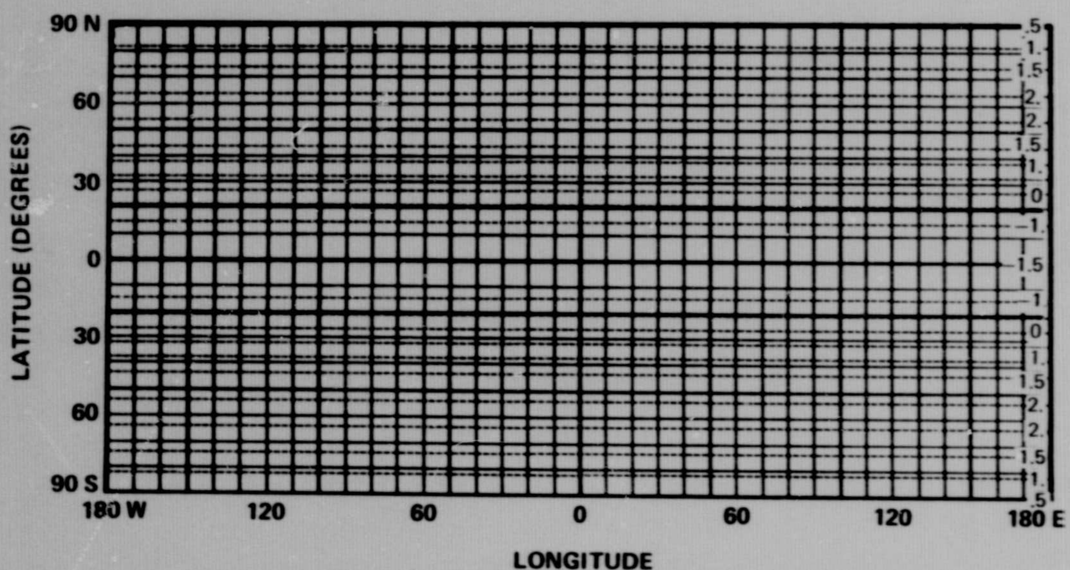
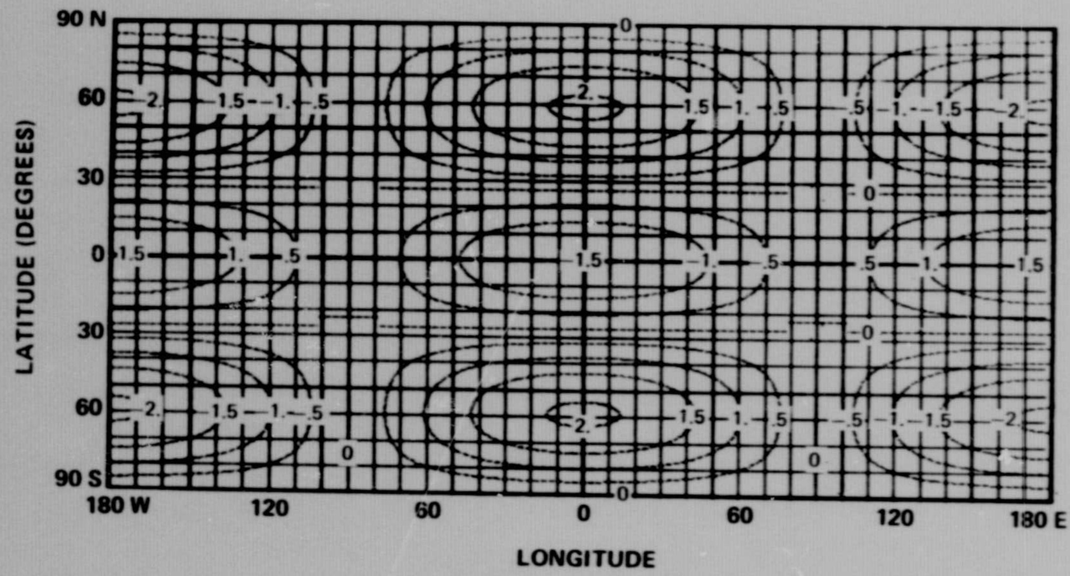


NORTH ACCELERATION

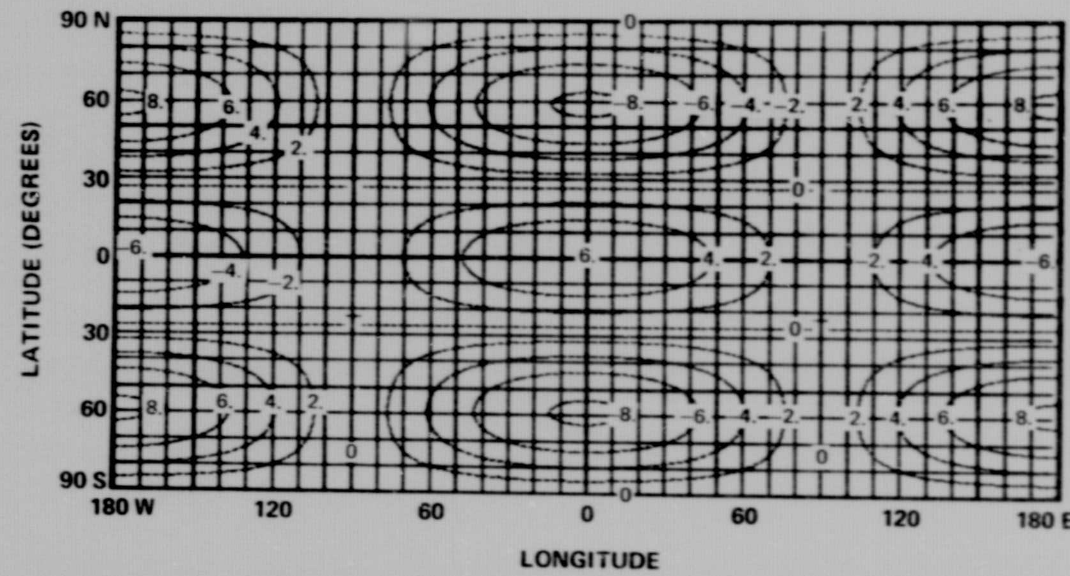
ZERO

EAST ACCELERATION

FIGURE 4 - EFFECTS OF C_{30}



POTENTIAL EQUAL SURFACES



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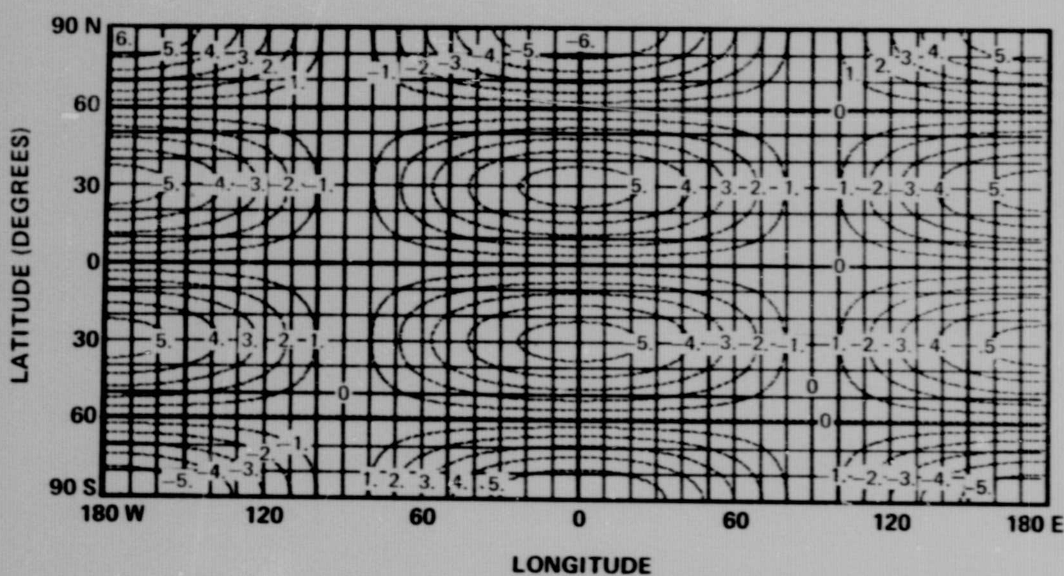

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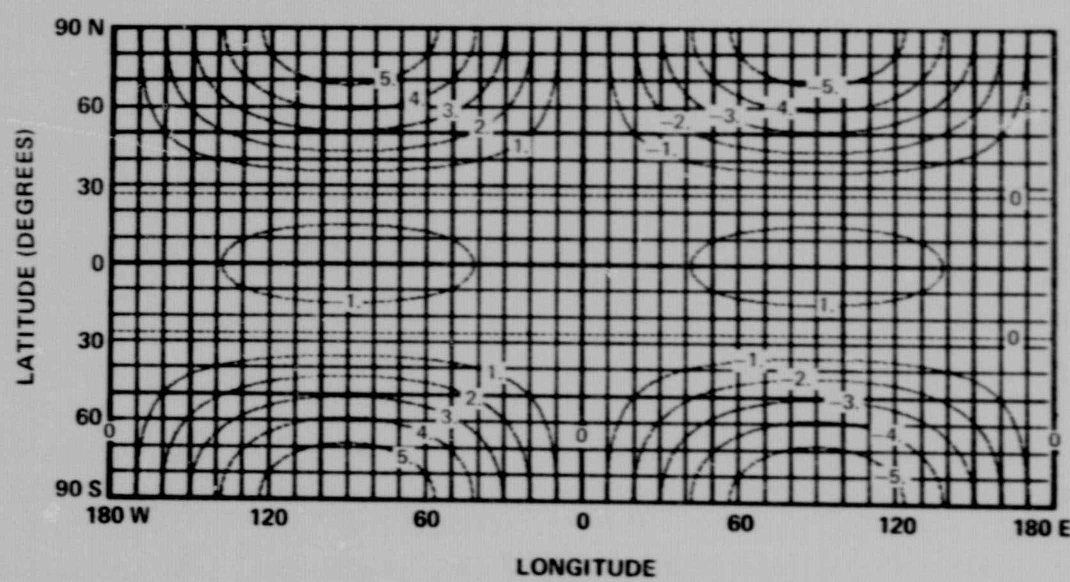


EAST ACCELERATION

FIGURE 5 - EFFECTS OF C_{31}

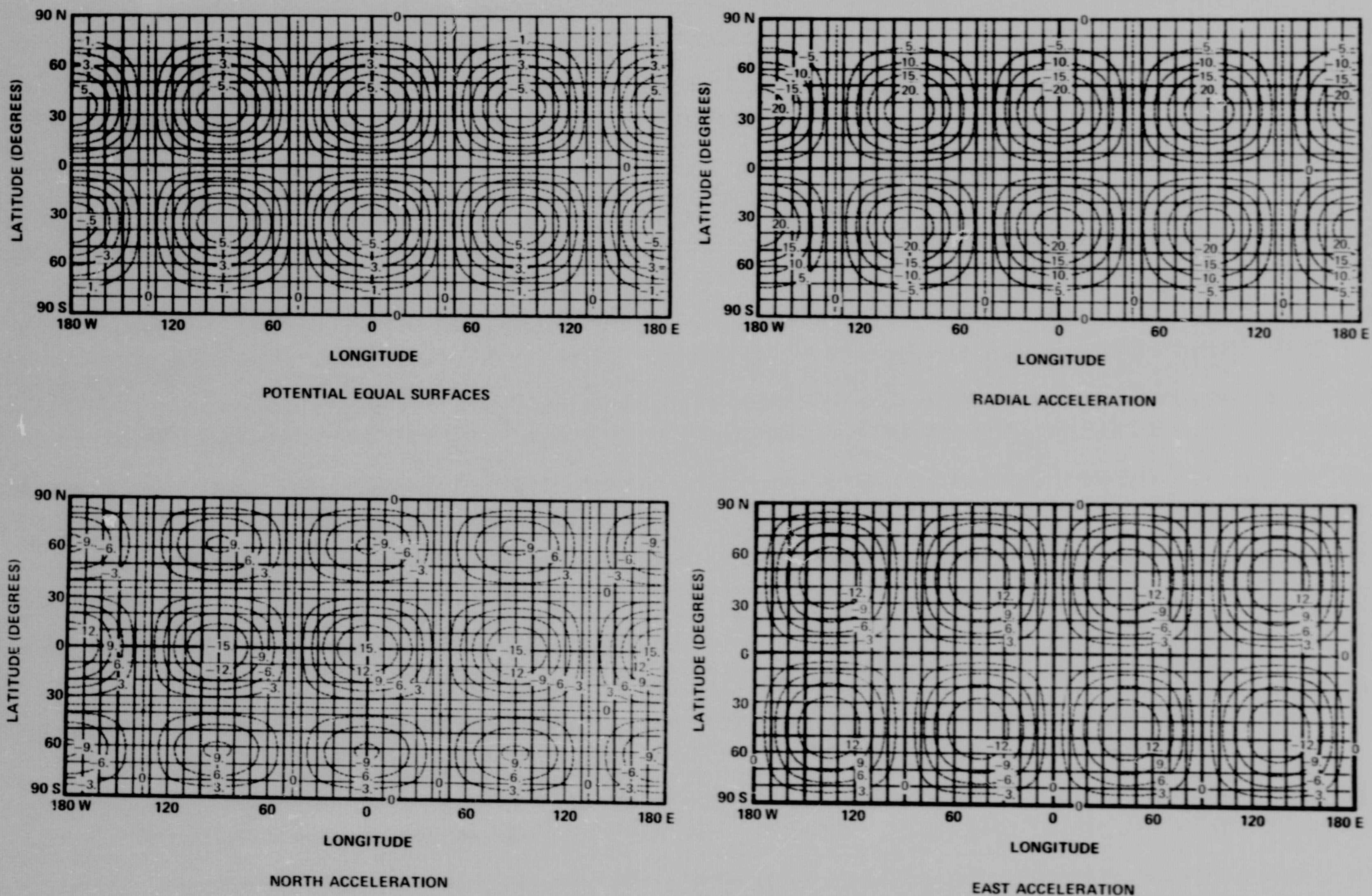
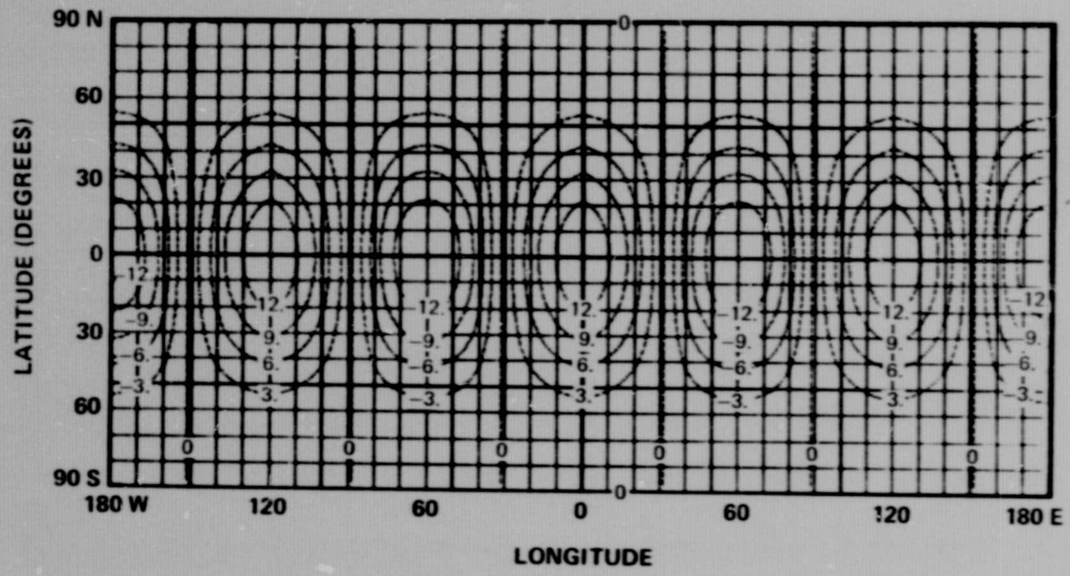


FIGURE 6 - EFFECTS OF C_{32}



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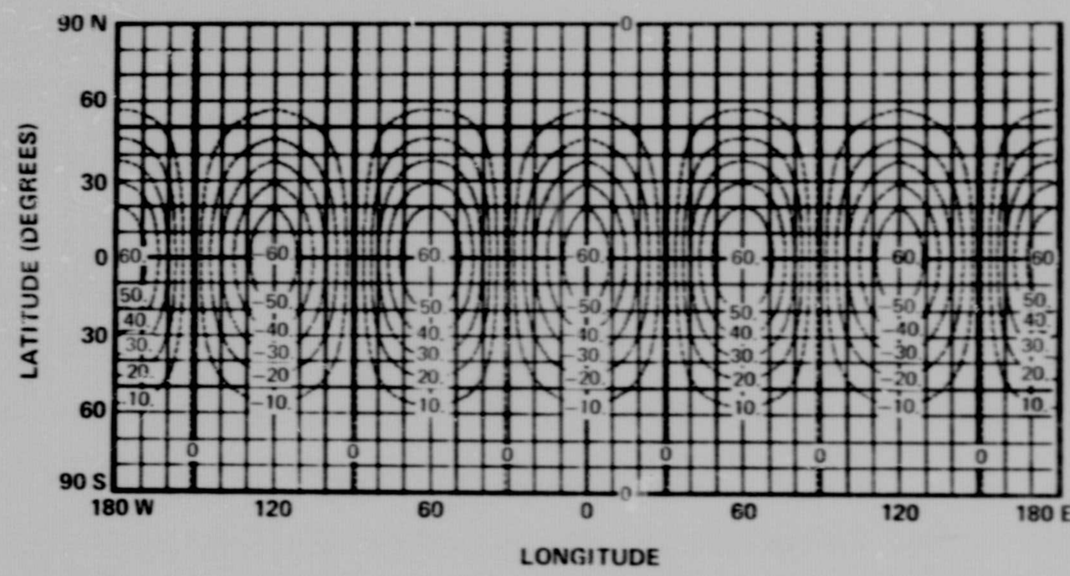

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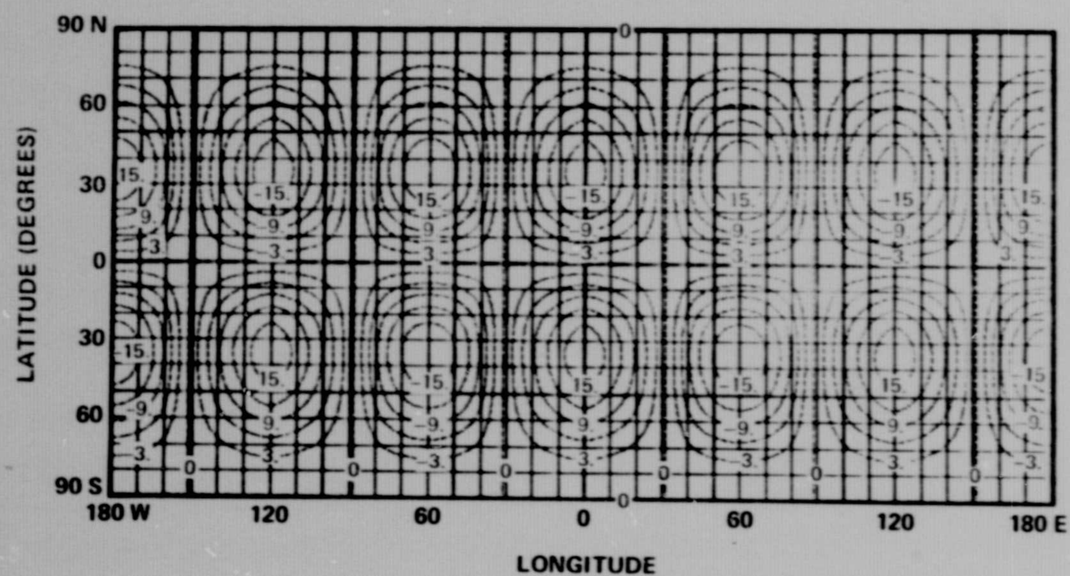
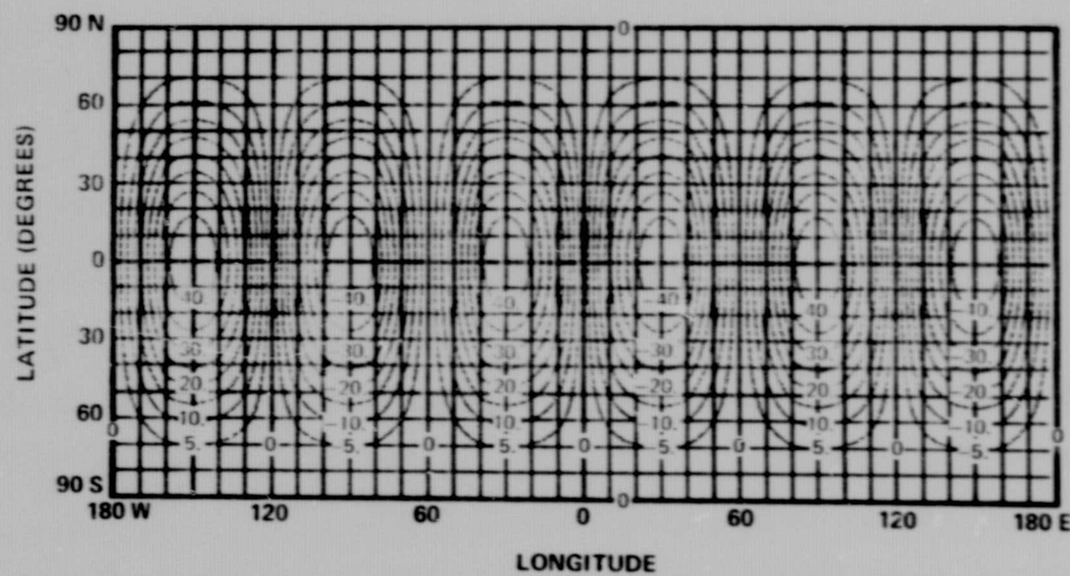
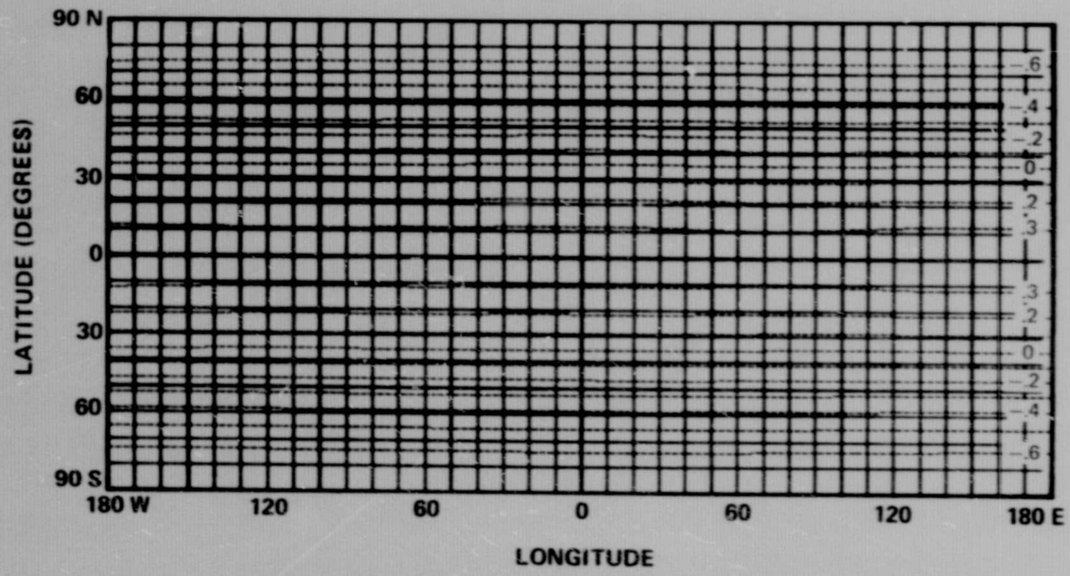


NORTH ACCELERATION



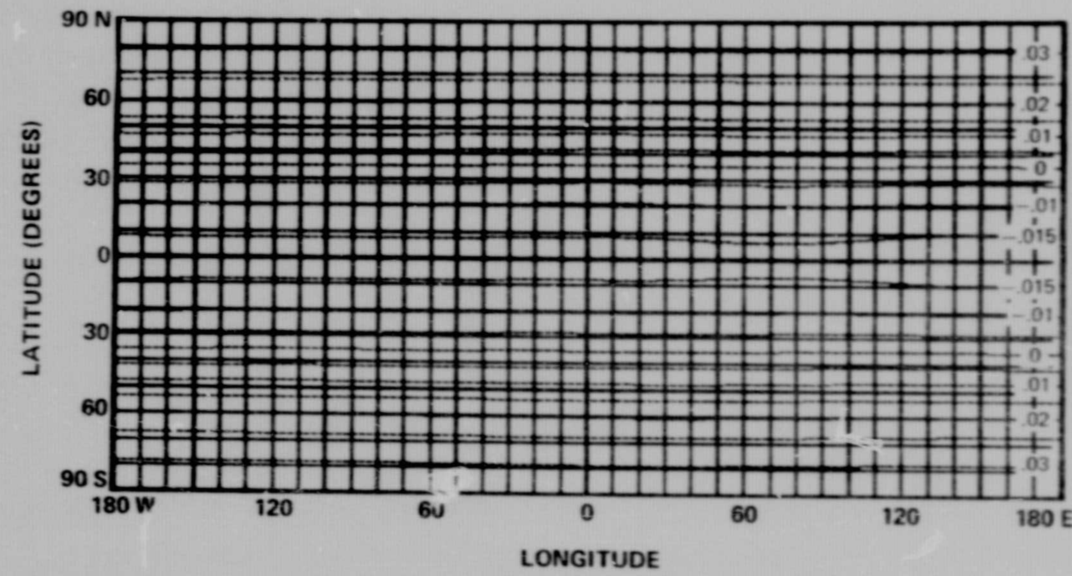
EAST ACCELERATION

FIGURE 7 - EFFECTS OF C_{33}



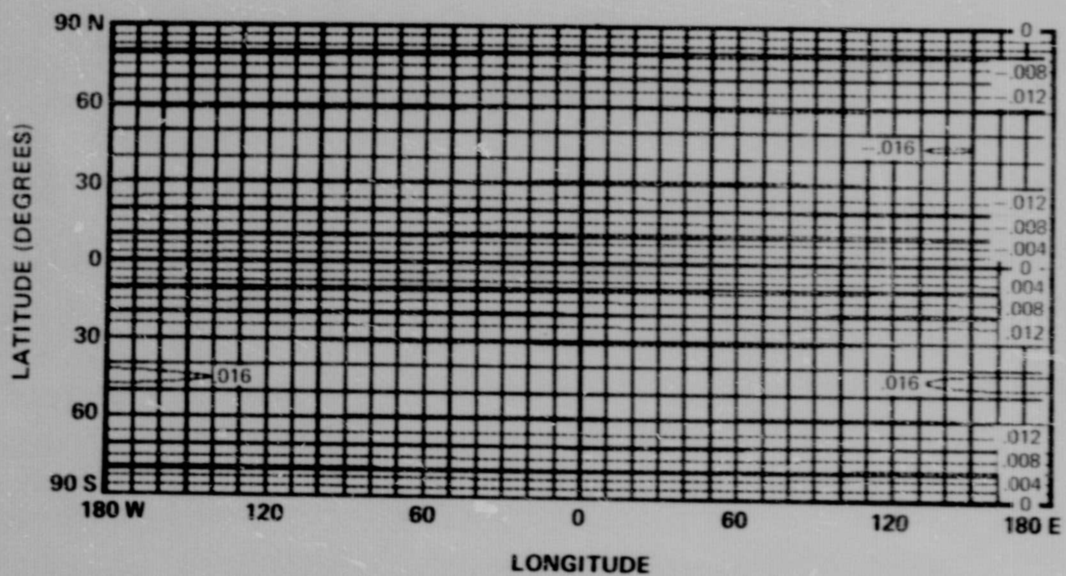
POTENTIAL EQUAL SURFACES (M^2/SEC^2)

(ADD $\frac{\mu}{R}$ = 6.249E7 M^2/SEC^2)

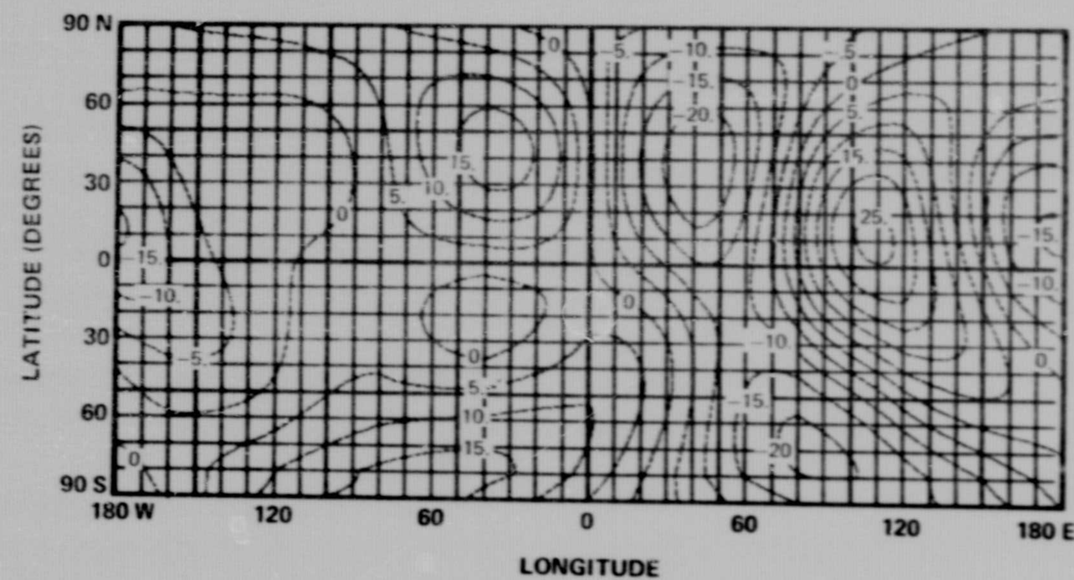


RADIAL ACCELERATION (M/SEC^2)

(ADD $\frac{\mu}{R^2}$ = 9.798 M/SEC^2)

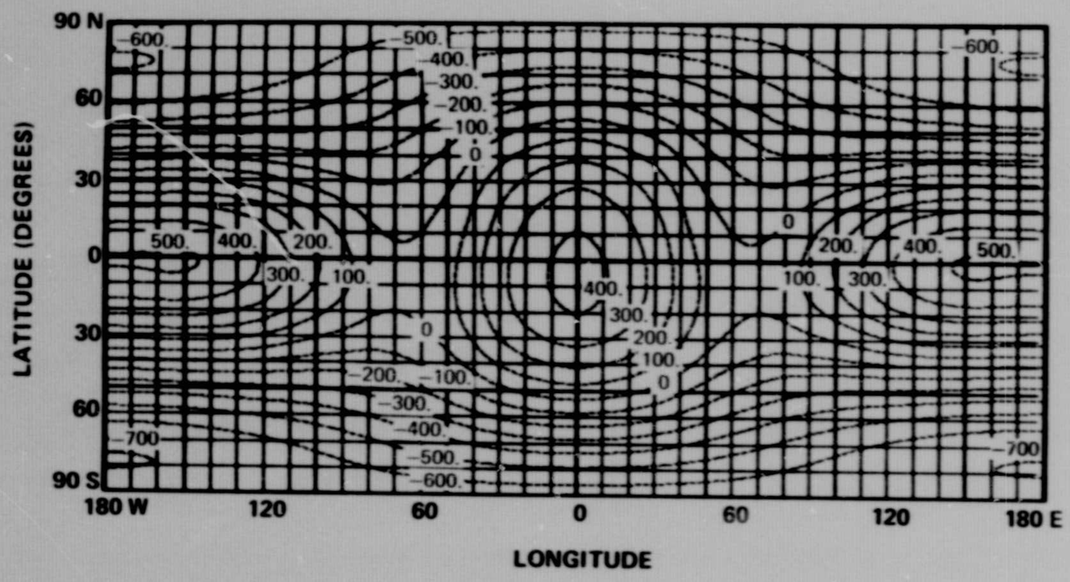


NORTH ACCELERATION (M/SEC^2)



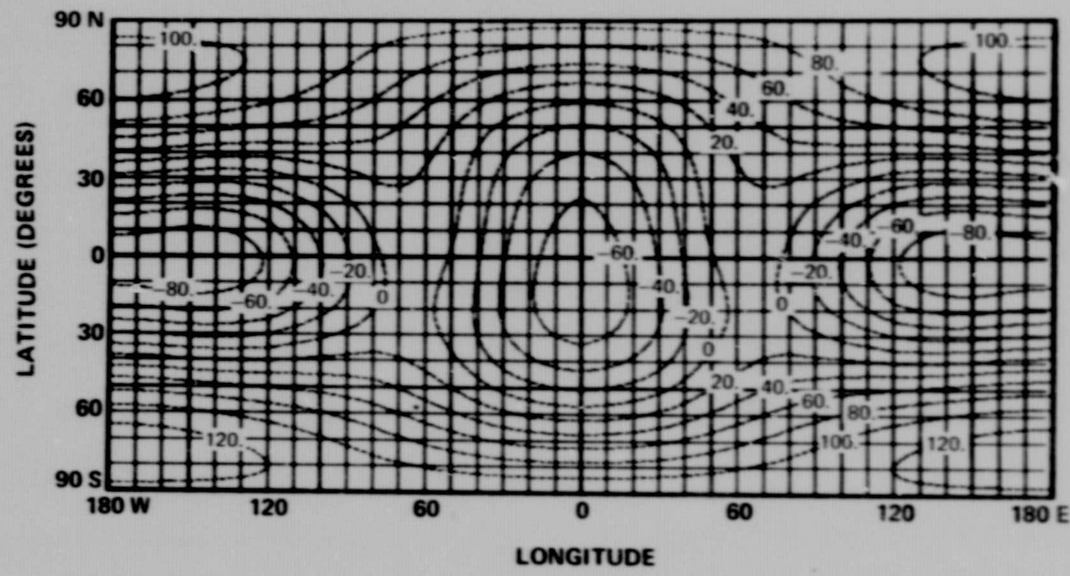
EAST ACCELERATION (10^{-5} M/SEC^2)

FIGURE 8 - THE EARTH POTENTIAL AND GRAVITY
(SEE TABLE 2)



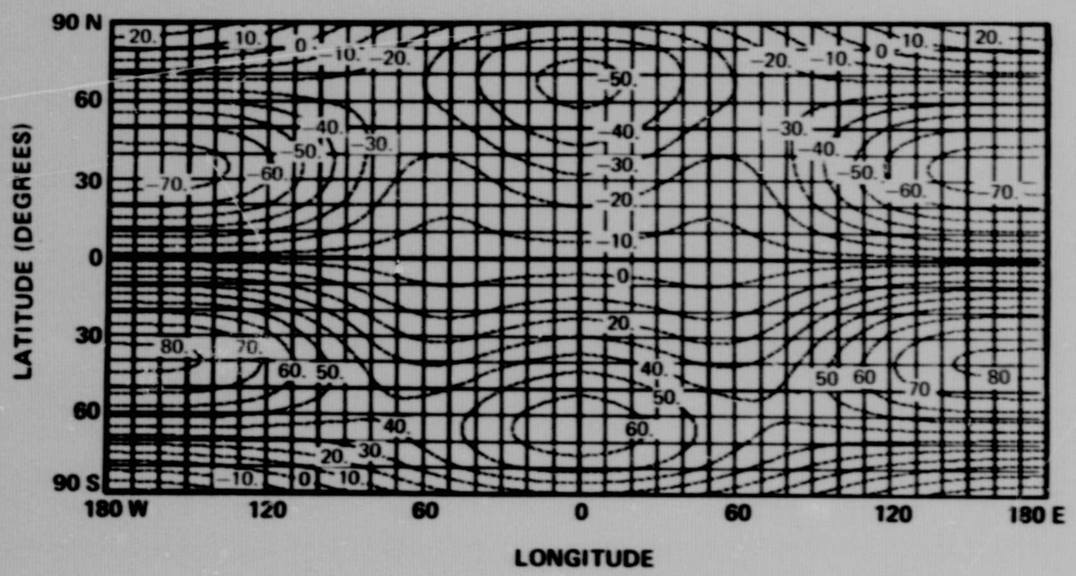
POTENTIAL EQUAL SURFACES (M^2/SEC^2)

(ADD $\frac{\mu}{R}$ = 2.821E6 M^2/SEC^2)

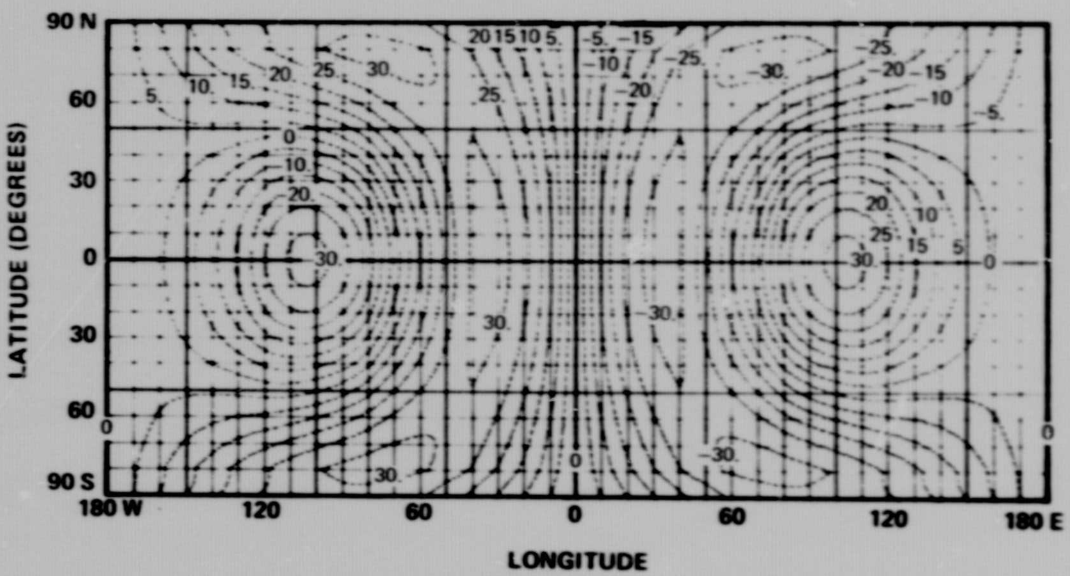


RADIAL ACCELERATION (10^{-5} M/SEC^2)

(ADD $\frac{\mu}{R^2}$ = 1.623 M/SEC^2)



NORTH ACCELERATION (10^{-5} M/SEC^2)



EAST ACCELERATION (10^{-5} M/SEC^2)

FIGURE 9 - THE MOON'S POTENTIAL AND GRAVITY
(SEE TABLE 2)

REFERENCES

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