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SOLUTIONS TO A PROBLEM IN POWER SERIES REVERSION*

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Abstract. This paper presents the general solution of the following problem in two forms. Let $f(x, y)$ be defined by the formal power series $f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn}x^m y^n$ with $f_{00} \neq 0$. If v satisfies $v(x, y) = f(xv^a, yv^b)$, where a and b are constants, then find the formal power series expansion of $v^c(x, y)$, where c is also a constant.

A special case of this problem, which occurs in a paper by R. A. Handelsman and J. S. Lew [1], has been proposed as a problem to be solved by computer using a symbolic algebra system [2].

1. Introduction and summary. In this paper we give two formulations of the answer to the following problem.

Let $f(x, y)$ be defined by

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn}x^m y^n$$

with $f_{00} \neq 0$. If v satisfies

$$v(x, y) = f(xv^a, yv^b), \tag{1}$$

then find the formal power series expansion of $v^c(x, y)$ for arbitrary c.

First, we show that

$$(v^c)_{mn} = \begin{cases} (1 + c \cdot \ln f)_{mn}, & am + bn + c = 0, \\ \dfrac{c}{am + bn + c}(f^{am+bn+c})_{mn}, & am + bn + c \neq 0, \end{cases} \tag{2}$$

where the notation $(g)_{mn}$ denotes the coefficient of $x^m y^n$ in the series expansion of $g(x, y)$.

Equation (2) is conceptually simple but computationally difficult. To provide a formula more amenable to computation we show that (2) can be rewritten as

$$(v^c)_{mn} = \begin{cases} f_{00}^c, & m = n = 0, \\ cf_{00}^{am+bn+c} \sum_{k=1}^{m+n} F_k(m, n)(am + bn + c - 1)_{k-1} f_{00}^{-k}, & m + n > 0, \end{cases} \tag{3}$$

where $(w)_k$ is the falling factorial defined by

$$(w)_0 = 1,$$

$$(w)_k = w(w - 1) \cdots (w - k + 1), \quad k > 0,$$

and F_k is defined by the generating function

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_k(m, n)x^m y^n z^k = e^{(f(x,y) - f(0,0))z} \tag{4}$$

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We then derive the following recursive formula for the computation of $F_k(m, n)$ for all k, m and n:

$$\begin{aligned} &F_0(0,0) = 1, && \\ &F_0(m,n) = 0, && m+n>0, \\ (5)\qquad &F_k(m,n) = 0, && m+n<k, \\ &F_{k+1}(m,n) = \frac{1}{m+n}\sum_{i=0}^{m}\sum_{j=0}^{n}(i+j)f_{ij}F_k(m-i,n-j), && m+n>0. \end{aligned}$$

2. Derivation of formula (2). In order to illustrate a technique that may be applicable to other similar problems, we give a derivation based on the residue operator for formal power series[1] given in [3]. Proofs based on complex variable theory can be obtained for the one-variable case from Lagrange's theorem [4, §7.32], and for the two-variable case from the generalization of that theorem by I. J. Good in [5].

We shall briefly summarize the relevant results of [3]. Let $h(x_1, \cdots, x_r)$ be a formal power series in r variables (i.e., no convergence restrictions) of the form

$$h = \sum_{n_1=k_1}^{\infty} \cdots \sum_{n_r=k_r}^{\infty} h_{n_1 \cdots n_r} x_1^{n_1} \cdots x_r^{n_r},$$

where $k_1, \cdots, k_r$ are finite but may be negative. The sum, difference, product and partial derivatives of formal power series are defined in the usual way and have the usual properties. For exponentiation, let h have a nonzero constant term h_0 and no negative exponents. Thus

$$h = h_0(1 + H(x_1, \cdots, x_n)), \tag{6}$$

where H has a zero constant term and no negative exponents. Then h^α can be defined by

$$h^\alpha = h_0^\alpha\left(\sum_{k=0}^{\infty} \frac{(\alpha)_k H^k}{k!}\right)$$

and this exponentiation has all the usual properties. For h as in (6), we define

$$\begin{aligned} \ln h &= \ln h_0 + \ln(1+H) \\ &= \ln h_0 + \sum_{k=1}^{\infty} (-1)^{k+1} H^k/k \end{aligned}$$

and

$$\begin{aligned} e^h &= e^{h_0} e^{h_0 H} \\ &= e^{h_0} \sum_{k=0}^{\infty} (h_0 H)^k / k!. \end{aligned}$$

These have the usual inverse and differentiation properties.

The basic requirement in the manipulation of formal power series is a finiteness condition: If the manipulation of the operands $g, h, \cdots$ (which are formal power series) results in a formal power series f, then the coefficient of any term in f may

[1] I. Niven gives an excellent survey of formal power series in [6].

not involve more than a finite (but possibly unbounded) number of coefficients of the operands $g, h, \cdots$.

The residue operator applied to any h is defined by

$$R(h) = \text{coefficient of } x_1^{-1} \cdots x_r^{-1} \text{ in } h.$$

As a consequence,

$$R(h/x_1^{n_1+1} \cdots x_r^{n_r+1}) = \text{coefficient of } x_1^{n_1} \cdots x_r^{n_r} \text{ in } h. \tag{7}$$

The main result for this operator deals with substitutions [3]. Let

$$g_i = x_1^{m_{i1}} \cdots x_r^{m_{ir}} G_i(x_1, \cdots, x_r), \qquad i = 1, \cdots, r,$$

where G_i has no negative exponents and a nonzero constant term and $m_{ij} \geqq 0$ with $\sum_{j=1}^{r} m_{ij} > 0$. Then

$$\det(m_{ij})\, R(h) = R\left(h(g_1, \cdots, g_r)\frac{\partial(g_1, \cdots, g_r)}{\partial(x_1, \cdots, x_r)}\right), \tag{8}$$

where the Jacobian is defined by

$$\frac{\partial(g_1, \cdots, g_r)}{\partial(x_1, \cdots, x_r)} = \det\left(\frac{\partial g_i}{\partial x_j}\right).$$

The fact that $h(g_1, \cdots, g_r)$ is well-defined can be shown by observing that each coefficient in its formal series will involve only a finite number of the coefficients from the g_i (see [7] for a detailed proof).

For our problem, let us first make a change of variable from x, y to s, t, so that (1) becomes

$$v(s, t) = f(sv^a(s, t), tv^b(s, t)),$$

where $f_{00} \neq 0$ and therefore $v_{00} = f_{00}$. If

$$x = sv^a(s, t),$$

$$y = tv^b(s, t),$$

then

$$v(s, t) = f(x, y)$$

giving

$$s = xf^{-a}(x, y),$$

$$t = yf^{-b}(x, y),$$

$$v(xf^{-a}(x, y), yf^{-b}(x, y)) = f(x, y),$$

which is the dual of equation (1).

Now by (7),

$$(v^c)_{mn} = R(v^c(s, t)/s^{m+1}t^{n+1}),$$

and so the substitution theorem (8) yields

$$(v^c)_{mn} = R\left(\frac{v^c(xf^{-a}, yf^{-b})}{x^{m+1}y^{n+1}} f^{am+a+bn+b} \frac{\partial(xf^{-a}, yf^{-b})}{\partial(x, y)}\right)$$

$$= R\left(\frac{f^{am+bn+a+b+c}}{x^{m+1}y^{n+1}} \cdot \frac{\partial(xf^{-a}, yf^{-b})}{\partial(x, y)}\right).$$

The Jacobian has the value

$$f^{-a-b}\left(1 - axf^{-1}\frac{\partial f}{\partial x} - byf^{-1}\frac{\partial f}{\partial y}\right)$$

giving (with $d = am + bn + c$)

$$(v^c)_{mn} = R\left(f^d\left(1 - axf^{-1}\frac{\partial f}{\partial x} - byf^{-1}\frac{\partial f}{\partial y}\right)\Big/x^{m+1}y^{n+1}\right)$$

$$= R(f^d/x^{m+1}y^{n+1}) - aR\left(f^{d-1}\frac{\partial f}{\partial x}\Big/x^m y^{n+1}\right) - bR\left(f^{d-1}\frac{\partial f}{\partial y}\Big/x^{m+1}y^n\right).$$

Now for any h,

$$R\left(\frac{\partial h}{\partial x}\Big/x^m y^{n+1}\right) = mR(h/x^{m+1}y^{n+1}). \tag{9}$$

Thus if $d \neq 0$, the second term is

$$\frac{-a}{d} R\left(\frac{\partial f^d}{\partial x}\Big/x^m y^{n+1}\right) = \frac{-am}{d} R(f^d/x^{m+1}y^{n+1})$$

and similarly the third term is $(-bn/d)R(f^d/x^{m+1}y^{n+1})$. Replacing d by $am + bn + c$ and adding, we have

$$(v^c)_{mn} = \frac{c}{am + bn + c}(f^{am+bn+c})_{mn},$$

which is the second part of (2).

If $d = 0$, the second term is

$$-aR\left(f^{-1}\frac{\partial f}{\partial x}\Big/x^m y^{n+1}\right) = -aR\left(\frac{\partial \ln f}{\partial x}\Big/x^m y^{n+1}\right)$$

$$= -amR(\ln f/x^{m+1}y^{n+1}),$$

and similarly the third is

$$-bnR(\ln f/x^{m+1}y^{n+1}).$$

Therefore, since $d = 0$ we have $c = -am - bn$ and

$$(v^c)_{mn} = (1 + c \cdot \ln f)_{mn},$$

which is the first part of (2).

3. Derivation of formulas (3) and (5). Formula (3) is a special case of the following more general result: If $h(x, y) = g(u)$ and $u = f(x, y)$ where f is a given formal power series with no negative exponents and $g(u)$ has a series expansion around the point $u_0 = f(0, 0) = f_{00}$, then the coefficients of the formal series for h are given by:

$$h_{00} = g(f(0, 0)),$$

(10)
$$h_{mn} = \sum_{k=1}^{m+n} F_k(m, n) \left. \frac{d^k g}{du^k} \right|_{u=f(0,0)}, \qquad m + n > 0,$$

where, as before, the $F_k(m, n)$ are defined by

(11)
$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_k(m, n) x^m y^n z^k = e^{(f(x,y) - f(0,0))z}.$$

Using (10) we obtain (3) in the case $am + bn + c \neq 0$ by just specializing $g(u)$ to $(c/\gamma)u^\gamma$ so that $h(x, y) = (c/\gamma)f^\gamma$. Then (10) gives

$$\left(\frac{c}{\gamma} f^\gamma\right)_{00} = \frac{c}{\gamma} f_{00}^\gamma,$$

$$\left(\frac{c}{\gamma} f^\gamma\right)_{mn} = c \sum_{k=1}^{m+n} F_k(m, n)(\gamma)_{k-1} f_{00}^{\gamma-k}, \qquad m + n > 0.$$

Setting $\gamma = am + bn + c$ and substituting these into the lower part of (2), we arrive at (3).

To obtain (3) for the case $am + bn + c = 0$, we specialize $g(u)$ to $1 + \gamma \cdot \ln u$, so that $h(x, y) = 1 + \gamma \cdot \ln f$. Then (10) gives

$$(1 + \gamma \cdot \ln f)_{00} = 1 + \gamma \cdot \ln f_{00},$$

$$(1 + \gamma \cdot \ln f)_{mn} = \gamma \sum_{k=1}^{m+n} F_k(m, n)(-1)_{k-1} f_{00}^{-k}, \qquad m + n > 0.$$

Setting $\gamma = c = -am - bn$ and substituting these into the upper part of (2), we obtain

$$(v^c)_{mn} = \begin{cases} 1, & m = n = c = 0, \\ c \sum_{k=1}^{m+n} F_k(m, n)(-1)_{k-1} f_{00}^{-k}, & m + n > 0, \end{cases}$$

which agrees with (3) when $am + bn + c = 0$.

Now to prove (5) and (10), we first equate the coefficients of z^k in (11) to obtain

(12)
$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_k(m, n) x^m y^n = \begin{cases} 1, & k = 0, \\ \dfrac{(f(x, y) - f(0, 0))^k}{k!}, & k > 0. \end{cases}$$

From this, we see immediately that

$$F_0(0,0) = 1,$$
$$F_0(m,n) = 0, \quad m+n > 0,$$
$$F_k(m,n) = 0, \quad m+n < k,$$

which are the first three parts of (5).

If we expand $h(x, y)$ in a series around the point u_0, we have

$$h(x,y) = \sum_{k=0}^{\infty} \frac{d^k g}{du^k}\bigg|_{u=u_0} \frac{(u-u_0)^k}{k!}.$$

If $u = f(x, y)$ and $u_0 = f(0, 0)$, then

$$h(x,y) = \sum_{k=0}^{\infty} \frac{d^k g}{du^k}\bigg|_{u=f(0,0)} \frac{(f(x,y)-f(0,0))^k}{k!},$$

whence by (12),

$$h(x,y) = \sum_{k=0}^{\infty} \frac{d^k g}{du^k}\bigg|_{u=f(0,0)} \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_k(m,n)x^m y^n\right).$$

Interchanging the order of summation and equating coefficients of x and y we obtain (10).

It remains to prove the last formula in (5). First differentiate (11) with respect to x to obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F_k(m,n) m x^{m-1} y^n z^k = e^{(f(x,y)-f(0,0))z}\left(z \frac{\partial}{\partial x} f(x,y)\right)$$
$$= \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F_k(m,n)x^m y^n z^{k+1}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{mn} m x^{m-1} y^n\right).$$

Equating coefficients of z^{k+1} we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_{k+1}(m,n) m x^{m-1} y^n = \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_k(m,n) x^m y^n\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{mn} m x^{m-1} y^n\right).$$

Applying the convolution formula for the product of two power series and equating the coefficients of $x^{m-1}y^n$, we obtain

$$mF_{k+1}(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} i f_{ij} F_k(m-i, n-j).$$

Similarly, differentiating (11) with respect to y, we obtain

$$nF_{k+1}(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} j f_{ij} F_k(m-i, n-j).$$

Finally, adding these two equations and dividing by $m + n$ we obtain the last formula in (5).

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