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TELEPHONE AND POWER TRANSMISSION

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GENERAL INTRODUCTION

It is now many years since the principles which govern the propagation of electric waves along lines were discovered, and there are a number of excellent treatises in existence dealing with the various aspects of the subject. In particular their application to telephony is very fully and ably dealt with by Mr. J. G. Hill,* and also by Professor Fleming,† and it is therefore evident that any further attempt to cover the same ground needs some explanation.

The present book, however, is not so much a treatise as a series of notes on lectures delivered by the authors to students of the London University, and the object specially aimed at is to present the subject in as concise, simple, and practical a form as possible. Their experience is that the books already in existence are apt, by their very completeness, to give the impression that the matter is one of considerable complexity and beyond the grasp of practical engineers not well equipped with a thorough mathematical training. This is not the case. The general theory is simple, and the only stumbling block is the use of complex hyperbolic functions without the aid of which the problems are practically insoluble. But this difficulty is far more apparent than real, since it is quite unnecessary to form any mathematical conception of these

* "Telephonic Transmission," J. G. Hill.

† "Propagation of Electric Currents," J. A. Fleming.

functions in order to be able to use them for the purpose in hand. Once it is realised that they are far more formidable in appearance than in reality, and are merely a useful tool requiring no great skill in manipulation, the difficulty largely vanishes.

It has been the authors' aim, therefore, to write for that vast majority of students and practical engineers who are engineers first and mathematicians only in so far as their profession demands. They have therefore assumed a minimum of mathematical knowledge, and have commenced with an introductory chapter in which the use of Vectors and Hyperbolic Function is explained. Once this chapter is thoroughly mastered there should be no difficulty in grasping the theory of Transmission dealt with in Chapter II., and the reader is then in a position to apply his knowledge to any of the practical directions in which it is now so valuable.

The first of these applications is to the problems arising in long-distance telephone transmission, and the remainder of Part I. is devoted to this subject. It was, indeed, for this purpose that the theory was originally developed, and Telephone Engineers were amongst the first to recognise its extreme practical importance. It is no exaggeration to say that the result has been a complete revolution in the methods of long-distance telephony, accompanied by a vast reduction in cost and gain in efficiency. Formerly long-distance circuits were composed of heavy gauge aerial lines, expensive to erect, costly and difficult to maintain, and subject to frequent and inevitable breakdown, owing to climatic conditions. Moreover, the number of circuits which can be provided on any one air route is obviously strictly limited. The modern method, which is a direct application of the theory, consists in the employment of loaded underground cables of comparatively light gauge with their attendant advantages of reliability and ease of maintenance coupled with the all-important fact that an unlimited number of circuits between important points can be provided. The invention of telephone repeaters has enabled the gauge of circuits to be reduced still further, and a combination of the two methods, that is, a loaded underground cable with repeaters at suitable intervals, enables long-distance circuits to be provided with a maximum of efficiency and a minimum of cost.

Loading is not confined to underground cables, but is applied with equal success to submarine lines. Both lump and series loading are used, and the resulting advantages are very great indeed. Before the introduction of loading the conductors required to be of comparatively heavy gauge and the number which would be provided in a single cable was very small. Moreover, the capacity of submarine cables is large, and the distortion of speech transmitted by them was accordingly great. The addition of loading not only permits of more wires of lighter gauge to be included in the cable, but has also improved the quality of the speech transmitted to an enormous extent. When the great cost of submarine cables is borne in mind the advantages accruing from loading will be appreciated.

The second part of the book includes the application of the theory to the transmission of power along single and three-phase transmission lines. In the past this aspect of the subject has not received much attention in England, the voltages and distances of transmission being such as to make the various approximate methods neglecting line capacity sufficiently accurate. With the increase of long-distance high-voltage power transmission, however, the accurate calculation of voltage drop by the application of the general theory of transmission will become of increasing importance.

The attention devoted to the theory of travelling waves needs little apology, since the extreme importance of these transient phenomena in power transmission is now well recognised. The usual simplified treatment is given in Chapter VII., and is supplemented by a fuller mathematical discussion in Chapter IX. The latter treatment is necessarily somewhat incomplete owing to the necessity of avoiding higher mathematics; but, even so, its inclusion will, it is hoped, serve to give increased confidence in the results arrived at by the simplified process. The authors desire to express their appreciation of the assistance rendered by Professor G. S. Le Beau, Professor of Mathematics at East London College, in the preparation of this chapter.

The preparation of the book has been divided between the two authors, Mr. Bradfield being responsible for Part I., and Mr. John for Part II. They desire to acknowledge their indebtedness to Mr. J. G. Hill for permission to reproduce Tables I. and II. (pages 227 and 228) from his book, "Telephonic Transmission," * already referred to, and also to Professor A. E. Kennelly, who kindly allowed the shortened Tables of Complex Hyperbolic Functions to be included. These are inserted in order to familiarise the reader with the use of the full tables to which reference is made later, and which are, of course, essential for the solution of practical cases.

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PART I.

TELEPHONE TRANSMISSION.

CHAPTER I.

MATHEMATICAL INTRODUCTION.

It is impossible to read any treatise on mathematics without encountering immediately the term **function**—a term the meaning of which it is essential to grasp. It may be defined as follows. One variable quantity is said to be a function of another when the relationship between them is such that the value of one is determined by the value of the other. Thus, if

$$y = \log_e x, y = x^2, y = e^x,$$

then, for every value of x, there is, in each case, a corresponding value of y, and y is a function of x in all three instances. In general, when such relationship exists, we write

$$y = \phi(x)$$

or

$$y = f(x),$$

which means that y is some function or other, at present unknown, of x.

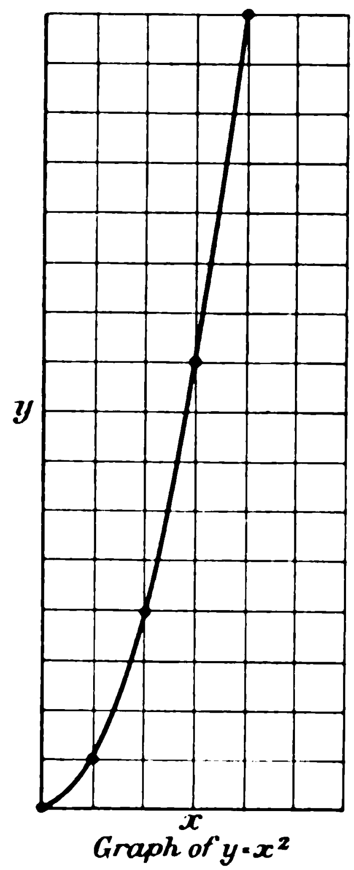


FIG. 1.

If we take a piece of squared paper and plot values of y against corresponding values of x, we obtain a curve which is the graphic representation of the function. Thus Fig. 1 shows the graph of the function $y = x^2$.

Now observe the curve drawn in Fig. 2.

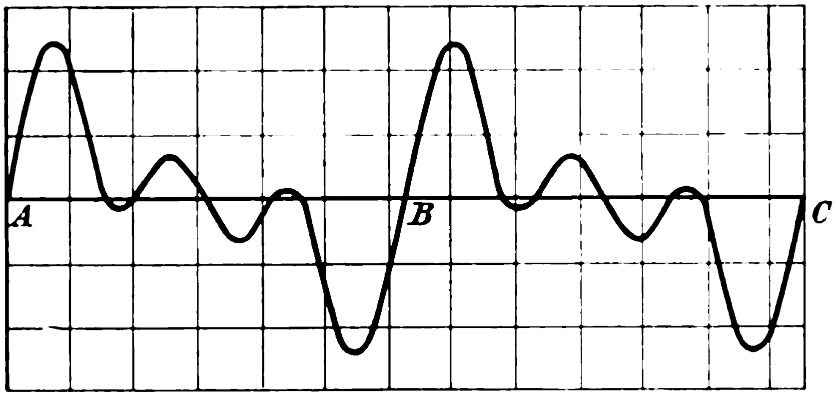


FIG. 2.

This curve is the graph of some function of x, but it has this special property, that the section from B to C is an exact reproduction of the section from A to B, and the curve goes on repeating itself in this manner indefinitely. Such a function is called a **periodic function,** and its **period** is AB. Amongst the best known of such periodic functions are the ordinary Circular Functions sin x, cos x, etc., which repeat themselves at intervals of 2π.

Now, it has been proved that *any* single-valued periodic function can be replaced by a series of the form

$$\begin{aligned} &A + B \sin x + C \cos x \\ &\quad + D \sin 2x + E \cos 2x \\ &\quad + F \sin 3x + G \cos 3x \\ &\quad + \ldots \end{aligned}$$

provided the constants A, B, C . . . are suitably chosen.* This series is known as **Fourier's series,** and is of fundamental importance for the following reason. It permits an irregular periodic curve of any shape which would otherwise be difficult or impossible to treat mathematically to be resolved into a series of sines and cosines which are easily susceptible of separate mathematical treatment. It gives us, moreover, a

* The constant term A in the series determines the position of the curve above or below the axis of x. If the curve is symmetrical about the axis of x this term of course disappears and this condition is that commonly encountered.

conception of the nature of such irregular curves which is of the utmost importance and should be fully grasped, since the whole theory of transmission is based upon it. It does not matter how irregular may be the form of the electric wave whose behaviour we are considering; all that is necessary is to split it into its component parts and examine what occurs to each separately. If we are able to do this we have the key to the behaviour of the wave as a whole.

It is not necessary here to examine the precise method by which a curve can be so resolved. The reverse process, however, illustrates in a rough manner the truth of the theorem. For example, the curve in Fig. 2 is built up, as shown in Fig. 3, from the following series :—

$$\sin x, \qquad \sin 2x, \qquad \sin 3x.$$

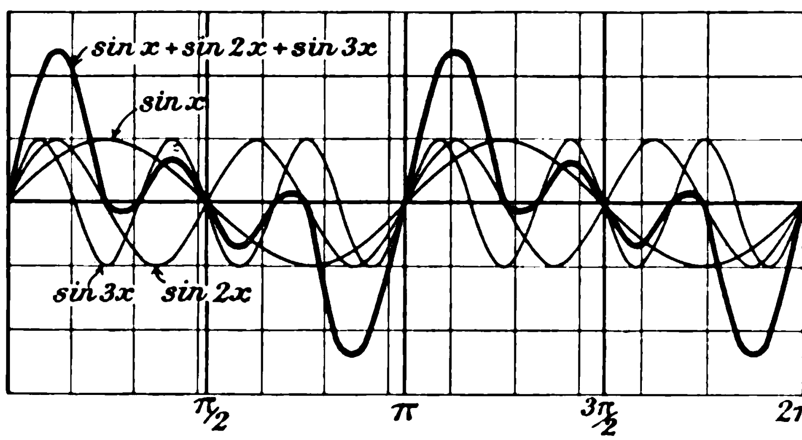


FIG. 3.

It is not, of course, necessary that all the terms of the series should be present. In the above instance, sin $4x$ and subsequent terms in addition to the cosine terms are missing; or, to regard it in another way, the coefficient of these terms is zero.

It will be noted that the first term of the series has the same period as that of the original curve. It is usual to speak of this term as the **fundamental** and of the other terms as **harmonics.** Thus, the term involving sin $2x$ is the **first harmonic,** and so on.

It is necessary here to observe that there is unfortunately a difference in practice as between physicists and electrical engineers, particularly power engineers, in the use of the term "harmonics." The definition given above is that employed by the former, but engineers frequently refer to the fundamental as the *first* harmonic, so that the term involving sin $2x$ is then the *second* harmonic. This method has the advantage of simplicity, since the term involving sin nx is then the nth and not the $(n - 1)$th harmonic.

VECTORS.

In physics, quantities are frequently met with possessing not only magnitude and sense (i.e. they can be either positive or negative), but which possess *direction* in addition. Thus, alternating sinusoidal currents or voltages are not fully described unless, in addition to their magnitude, their phase relationship is stated. Quantities such as these may be represented by **vectors** in the manner shown in Fig. 4.

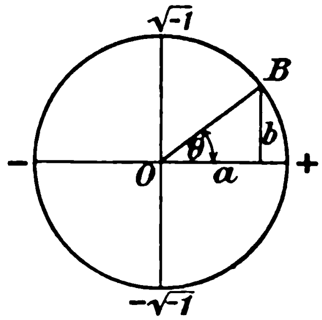


FIG. 4.

Here the vector OB represents a quantity whose magnitude is given by the length of OB, and whose direction is given by the angle θ which OB makes with the horizontal axis. If the length of OB is A, we may indicate the vector by the expression $A\underline{/\theta}$, or $\sqrt{a^2 + b^2}\underline{/\tan^{-1}\frac{b}{a}}$ where a and b are the projections of OB on the horizontal and vertical axes respectively. The portion A is called the **modulus,** while θ is called the **argument,** or, often, simply the **angle.**

We may regard projections on the horizontal axis to the right of O as positive, and to the left of O as negative. But what sign is to be given to projections on the vertical axis? Now, it requires one operation for a vector to travel from the positive horizontal direction to the vertical, and another exactly equal operation to reach the negative direction. Therefore we may regard the vertical axis above O as possessing the sign $\sqrt{-1}$, since, if we perform the operation denoted by $\sqrt{-1}$ twice, we arrive at the negative sign. Similarly, the vertical axis below A is given the sign $-\sqrt{-1}$, since, if we perform the operation denoted by $\sqrt{-1}$ once again, we arrive at the positive sign.

A quantity having the sign $\sqrt{-1}$, or, as it is usually written, j, is frequently referred to as an **unreal** or **imaginary** quantity. It is not necessary here to enter into any discussion as to the exact significance of the sign, but only to remember what operation it represents, and also to remember that all quantities bearing this sign must be treated separately from those which are **real,** i.e. which bear the ordinary positive or negative sign.

To return to the vector OB: if we now introduce the notation arrived at, we may write the horizontal projection as $+a$ and the vertical projection as $+jb$; and, since the vector is the resultant of these projections, it is completely fixed in magnitude and direction by the expression

$$a + jb.$$

Such an expression is called a **complex quantity,** since it consists of an "imaginary" component jb in addition to a "real" component a. Further, the projection $a = \text{A} \cos \theta$, and the projection $b = \text{A} \sin \theta$, so that

$$a + jb = \text{A}(\cos \theta + j \sin \theta).$$

Again, by De Moivre's Theorem referred to later,

$$\cos \theta + j \sin \theta = e^{j\theta},$$

so that $$\text{A}(\cos \theta + j \sin \theta) = \text{A}e^{j\theta}.$$

Hence we have the following five ways in which the vector OB may be denoted :—

$$\text{A}\underline{/\theta},$$

$$\sqrt{a^2 + b^2}\underline{/\tan^{-1}\frac{b}{a}},$$

$$a + jb,$$

$$\text{A}(\cos \theta + j \sin \theta),$$

$$\text{A}e^{j\theta}.$$

We have so far considered only a vector in the first quadrant, but there is no difficulty in extending the notation to vectors

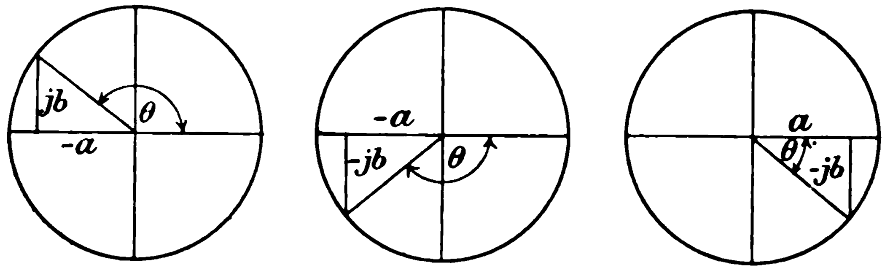


FIG. 5.

in other quadrants. Thus (Fig. 5) a vector in the second quadrant is

$$A\underline{/\theta} \quad \text{or} \quad -a + jb.$$

A vector in the third quadrant is $A\underline{/-\theta}$, which is usually written $A\overline{\diagdown\theta}$ to denote that the angle is negative. Or, alternatively, it may be written $-a - jb$. Finally, a vector in the fourth quadrant is $A\overline{\diagdown\theta}$ or $a - jb$.

The various modes of denoting a vector are easily interchangeable, and it is necessary to gain facility in the process. For example, if it is required to express the vector $10\underline{/30^\circ}$ in the form $a + jb$, we proceed as follows :—

$$a = A\cos 30^\circ = 10 \times \cdot 866 = 8\cdot 66.$$
$$b = A\sin 30^\circ = 10 \times \cdot 5 = 5.$$
$$\therefore\ 10\underline{/30^\circ} = 8\cdot 66 + j5.$$

Again,

$$\begin{aligned} 5\cdot 7\underline{/118^\circ} &= 5\cdot 7\cos 118^\circ + j5\cdot 7\sin 118^\circ \\ &= 5\cdot 7(-\cdot 469) + j5\cdot 7 \times \cdot 883 \\ &= -2\cdot 67 + j5\cdot 03. \end{aligned}$$

The reverse process is equally simple. Thus, to put $4 + j3$ into the form $A\underline{/\theta}$ we have only to remember that

$$A = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5,$$
$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}(\cdot 75) = 36^\circ\ 53',$$
$$\therefore\ 4 + j3 = 5\underline{/36^\circ\ 53'}.$$

It now remains to show how the ordinary processes of addition, multiplication, etc., can be performed on vector quantities, and it is largely to facilitate these operations that the various forms by which a vector may be denoted have been derived.

Addition and Subtraction.—The vectors must first be put into the form $a + jb$, $a' + jb'$ . . .

Then **add or subtract the real and imaginary parts separately,** thus

$$(a + a' + \ldots) + j(b + b' + \ldots).$$

Example.—

$$\begin{aligned}&(10 + j5) + (6 - j4) - (8 + j10)\\ &= (10 + 6 - 8) + j(5 - 4 - 10)\\ &= 8 - j9.\end{aligned}$$

Multiplication.—The vectors must be put into the form

$$A\underline{/\theta},\ A'\underline{/\theta'},\ \ldots$$

Then **multiply the moduli and add the angles,*** thus,

$$(A \times A' \times \ldots)\underline{/\theta + \theta' + \ldots}$$

Example.—

$$15\underline{/57^\circ} \times 10\overline{\backslash 30^\circ} = 15 \times 10\underline{/57^\circ - 30^\circ} = 150\underline{/27^\circ}.$$

Division is performed in a manner exactly similar to multiplication, the rule being, **divide the moduli and subtract the angles.**

Example.—

$$\frac{4\underline{/70^\circ}}{8\underline{/30^\circ}} = \frac{4}{8}\underline{/70^\circ - 30^\circ} = \cdot 5\underline{/40^\circ}.$$

Powers of a Vector.—This is only a special case of multiplication. Thus

$$(A\underline{/\theta})^n = (A \times A \times \ldots n \text{ times})\underline{/\theta + \theta + \ldots \text{to } n \text{ terms}} = A^n\underline{/n\theta}.$$

* Proof. Let the vectors be put into the form $Ae^{j\theta}$, $A'e^{j\theta'}$, ... Multiplying we have

$$(A \times A' \times \ldots)e^{j(\theta + \theta' + \ldots)} = (A \times A' \times \ldots)\underline{/\theta + \theta' + \ldots}$$

Example.—

$$(3\angle 80^\circ)^3 = 27\angle 240^\circ = 27\overline{\setminus 120^\circ}.$$

A particular case is the **root of a vector.** Thus

$$\sqrt{A\angle\theta} = (A\angle\theta)^{\frac{1}{2}} = A^{\frac{1}{2}}\angle\theta/2.$$

Example.—

$$\sqrt{4\angle 30} = 2\angle 15^\circ.$$

Logarithm of a Vector.—This operation is performed as follows :—

$$\log_e (A\angle\theta) = \log_e (Ae^{j\theta}) = \log_e A + \log_e e^{j\theta}.$$

But, from the definition of a logarithm,

$$\log_e e^{j\theta} = j\theta,$$

$$\therefore \log_e A\angle\theta = \log_e A + j\theta.$$

where θ is in *radians.*

Example.—

$$\begin{aligned}\log_{10} (4\angle 86^\circ) &= \cdot 434 \times \log_e (4\angle 86^\circ) \\ &= \cdot 434 \times (\log_e 4 + j1\cdot 5) \text{ since } 86^\circ = 1\cdot 5 \text{ radians} \\ &= \cdot 434 \times (1\cdot 386 + j1\cdot 5) \\ &= \cdot 434 \times 2\cdot 04\angle 47^\circ\ 17' \\ &= \cdot 885\angle 47^\circ\ 17'.\end{aligned}$$

The essential point to remember in these operations is the necessity to reduce the complex quantities to the appropriate forms before commencing. Thus $\sqrt{a + jb}$ must first be reduced to the form $\sqrt{A\angle\theta}$ before it can be evaluated, and so on.

Equations Involving Complex Quantities.—If two complex quantities are equal, that is, if

$$a + jb = c + jd,$$

then one fact emerges which is of great importance, namely, that **the real parts and the imaginary parts separately must be equal.** A very little consideration will show that this must be the case. For, any vector $A\angle\theta$ has a definite projection, a, on the horizontal axis, and a definite projection, b, on the vertical axis. Any vector which is equal to it must obviously have the same modulus, the same angle, and the same projections. Hence, from the above equation, we can immediately assert that

$$a = c$$

and $$b = d.$$

Whence we have the following rule for equations involving complex quantities. **Equate the real and imaginary parts separately.**

Example.—Find x and y from the equation

$$x + jy = \frac{1}{a - jb}.$$

First put the right-hand side into the same form as the left. Thus

$$\frac{1}{a - jb} = \frac{a + jb}{(a - jb)(a + jb)} = \frac{a + jb}{a^2 + b^2} = \frac{a}{a^2 + b^2} + j\left(\frac{b}{a^2 + b^2}\right)$$

whence $$x = \frac{a}{a^2 + b^2} \quad \text{and} \quad y = \frac{b}{a^2 + b^2}.$$

Application to Electrical Measurements.

Suppose we have an alternating sinusoidal current whose instantaneous value is given by

$$i = \mathscr{I} \sin \omega t.$$

This may be represented by a vector I whose magnitude is the maximum value of i, that is, $\mathscr{I}$. If this current traverse a non-inductive resistance R, the voltage across R at any instant is R $\times$ $\mathscr{I} \sin \omega t$, that is to say, the voltage is a vector E of magnitude I $\times$ R in phase with the current. Putting this in the form of an equation, we have

$$\frac{\mathrm{E}}{\mathrm{I}} = \mathrm{R}$$

where the " impedance " R is obviously a real quantity, since there is no difference in angle between E and I.

If, however, the same current traverse an inductance L, the voltage across the inductance at any instant, from first principles, is given by the equation

$$e = L\frac{di}{dt} = L\frac{d}{dt}(\mathcal{I}\sin \omega t) = L \times \mathcal{I}\omega \cos \omega t.$$

Now, $\mathcal{I} \cos \omega t$ is a vector exactly 90° out of phase with the current, and may therefore be written jI since the operation denoted by j is that of turning the vector through 90°. Hence the voltage in this case may be written $j\omega$LI, and we have the relation

$$\frac{E}{I} = j\omega L.$$

That is to say, the impedance which the inductance L offers to a current of frequency ω is represented by the imaginary quantity $j\omega$L.

Again, if an alternating sinusoidal voltage whose instantaneous value is $\mathcal{E} \sin \omega t$ is applied to a condenser of capacity C, the current through the condenser may be calculated from first principles in the following manner.

The quantity of electricity, q, stored at any instant in a condenser of capacity C is equal to the product of C and of the E.M.F., e, existing across the plates of the condenser. That is,

$$q = C \times e.$$

$$\therefore \frac{dq}{dt} = C\frac{de}{dt}.$$

But $\frac{dq}{dt}$ is, by definition, the current through the condenser

Therefore

$$I = C\frac{d}{dt}(\mathcal{E} \sin \omega t) = C\,\mathcal{E}\,\omega \cos \omega t = C \times j\omega E$$

in the same way as before.

Whence $$\frac{E}{I} = \frac{1}{j\omega C} = -\frac{j}{\omega C}.$$

indicating that the impedance offered by the condenser is an imaginary quantity of amount $-\frac{j}{\omega C}$.

Now, if we have a circuit such as that shown in Fig. 6, which combines all three, we have merely to add the impedance

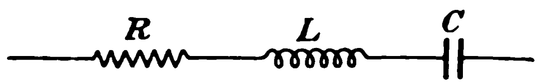


FIG. 6.

of each part just as though there were three resistances in series, so that the impedance of the whole circuit is

$$R + j\omega L - \frac{j}{\omega C},$$

or
$$R + j\left(\omega L - \frac{1}{\omega C}\right).$$

which is itself a complex quantity.

UNITS.

The same units must, of course, be used throughout. Thus R is reckoned in **ohms,** L in **henries,** and C in **farads.** The angular velocity ω is in **radians per second,** i.e. $2\pi \times$ **frequency.** The current arrived at will be in **amperes** and the E.M.F. in **volts.**

The method affords an extremely simple means of calculating the impedance of any circuit, however complicated. For example, a circuit consisting of a capacity and an inductance in parallel, as in Fig. 7,

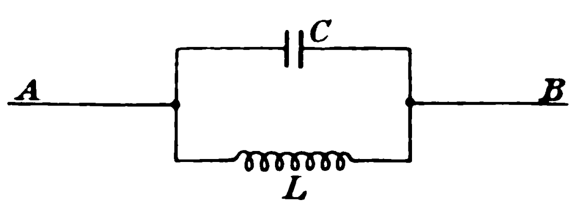


FIG. 7.

is treated exactly as though we were dealing with two resistances in parallel. Thus the conductance of the condenser $= \dfrac{1}{-\dfrac{j}{\omega C}} = j\omega C;$

the conductance of the inductance $= \dfrac{1}{j\omega L},$

therefore the conductance of the whole $= j\omega C + \dfrac{1}{j\omega L},$

and the impedance of the whole

$$= \frac{1}{j\omega C + \dfrac{1}{j\omega L}} = \frac{j\omega L}{j^2\omega^2 CL + 1} = \frac{j\omega L}{1 - \omega^2 LC}.$$

A special case of great importance occurs when $w^2LC = 1$ or $\omega = \dfrac{1}{\sqrt{LC}}$. The impedance of the circuit from A to B then becomes infinite, but if we consider the impedance offered to an idle current circulating within the loop composed of the capacity and inductance this is

$$j\omega L - \frac{j}{\omega C} = j\left(\frac{\omega^2 LC - 1}{\omega C}\right) = 0.$$

This is the well-known case of **resonance** in which the circulating current would become infinite were it not for the fact that certain losses always occur both in the condenser and the inductance which limit its amount.

A numerical example will perhaps assist in showing the application of the method to actual problems Suppose an alternating E.M.F. of 50 volts and frequency 50 cycles is applied to a coil having an inductance of one henry and a resistance of 50 ohms.

Impedance of the coil $= R + j\omega L$

where $R = 50$

$\omega = 50 \times 2\pi = 314$

$L = 1$

Impedance $= 50 + j314 = 318\underline{/80^\circ\ 57'}.$

The current in the coil is, therefore,

$$\frac{\text{E}}{\text{Impedance}} = \frac{50}{318\underline{/80^\circ\ 57'}}, = \cdot 1572\overline{\diagdown 80^\circ\ 57'}.$$

That is to say, the current is ·1572 amperes and, since its angle is negative, it lags behind the voltage by the angle 80° 57′.

HYPERBOLIC FUNCTIONS.

In any book of Trigonometry will be found the proof of the following theorem, known as De Moivre's Theorem :—

$$e^{j\theta} = \cos\theta + j\sin\theta,$$

and, similarly,

$$e^{-j\theta} = \cos\theta - j\sin\theta.$$

Whence, by subtraction $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$ . . . (1)

and, by addition $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$ . . . (2)

Now there exists another set of functions bearing a relationship to the rectangular hyperbola similar to that which the circular functions bear to the circle, and known as the **hyperbolic functions.** It is entirely outside the scope of this short book to enter into the mathematical interpretation attaching to them, but, fortunately, from the standpoint of an engineer this is wholly unnecessary. For our purpose they are merely a tool, and their usefulness lies in the simplification which their employment introduces in the handling of mathematical expressions which would otherwise be extremely difficult and cumbersome of manipulation.

Just as the circular functions $\sin\theta$ and $\cos\theta$ may be expressed in terms of the "exponential function," e, as shown above, so the hyperbolic sine and cosine—or, as they are written, sinh and cosh—may also be expressed in a similar manner thus—

$$\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2} \quad . \quad . \quad . \quad . \quad (3)$$

$$\cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2} \quad . \quad . \quad . \quad . \quad (4)$$

It follows from (1) that

$$\sin j\theta = \frac{e^{j^2\theta} - e^{-j^2\theta}}{2j} = j\frac{e^{\theta} - e^{-\theta}}{2} = j \sinh \theta,$$

and from (2)

$$\cos j\theta = \frac{e^{j^2\theta} + e^{-j^2\theta}}{2} = \frac{e^{\theta} + e^{-\theta}}{2} = \cosh \theta.$$

Hence, the two sets of functions are related by the identities

$$\sin j\theta = j \sinh \theta \quad . \quad . \quad . \quad . \quad (5)$$

$$\cos j\theta = \cosh \theta \quad . \quad . \quad . \quad . \quad (6)$$

There are in all six hyperbolic functions, the remaining four being derived from the sinh and cosh precisely as in the case of the circular functions, thus—

$$\operatorname{sech} \theta = \frac{1}{\cosh \theta} = \frac{2}{e^{\theta} + e^{-\theta}} \quad . \quad . \quad . \quad (7)$$

$$\operatorname{cosec} \theta = \frac{1}{\sinh \theta} = \frac{2}{e^{\theta} - e^{-\theta}} \quad . \quad . \quad . \quad (8)$$

$$\tanh \theta = \frac{\sinh \theta}{\cosh \theta} = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} \quad . \quad . \quad . \quad (9)$$

$$\coth \theta = \frac{\cosh \theta}{\sinh \theta} = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}} \quad . \quad . \quad . \quad (10)$$

Other well-known identities which exist in the circular functions have their analogy in the hyperbolic functions. Thus, if $\phi = j\theta$, we have

$$\sin^2 \phi + \cos^2 \phi = 1 ;$$

but $\sin^2 \phi = \sin^2 (j\theta) = (j \sinh \theta)^2 = - \sinh^2 \theta$

and $\cos^2 \phi = \cos^2 (j\theta) = \cosh^2 \theta.$

Hence

$$\cosh^2\theta - \sinh^2\theta = 1 \quad . \quad . \quad . \quad (11)$$

Again,

$$\sinh\theta + \cosh\theta = \frac{e^\theta - e^{-\theta}}{2} + \frac{e^\theta + e^{-\theta}}{2} = e^\theta \quad . \quad (12)$$

In a similar manner it may be shown that

$$\sinh(\theta + \theta') = \sinh\theta\cosh\theta' + \cosh\theta\sinh\theta' \quad . \quad (13)$$

$$\sinh(\theta - \theta') = \sinh\theta\cosh\theta' - \cosh\theta\sinh\theta' \quad . \quad (14)$$

$$\cosh(\theta + \theta') = \cosh\theta\cosh\theta' + \sinh\theta\sinh\theta' \quad . \quad (15)$$

$$\cosh(\theta - \theta') = \cosh\theta\cosh\theta' - \sinh\theta\sinh\theta' \quad . \quad (16)$$

$$\tanh(\theta + \theta') = \frac{\tanh\theta + \tanh\theta_1}{1 + \tanh\theta\tanh\theta'} \quad . \quad . \quad (17)$$

$$\tanh(\theta - \theta_1) = \frac{\tanh\theta - \tanh\theta'}{1 - \tanh\theta\tanh\theta'} \quad . \quad . \quad (18)$$

$$\sinh 2\theta = 2\sinh\theta\cosh\theta \quad . \quad . \quad (19)$$

$$\cosh 2\theta = 2\cosh^2\theta - 1 \quad . \quad . \quad (20)$$

$$\tanh 2\theta = \frac{2\tanh\theta}{1 + \tanh^2\theta} \quad . \quad . \quad . \quad (21)$$

In general, therefore, hyperbolic functions can be used in engineering problems in a manner precisely similar to the circular functions, and tables* of their values are published similar to the ordinary tables of the circular functions.

Complex Hyperbolic Functions.

In their application to transmission problems, however, we encounter them usually, not simply as the functions of *real* quantities, as hitherto, but of *complex quantities*, such as sinh $(a + jb)$ or sinh $A\underline{/\theta}$, and it is therefore necessary to show how the value of such a **complex hyperbolic function** can be calculated from the circular and hyperbolic functions of real quantities, the values of which can be found in tables. We proceed as follows :—

* "Smithsonian Math. Tables—Hyperbolic Functions," published by Smithsonian Institution, City of Washington (London agent, W. Wesley & Son, 28 Essex Street, Strand).

$$\sinh(a + jb) = \sinh a \cosh jb + \cosh a \sinh jb\,;$$

but $$\cosh jb = \cos(j \times jb) = \cos(-b) = \cos b$$

and $$\sinh jb = \frac{\sin(j \times jb)}{j} = \frac{\sin(-b)}{j} = j \sin b.$$

Hence

$$\sinh(a + jb) = \sinh a \cos b + j \cosh a \sin b \qquad . \quad (22)$$

which is itself, as we should expect, a complex quantity.

Similarly,

$$\cosh(a + jb) = \cosh a \cos b + j \sinh a \sin b \qquad . \quad (23)$$

$$\sinh(a - jb) = \sinh a \cos b - j \cosh a \sin b \qquad . \quad (24)$$

$$\cosh(a - jb) = \cosh a \cos b - j \sinh a \sin b \qquad . \quad (25)$$

$$\tanh(a + jb) = \frac{\tanh a + j \tan b}{1 + j \tanh a \tan b} \quad . \quad . \quad . \quad (26)$$

$$\tanh(a - jb) = \frac{\tanh a - j \tan b}{1 - j \tanh a \tan b} \quad . \quad . \quad . \quad (27)$$

Example.—Find $\cosh(2\overline{\diagdown 60^\circ})$.

$$\cosh(2\overline{\diagdown 60^\circ}) = \cosh(2 \cos 60^\circ - j2 \sin 60^\circ)$$
$$= \cosh(1 - j\,1{\cdot}732)$$
$$= \cosh(1)\cos(1{\cdot}732) - j \sinh(1)\sin(1{\cdot}732).$$

Remember that 1 and 1·732 are *radians*.

$$\cosh(1) = 1{\cdot}543$$
$$\sinh(1) = 1{\cdot}175$$
$$\cos(1{\cdot}732) = -\,{\cdot}160$$
$$\sin(1{\cdot}732) = {\cdot}987.$$

$$\therefore \cosh(2\overline{\diagdown 60^\circ}) = -\,1{\cdot}543 \times {\cdot}160 - j\,1{\cdot}175 \times {\cdot}987.$$
$$= -\,{\cdot}2465 - j\,1{\cdot}160.$$
$$= 1{\cdot}18\overline{\diagdown 102^\circ}.$$

Tables * have, however, been published giving the values of complex hyperbolic functions over a wide range, and, if a number of calculations have to be made, the labour is greatly reduced if such tables are available.

* "Tables of Complex Hyperbolic and Circular Functions," A. E. Kennelly. Harvard University Press.

EXAMPLES ON CHAPTER I.

1. Express $3 + j4$ in the form $A\underline{/\theta}$.
   Answer: $5\underline{/53^\circ\, 5'}$.
2. Express $7 - 6j$ in the form $A\underline{/\theta}$.
   Answer: $9{\cdot}2\overline{\diagdown 40^\circ\, 36'}$.
3. Express $5{\cdot}3\underline{/36^\circ}$ in the form $a + jb$.
   Answer: $4{\cdot}28 + j3{\cdot}11$.
4. Express $2{\cdot}6\overline{\diagdown 21^\circ}$ in the form $a + jb$.
   Answer: $2{\cdot}43 - j\,{\cdot}932$.
5. Add $(2{\cdot}3 + j3{\cdot}6)$ and $(4{\cdot}6 - j1{\cdot}7)$ and express result in form $A\underline{/\theta}$.
   Answer: $7{\cdot}15\underline{/15^\circ\, 26'}$.
6. Add $2\underline{/30^\circ}$ and $6\overline{\diagdown 45^\circ}$.
   Answer: $5{\cdot}98 - j3{\cdot}24$.
7. Multiply $4\underline{/22^\circ}$ by $5\underline{/34^\circ}$.
   Answer: $20\underline{/56^\circ}$.
8. Divide $1{\cdot}5\underline{/10^\circ}$ by $6\underline{/70^\circ}$.
   Answer: ${\cdot}25\overline{\diagdown 60^\circ}$.
9. Multiply $1{\cdot}5\underline{/67^\circ}$ by $2{\cdot}3\underline{/113^\circ}$.
   Answer: $-3{\cdot}45$.
10. Find square root of $3 + j4$.
    Answer: $2{\cdot}24\underline{/26^\circ\, 34'}$.
11. Find $\log_e (2 + j3)$.
    Answer: $1{\cdot}28 + j\,{\cdot}98$.
12. If $1{\cdot}09 + j3x = 1{\cdot}2\underline{/25^\circ}$, find x.
    Answer: $x = {\cdot}168$.
13. Evaluate $\sinh (3 + j4)$ in the form $a + jb$.
    Answer: $-6{\cdot}548 - j7{\cdot}620$.
14. Evaluate $\cosh (1 + j1)$ in the form $a + jb$.
    Answer: ${\cdot}834 + j{\cdot}99$.
15. A voltage of 28·3 volts of periodicity $\omega = 1000$ is applied to a network consisting of a capacity of 40 mfs., an inductance of 15 millihenries, and a resistance of 10 ohms in series. Find the current passing and its phase relationship to the voltage.
    Answer: 2 amps. leading by 45°.

CHAPTER II.

GENERAL THEORY OF TRANSMISSION.

ANY circuit used for the transmission of electric currents, whether consisting of aerial wires or underground cable, possesses the following four **primary constants** :—

Firstly, it has a certain **resistance,** denoted by the symbol R and stated in ohms per mile of circuit or "loop."

Secondly, since the insulation of the circuit, however good, can never be perfect, there must be a certain **leakance** from wire to wire, denoted by the symbol G and stated in "mhos," or sometimes in "micro-mhos" per mile of loop, these units being the reciprocal of the ohm and megohm respectively.

Thirdly, the circuit possesses a certain **inductance,** denoted by the symbol L, and usually stated in henries per mile of loop.

Finally, a **capacity** exists between wire and wire, denoted by the symbol C and stated in micro-farads per mile of loop.

These constants can, in general, be calculated with a fair degree of accuracy from the cross-section and material of the wires and their distance apart ; but a full discussion of them, together with a description of the methods by which they can be both calculated and measured experimentally, will be found in Chapter IV.

Although, throughout this chapter, only circuits consisting of a pair of wires are considered, the calculations apply equally to a circuit made up of a single wire with an earth return. In this case the primary constants are, of course, stated per mile of single wire, an allowance being made for earth resistance.

DIRECT CURRENT CASE.

The problem of the transmission of alternating currents is very much simplified by a preliminary examination of the much simpler problem of the transmission of direct currents, since there is a close analogy between the two. Moreover, the solution of the latter problem is useful in itself, since it occurs in practice in such cases as long telegraph lines, etc.

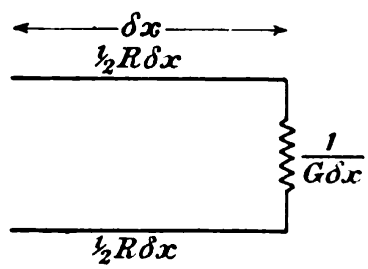


FIG. 8.

The method of procedure is to consider first what occurs in a very short length δx of a line whose primary constants are R, G, L, and C (see Fig. 8).

Since the resistance of the loop is R ohms per mile, the resistance of each side of the section under consideration is clearly $\frac{1}{2}R\delta x$. Similarly, the resistance of the leakage path across the section is $\frac{1}{G\delta x}$. Since we are dealing with direct currents only, the capacity and inductance of the line will have no effect, and they need not, therefore, be considered.

Suppose, then, that a current I is traversing the section. The drop in voltage, $-\delta v$, across it will be the product of the current and the resistance of the section. That is

$$-\delta V = I \times R\delta x.$$

Or, if the section be infinitely short,

$$-\frac{dV}{dx} = IR \quad . \quad . \quad . \quad . \quad (28)$$

Again, suppose the voltage across the circuit at this point is V, then the current which passes through the leakage path is $V \times G\delta x$, that is to say, the drop in current, $-\delta I$, in the section is

$$-\delta I = V \times G\delta x.$$

Or, if the section be infinitely short,

$$-\frac{dI}{dx} = VG \quad . \quad . \quad . \quad . \quad (29)$$

Differentiating (28) again,

$$\frac{d^2V}{dx^2} = -\frac{dI}{dx}R = GRV.$$

And differentiating (29) again,

$$\frac{d^2I}{dx^2} = -\frac{dV}{dx}G = GRI.$$

Writing $\gamma = \sqrt{GR}$ we have

$$\frac{d^2V}{dx^2} = \gamma^2V \quad . \quad . \quad . \quad . \quad (30)$$

$$\frac{d^2I}{dx^2} = \gamma^2I \quad . \quad . \quad . \quad . \quad (31)$$

These two differential equations are obviously precisely similar, and the general solution of (30) may easily be shown to be

$$* \; V = Ae^{\gamma x} + Be^{-\gamma x} \quad . \quad . \quad . \quad (32)$$

where A and B are arbitrary constants.

The next step is to determine the value of the constants A and B, and to do this we proceed as follows :—

From the values of $\cosh \gamma x$ and $\sinh \gamma x$ given in (3) and (4) we obtain by addition $e^{\gamma x} = \cosh \gamma x + \sinh \gamma x$
and by subtraction $e^{-\gamma x} = \cosh \gamma x - \sinh \gamma x$.

Whence (32) may be written

$$V = (A + B) \cosh \gamma x + (A - B) \sinh \gamma x \quad . \quad (33)$$

We may now insert the " end conditions " † in order to find

* While the scope of this book does not admit of a discussion as to the method of arriving at this solution, the reader may at least convince himself that it does satisfy the equation by differentiating the right-hand side twice, thus

$$\frac{d^2V}{dx^2} = \gamma^2(Ae^{\gamma x} + Be^{-\gamma x}) = \gamma^2V.$$

† In the absence of knowledge as to these, the equation is obviously indeterminate.

the constants in terms of known quantities. Thus, let us suppose that a voltage V_o is applied to the line at the sending end, that is, at the point where $x = 0$. We then have

$$V_o = (A + B)\cosh 0 + (A - B)\sinh 0$$

But $$\sinh 0 = 0.$$
and $$\cosh 0 = 1.$$
$$\therefore A + B = V_o.$$

Again, if we differentiate (32) once, we have

$$\frac{dV}{dx} = \gamma(Ae^{\gamma x} - Be^{-\gamma x}).$$

But $$\frac{dV}{dx} = -IR,$$

$$\therefore\ -\frac{IR}{\gamma} = -I\sqrt{\frac{R}{G}}$$
$$= Ae^{\gamma x} - Be^{-\gamma x}$$
$$= (A + B)\sinh\gamma x - (B - A)\cosh\gamma x.$$

Now suppose the current at the sending end is I_o, then, since $x = 0$,

$$I_o\sqrt{\frac{R}{G}} = B - A.$$

Writing $Z_o = \sqrt{\frac{R}{G}}$ we arrive at the general equation,

$$V = V_o\cosh\gamma x - I_oZ_o\sinh\gamma x \quad . \quad . \quad (34)$$

which is the **general equation for the voltage along the line.**

Similarly,

$$IZ_o = (B - A)\cosh\gamma x - (A + B)\sinh\gamma x.$$

$$\therefore\ I = I_o\cosh\gamma x - \frac{V_o}{Z_o}\sinh\gamma x \quad . \quad . \quad . \quad (35)$$

which is the **general equation for the current along the line.**

It should be observed that these equations are determined by the values of V_o and I_o. In general, one of these is known, usually V_o, the voltage applied at the sending end. The other, however, becomes determinate as soon as the length of the line and the conditions at the distant end are fixed, i.e. the distant end may be open, short-circuited, or closed through some known resistance.

Thus, if the line is of length l and the distant end is **open,** it is obvious that, at the open end, the current is zero ; that is, when $x = l$, $I = 0$. Putting these values in (35), we have

$$I_o \cosh \gamma l = \frac{V_o}{Z_o} \sinh \gamma l,$$

whence, if V_o is known, I_o is given by

$$I_o = \frac{V_o}{Z_o} \tanh \gamma l,$$

and, therefore, the value of the current and voltage along the line are given by

$$I = \frac{V_o}{Z_o} (\tanh \gamma l \cosh \gamma x - \sinh \gamma x),$$
$$V = V_o (\cosh \gamma x - \tanh \gamma l \sinh \gamma x).$$

Again, if the line is **closed** at the distant end, then, when $x = l$, $V = 0$, so that, from (34), we have

$$V_o \cosh \gamma l = I_o Z_o \sinh \gamma l.$$

Hence $$I_o = \frac{V_o}{Z_o} \coth \gamma l.$$

Hence the current and voltage along the line in this case are given by

$$I = \frac{V_o}{Z_o} (\coth \gamma l \cosh \gamma x - \sinh \gamma x),$$
$$V = V_o (\cosh \gamma x - \coth \gamma l \sinh \gamma x).$$

Line Infinite.

A special case of great interest occurs when the line is **infinite** in length. In this event, it is obvious that, when

$x = \infty$, both the current and the voltage must be zero. Putting these values in the original equation,

$$V = Ae^{\gamma x} + Be^{-\gamma x},$$

it becomes immediately evident that $A = o$. Also, when $x = 0$, $V = V_0$, hence $B = V_0$, so that the equation is simplified into the form

$$V = V_0 e^{-\gamma x}.$$

Again, the current

$$I = -\frac{dV}{dx} \times \frac{1}{R}.$$

$$= -\frac{1}{R}\frac{d}{dx}(V_0 e^{-\gamma x}),$$

$$= \frac{V_0}{R}\gamma e^{-\gamma x},$$

$$= V_0\sqrt{\frac{G}{R}}e^{-\gamma x}.$$

$$\therefore\ I = \frac{V_0}{Z_0}e^{-\gamma x}.$$

At the sending end, when $x = 0$, $I = I_0$.

$$\therefore\ I_0 = \frac{V_0}{Z_0},$$

or

$$Z_0 = \frac{V_0}{I_0}.$$

That is to say, Z_0 is the **sending-end impedance** of the line when it is infinite in length, and is termed the **characteristic impedance.**

Alternating Current Case.

Turning now to the more difficult problem of alternating current transmission, it should be observed, in the first place, that the method employed is based on the fact previously mentioned,* that all periodic functions can be split into a

* See Chap. I., p. 2.

series of pure harmonic quantities. The procedure, therefore, is to deduce the laws governing the propagation of voltages and currents of pure sine-wave form; since, if the behaviour of the component parts of the original wave can be calculated, the behaviour of the whole is known.

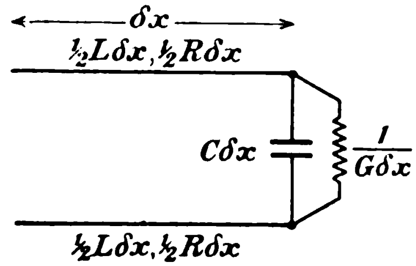


FIG. 9.

As in the direct current case, we commence by considering a very short section of line δx (Fig. 9), but, since the current and voltage are now alternating, account has to be taken of all four primary constants.

Suppose the current in the section to be

$$i = \mathcal{I} \sin \omega t,$$

which may be represented by the vector I. The impedance of the section is clearly

$$\mathrm{R}\delta x + j\omega \mathrm{L}\delta x.$$

Hence the voltage drop, $-\delta\mathrm{V}$, along it is given by the equation

$$-\delta\mathrm{V} = \mathrm{I}(\mathrm{R}\delta x + j\omega\mathrm{L}\delta x).$$

or, if the section is infinitely short,

$$-\frac{d\mathrm{V}}{dx} = \mathrm{I}(\mathrm{R} + j\omega\mathrm{L}) \quad . \quad . \quad . \quad (36)$$

Again, suppose the voltage at this point is

$$v = \mathcal{V} \sin(\omega t + \phi),$$

which may also be represented by the vector V, which will, in general, be out of phase with the current vector I by some angle ϕ.

The conductance path between the wires is

$$\mathrm{G}\delta x - \frac{\omega\mathrm{C}\delta x}{j} = \mathrm{G}\delta x + j\omega\mathrm{C}\delta x,$$

so that the loss of current, $-\delta\mathrm{I}$, in the section is

$$-\delta\mathrm{I} = \mathrm{V}(\mathrm{G}\delta x + j\omega\mathrm{C}\delta x).$$

Or, if the section is infinitely short,

$$-\frac{dI}{dx} = V(G + j\omega C) \quad . \quad . \quad . \quad (37)$$

Differentiating (36) again,

$$\frac{d^2V}{dx^2} = -(R + j\omega L)\frac{dI}{dx} = V(R + j\omega L)(G + j\omega C),$$

and, differentiating (37) again,

$$\frac{d^2I}{dx^2} = -(G + j\omega C)\frac{dV}{dx} = I(R + j\omega L)(G + j\omega C).$$

Writing $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$,

We have

$$\frac{d^2V}{dx^2} = \gamma^2 V \quad . \quad . \quad . \quad . \quad (38)$$

$$\frac{d^2I}{dx^2} = \gamma^2 I \quad . \quad . \quad . \quad . \quad (39)$$

These equations will be seen to be identical with (30) and (31) in the D.C. case, and the solution of (38) is, therefore,

$$V = Ae^{\gamma x} + Be^{-\gamma x} \quad . \quad . \quad . \quad (40)$$

In order to find the value of the constants, we proceed exactly as in the D.C. case, first putting (38) into the form

$$V = (A + B)\cosh\gamma x + (A - B)\sinh\gamma x,$$

when, as before,

$$A + B = V_0$$

$$B - A = I_0\sqrt{\frac{R + j\omega L}{G + j\omega C}},$$

$$= I_0 Z_0$$

where $Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$ and is the **characteristic impedance** of the line. Hence

$$V = V_0\cosh\gamma x - I_0 Z_0\sinh\gamma x \quad . \quad . \quad (41)$$

$$I = I_0\cosh\gamma x - \frac{V_0}{Z_0}\sinh\gamma x \quad . \quad . \quad (42)$$

which are the general equations for the current and voltage at any point along the line.

If the line is infinite we obtain, as in the D.C. case,

$$V = V_o e^{-\gamma x} \quad . \quad . \quad . \quad . \quad (43)$$

$$I = \frac{V_o}{Z_o} e^{-\gamma x} \quad . \quad . \quad . \quad . \quad (44)$$

PHYSICAL INTERPRETATION.

It is interesting, at this point, to indicate what really is the physical interpretation of these equations, and, since the infinite line is simpler, let us examine this first. The voltage distribution along the line is, in this case, the product of a vector or harmonic quantity V_o, and the exponential term $e^{-\gamma x}$. But the index of the exponential is itself complex, since γ is, in general, of the form $\beta + j\alpha$. Therefore the exponential term may be written

$$e^{-(\beta + j\alpha)x} = e^{-\beta x} \times e^{-j\alpha x} = e^{-\beta x}\overline{\diagdown \alpha x},$$

that is, the amplitude of the voltage at any point along the line is $V_o \times e^{-\beta x}$, while it differs in phase from the voltage, V_o, at the sending end by the angle $-\alpha x$.

To look at the same thing from another point of view, the exponential term $e^{-\beta x}\overline{\diagdown \alpha x}$ is itself a vector or harmonic quantity. Hence we have the combination of two waves, the resultant of which may be shown to be a wave travelling along the line.

This is a well-known phenomenon in physics, and will perhaps be made clear by reference to the familiar example of waves on the surface of water. If an agitation is set up at any point in the water, as by dropping a stone into it, the surface of the water at this point is set into harmonic motion in a perpendicular direction. This perpendicular motion is communicated in turn to successive surface layers of water more and more remote from the original point of disturbance, with a small but regular " time lag " from layer to layer, until a large area of the pond is in vibration. But although the wave motion of each particular section of water is confined to the perpendicular direction and there is no lateral movement at all, the phenomenon presents itself to the eye as a wave travelling *along* the surface in a horizontal direction. It is easily verified that the water does not move laterally by watching the motion of any body floating thereon, which will be observed merely to oscillate up and down ; yet, although the medium is in perpendicular vibration only, the resultant wave upon its surface *does* travel horizontally at a speed determined by the time taken by each layer of water to communicate its motion to the next.

Let us now examine the more general case of the finite line, the equation for which is

$$V = Ae^{\gamma x} + Be^{-\gamma x}.$$

We have seen that the second term denotes a wave travelling along the line with decreasing amplitude. When this reaches the distant end, it is *reflected* there, and the first term represents the reflected wave travelling *back* along the line decreasing in amplitude as it approaches the sending end. If the line is infinite, there can obviously be no reflection from the distant end, and we should accordingly expect this term to be absent, which will be observed to be the case.

Wave-length and Velocity.

To return, then, to the voltage along the line, we have a stationary wave, V_o, whose amplitude or envelope is another wave, $e^{-\beta x}\overline{\diagdown \alpha x}$, and the resultant is a wave decreasing in amplitude as it travels along the line in accordance with the exponential term $e^{-\beta x}$, and whose wave-length and rate of travel are determined by αx in the following manner. It is obvious that, at a distance l along the line, such that

$$\alpha l = 2\pi,$$

the stationary wave will again be in phase with the stationary wave at the sending end. That is to say, the horizontal wave will have completed its period at this point, and its **wave-length** is therefore

$$l = \frac{2\pi}{\alpha} \quad . \quad . \quad . \quad . \quad (45)$$

Moreover, if the periodicity of the voltage V_0 is n per second, the whole cycle of operations will take $\frac{1}{n}$ seconds, and, therefore, the **velocity** of the wave

$$v = \frac{l}{\frac{1}{n}} = \frac{2\pi n}{\alpha} \quad . \quad . \quad . \quad (46)$$

Secondary Constants.

Since it is the factor β which determines the decrease in amplitude or attenuation of the wave, it is called the **attenuation constant,** while, since α similarly determines the wave-length, it is called the **wave-length constant.** Further, since both of these go to make up the complex quantity γ, the latter completely governs the propagation of the wave, and is therefore called the **propagation constant.** These constants are known as the **secondary** or **transmission constants** of the line.

Again, since the propagation constant γ is equal to $\sqrt{(R + j\omega L)(G + j\omega C)}$, it is evident that it is dependent, not only upon the primary constants of the line, R, L, G, and C, but also upon the angular velocity, ω, of the particular wave being transmitted, and since the attenuation and wave-length constants are derived from it, these are also dependent upon ω. It follows, therefore, that, for the same transmission line, waves of different frequencies will, in general, be propagated at different velocities and with differing attenuation.

Line Open at Distant End.

Returning to the general equations (41) and (42), if the line is of length l and **open** at the distant end, then the current at the end is zero. That is

$$I = 0 \text{ when } x = l.$$

$$\therefore I_0 \cosh \gamma l = \frac{V_0}{Z_0} \sinh \gamma l.$$

$$\therefore I_0 = \frac{V_0}{Z_0} \tanh \gamma l \quad . \quad . \quad (47)$$

So that (41) and (42) become determinate, thus

$$V = V_0 (\cosh \gamma x - \tanh \gamma l \sinh \gamma x) \quad . \quad . \quad (48)$$

$$I = \frac{V_0}{Z_0} (\tanh \gamma l \cosh \gamma x - \sinh \gamma x) \quad . \quad . \quad (49)$$

and from these equations the current and voltage at any point along the line can be found.

Moreover, $\frac{V_0}{I_0}$ is clearly the apparent impedance or, as it is usually termed, the **sending-end impedance** of the line. This is commonly written Z_{open}, and we have

$$Z_{open} = \frac{V_0}{I_0} = Z_0 \coth \gamma l \quad . \quad . \quad . \quad (50)$$

If it is required to find the voltage at the distant end, this is, from (48),

$$V_r = V_0 (\cosh \gamma l - \tanh \gamma l \sinh \gamma l)$$

$$= V_0 \times \frac{\cosh^2 \gamma l - \sinh^2 \gamma l}{\cosh \gamma l}$$

$$\therefore V_r = \frac{V_0}{\cosh \gamma l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

LINE CLOSED AT DISTANT END.

Similarly, if the line is **closed** at the distant end,

$$V = 0 \text{ when } x = l.$$

$$\therefore V_0 \cosh \gamma l = I_0 Z_0 \sinh \gamma l.$$

$$\therefore I_0 = \frac{V_0}{Z_0} \coth \gamma l \quad . \quad . \quad . \quad (52)$$

so that (41) and (42) become determinate thus

$$V = V_o (\cosh \gamma x - \coth \gamma l \sinh \gamma x) \quad . \quad . \quad (53)$$

$$I = \frac{V_o}{Z_o} (\coth \gamma l \cosh \gamma x - \sinh \gamma x) \quad . \quad . \quad (54)$$

And the **sending-end impedance**

$$Z_{\text{closed}} = \frac{V_o}{I_o} = Z_o \tanh \gamma l \; . \quad . \quad . \quad (55)$$

Whence, from (50) and (55),

$$\sqrt{Z_{\text{open}} \times Z_{\text{closed}}} = Z_o \quad . \quad . \quad . \quad (56)$$

The current at the distant end of the closed line from (53) is

$$I_r = \frac{V_o}{Z_o}(\coth \gamma l \cosh \gamma l - \sinh \gamma l)$$

$$= \frac{V_o}{Z_o} \times \frac{\cosh^2 \gamma l - \sinh^2 \gamma l}{\sinh \gamma l}$$

$$\therefore I_r = \frac{V_o}{Z_o \sinh \gamma l} \quad . \quad . \quad . \quad . \quad . \quad (57)$$

whence the ratio of the received current to the sent current is

$$\frac{I_o}{I_r} = \frac{V_o}{Z_o} \coth \gamma l \times \frac{Z_o \sinh \gamma l}{V_o},$$

$$\therefore \frac{I_r}{I_o} = \frac{1}{\cosh \gamma l} \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Example.—The 100-lb. air-space paper-core telephone cable has the following primary constants :—

R = 17·6 ohms per mile.
L = ·001 henries per mile.
C = ·065 mfs. per mile.
G = 10^{-6} mhos. per mile.

At the standard frequency at which telephonic measurements are made, viz., $\omega = 5000$,

$$\gamma = \sqrt{(\mathrm{R} + j\omega\mathrm{L})(\mathrm{G} + j\omega\mathrm{C})}$$
$$= \sqrt{(17{\cdot}6 + j\,5)(10^{-6} + j\,3{\cdot}25 \times 10^{-4})}$$
$$= {\cdot}0771\underline{/52^\circ\,50'}.$$

$$Z_o = \sqrt{\frac{\mathrm{R} + j\omega\mathrm{L}}{\mathrm{G} + j\omega\mathrm{C}}}$$
$$= \sqrt{\frac{17{\cdot}6 + j\,5}{10^{-6} + j\,3{\cdot}25 \times 10^{-4}}}$$
$$= 237{\cdot}3\overline{\diagdown 36^\circ\,59'}.$$

Hence, if the line is 30 miles long and open at the distant end,

$$Z_{\text{open}} = Z_o \coth \gamma l$$
$$= \frac{Z_o}{\tanh \gamma l};$$

$$\tanh \gamma l = \tanh\,(2{\cdot}313\underline{/52^\circ\,50'})$$
$$= \tanh\,(1{\cdot}40 + j\,1{\cdot}84).$$

Quadranting,* this becomes $\tanh\,(1{\cdot}40 + j\,1{\cdot}171)$.

$$= 1{\cdot}110\overline{\diagdown 3^\circ\,30'}.\dagger$$

Hence
$$Z_{\text{open}} = \frac{237{\cdot}3\overline{\diagdown 36^\circ\,59'}}{1{\cdot}110\overline{\diagdown 3^\circ\,30'}}$$
$$= 214\overline{\diagdown 33^\circ\,29'}.$$

And if a voltage of 10 volts at $\omega = 5000$ is applied at the sending end, the voltage at the distant end is $\dfrac{V_o}{\cosh \gamma l}$.

$$\cosh \gamma l = \cosh\,(1{\cdot}40 + j\,1{\cdot}84)$$
$$= 1{\cdot}922\underline{/107^\circ\,12'}.\ddagger$$

∴ Voltage at distant end

$$= \frac{10}{1{\cdot}922\underline{/107^\circ\,12'}}$$
$$= 5{\cdot}20\overline{\diagdown 107^\circ\,12'}.$$

* See Appendix, p. 63.

† See Kennelly's "Tables," p. 127.

‡ *Ibid.*, p. 111.

Similarly, if the line is closed at the distant end,

$$\begin{aligned} Z_{\text{closed}} &= Z_0 \tanh \gamma l \\ &= 237\cdot3\,\overline{\diagdown 36^\circ\,59'} \times 1\cdot110\,\overline{\diagdown 3^\circ\,30'} \\ &= 263\cdot3\,\overline{\diagdown 40^\circ\,29'}. \end{aligned}$$

And with 10 volts applied as before, the current at the distant end is $\dfrac{V_0}{Z_0 \sinh \gamma l}$.

$$\begin{aligned} \sinh \gamma l &= \sinh (1\cdot40 + j\,1\cdot84) \\ &= 2\cdot132\,\underline{/103^\circ\,36'}.^* \end{aligned}$$

$$\begin{aligned} \therefore\ \text{Current} &= \frac{10}{237\cdot3\,\overline{\diagdown 36^\circ\,59'} \times 2\cdot142\,\underline{/103^\circ\,36'}} \\ &= \frac{10}{508\,\underline{/66^\circ\,37'}} \\ &= \cdot0196\,\overline{\diagdown 66^\circ\,37'}. \end{aligned}$$

The voltage or current at any intermediate point on the line can immediately be found from (48) and (49).

Thus, at a distance of 15 miles from the sending end, with the distant end open, the voltage may be found as follows :—

$$\begin{aligned} \text{voltage} &= V_0 (\cosh \gamma x - \tanh \gamma l \sinh \gamma x). \\ \cosh \gamma x &= \cosh 1\cdot156\,\underline{/52^\circ\,50'} \\ &= \cosh (\cdot70 + j\,\cdot92). \end{aligned}$$

Quadranting, this becomes cosh $(\cdot70 + j\,\cdot585)$,

$$= \cdot968\,\underline{/36^\circ\,36'}.$$

Similarly, $\sinh \gamma x = 1\cdot099\,\underline{/65^\circ\,12'}$.

$$\begin{aligned} \text{Voltage} &= 10(\cdot968\,\underline{/36^\circ\,36'} - 1\cdot110\,\overline{\diagdown 3^\circ\,30'} \times 1\cdot099\,\underline{/65^\circ\,12'}) \\ &= 9\cdot68\,\underline{/36^\circ\,36'} - 12\cdot20\,\underline{/61^\circ\,42'} \\ &= 7\cdot77 + j\,5\cdot77 - 5\cdot81 - j\,10\cdot73 \\ &= -\,1\cdot96 - j\,4\cdot96 \\ &= 5\cdot33\,\overline{\diagdown 111^\circ\,34'}. \end{aligned}$$

* See Kennelly's "Tables," p. 95.

Line Closed through Impedance.

The more general case, and one of great practical importance, is that in which a line of length l is closed at the distant end through some piece of apparatus whose impedance, Z_r, is known. If I_r and V_r are the current and voltage respectively at the distant end, then

$$I_r Z_r = V_r.$$

$$\therefore\ Z_r\left(I_o \cosh\gamma l - \frac{V_o}{Z_o}\sinh\gamma l\right) = V_o \cosh\gamma l - I_o Z_o \sinh\gamma l.$$

$$\therefore\ I_o(Z_r \cosh\gamma l + Z_o \sinh\gamma l) = V_o\left(\frac{Z_r}{Z_o}\sinh\gamma l + \cosh\gamma l\right)$$

$$= \frac{V_o}{Z_o}(Z_r \sinh\gamma l + Z_o \cosh\gamma l).$$

$$\therefore\ I_o = \frac{V_o}{Z_o} \times \frac{Z_r \sinh\gamma l + Z_o \cosh\gamma l}{Z_o \sinh\gamma l + Z_r \cosh\gamma l} \quad . \quad (59)$$

Whence

$$V = V_o\left(\cosh\gamma x - \frac{Z_r \sinh\gamma l + Z_o \cosh\gamma l}{Z_o \sinh\gamma l + Z_r \cosh\gamma l} \times \sinh\gamma x\right),$$

and

$$I = \frac{V_o}{Z_o}\left(\frac{Z_r \sinh\gamma l + Z_o \cosh\gamma l}{Z_o \sinh\gamma l + Z_r \cosh\gamma l} \times \cosh\gamma x - \sinh\gamma x\right).$$

Also, the **sending-end impedance** of the line so closed is

$$\frac{V_o}{I_o} = Z_o\frac{Z_o \sinh\gamma l + Z_r \cosh\gamma l}{Z_r \sinh\gamma l + Z_o \cosh\gamma l} \quad . \quad . \quad (60)$$

In particular, the current I_r through the impedance Z_r is

$$\frac{V_o}{Z_o}\left(\frac{Z_r \sinh\gamma l + Z_o \cosh\gamma l}{Z_o \sinh\gamma l + Z_r \cosh\gamma l} \times \cosh\gamma l - \sinh\gamma l\right)$$

$$= \frac{V_o}{Z_o} \times \frac{Z_r \sinh\gamma l \cosh\gamma l + Z_o \cosh^2\gamma l - Z_o \sinh^2\gamma l - Z_r \sinh\gamma l \cosh\gamma l}{Z_o \sinh\gamma l + Z_r \cosh\gamma l}.$$

$$\therefore\ I_r = \frac{V_o}{Z_o \sinh\gamma l + Z_r \cosh\gamma l} \quad . \quad . \quad (61)$$

And the ratio of the sent to the received current,

$$\frac{I_o}{I_r} = \frac{V_o}{Z_o} \times \frac{Z_r \sinh \gamma l + Z_o \cosh \gamma l}{Z_o \sinh \gamma l + Z_r \cosh \gamma l} \times \frac{Z_o \sinh \gamma l + Z_r \cosh \gamma l}{V_o}$$

$$= \cosh \gamma l + \frac{Z_r}{Z_o} \sinh \gamma l \quad . \quad . \quad . \quad . \quad . \quad (62)$$

It is interesting to note what occurs if $Z_r = Z_o$, that is to say, if the line is closed through an impedance equal to its own characteristic impedance. The voltage along the line is then

$$V = V_o\left(\cosh \gamma x - \frac{Z_o \sinh \gamma l + Z_o \cosh \gamma l}{Z_o \sinh \gamma l + Z_o \cosh \gamma l} \sinh \gamma x\right)$$

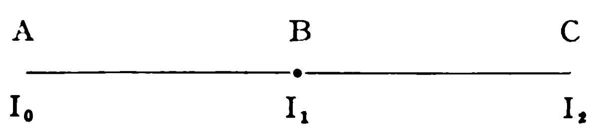
$$= V_o (\cosh \gamma x - \sinh \gamma x)$$

$$= V_o e^{-\gamma x},$$

i.e. the line behaves exactly as though it were infinite in length, and there is no reflection from the end. The result is what one would expect, and is valuable in certain cases where, for special reasons, it is desired to eliminate the reflected wave.

Composite Lines.

It frequently happens that a transmission line is not uniform throughout, but is made up of two or more sections, the line constants of which are different. Such lines may be dealt with as follows :—



Suppose the line to consist of a section AB of length l, propagation constant γ, and characteristic impedance Z_o, followed by a second section BC of length l', propagation constant γ', and characteristic impedance Z_o', and suppose the second section to be short-circuited at the end C. Then, by (55), the sending-end impedance of the section BC is

$$Z_o' \tanh \gamma' l'.$$

Hence we may consider the line as consisting of the first section only, closed through an impedance $Z_o' \tanh \gamma' l'$, and apply formulæ 59 to 62 by simply replacing Z_r by $Z_o \tanh \gamma' l'$.

Thus the current at the sending end,

$$I_0 = \frac{V_o}{Z_o} \times \frac{Z_o' \tanh \gamma' l' \sinh \gamma l + Z_o \cosh \gamma l}{Z_o \sinh \gamma l + Z_o' \tanh \gamma' l' \cosh \gamma l},$$

whilst the current at the end of the first section

$$I_1 = \frac{V_o}{Z_o \sinh \gamma l + Z_o' \tanh \gamma' l' \cosh \gamma l}.$$

Whence it follows from (58) that the current at the receiving end of the whole line,

$$I_2 = I_1 \times \frac{1}{\cosh \gamma' l'} = \frac{V_o}{Z_o \sinh \gamma l \cosh \gamma' l' + Z_o' \sinh \gamma' l' \cosh \gamma l}.$$

If the second section is closed through an impedance Z_r instead of being short-circuited, then the sending-end impedance of this section considered by itself is given by (60), and is

$$\frac{Z_o'(Z_o' \sinh \gamma' l' + Z_r \cosh \gamma' l')}{Z_r \sinh \gamma' l' + Z_o' \cosh \gamma' l'}.$$

Whence, by (61),

$$I_1 = \frac{V_o}{Z_o \sinh \gamma l + \dfrac{Z_o'(Z_o' \sinh \gamma' l' + Z_r \cosh \gamma' l') \cosh \gamma l}{Z_r \sinh \gamma' l' + Z_o' \cosh \gamma' l'}}.$$

And, by (62),

$$I_2 = \frac{I_1}{\cosh \gamma' l' + \dfrac{Z_r}{Z_o'} \sinh \gamma' l'}$$

$$= \frac{V_o}{\left\{Z_o \sinh \gamma l + \dfrac{Z_o'(Z_o' \sinh \gamma' l' + Z_r \cosh \gamma' l') \cosh \gamma l}{Z_r \sinh \gamma' l' + Z_o' \cosh \gamma' l'}\right\} \dfrac{(Z_o' \cosh \gamma' l' + Z_r \sinh \gamma' l')}{Z_o'}}$$

$$= \frac{V_o}{Z_o \sinh \gamma l \cosh \gamma' l' + \dfrac{Z_o Z_r}{Z_o'} \sinh \gamma l \sinh \gamma' l' + Z_o' \cosh \gamma l \sinh \gamma' l' + Z_r \cosh \gamma l \cosh \gamma' l'}.$$

Again, if the line consists of three sections, AB, BC, CD, of lengths l, l', and l'', propagation constants γ, γ', and γ'', and characteristic impedance Z_0, Z_0', and Z_0'', and short-circuited at D,

A B C D

I_0 I_1 I_2 I_3

then we proceed as before by considering the last section CD by itself first. The sending-end impedance of this section is Z_0'' tanh $\gamma' l''$. Taking now sections BC and CD together, we have in effect a line BC closed through an impedance Z_0'' tanh $\gamma'' l''$, and its sending-end impedance by (60) is

$$\frac{Z_0'(Z_0' \sinh \gamma' l' + Z_0'' \tanh \gamma'' l'' \cosh \gamma' l')}{Z_0'' \tanh \gamma'' l'' \sinh \gamma' l' + Z_0' \cosh \gamma' l'}.$$

We can now consider the line as a whole, for it is equivalent to a line AB closed through an impedance equal to the above expression. The current at B is therefore

$$I_1 = \frac{V_0}{Z_0 \sinh \gamma l + \dfrac{Z_0' \cosh \gamma l (Z_0' \sinh \gamma' l' + Z_0'' \tanh \gamma'' l'' \cosh \gamma' l')}{Z_0'' \tanh \gamma'' l'' \sinh \gamma' l' + Z_0' \cosh \gamma' l'}}.$$

Returning now to section BC, and remembering that this is equivalent to a line $l'\gamma' Z_0'$ closed through Z_0'' tanh $\gamma'' l''$, it follows from (62) that

$$I_2 = \frac{I_1}{\cosh \gamma' l' + \dfrac{Z_0'' \tanh \gamma'' l''}{Z_0'} \sinh \gamma' l'}.$$

Whilst considering the last section CD, we have from (58),

$$I_3 = \frac{I_2}{\cosh \gamma'' l''}.$$

Whence

$$I_3 = \frac{V_0}{\left\{ Z_0 \sinh \gamma l + \dfrac{Z_0' \cosh \gamma l (Z_0' \sinh \gamma' l' + Z_0'' \tanh \gamma'' l'' \cosh \gamma' l')}{Z_0'' \tanh \gamma'' l'' \sinh \gamma' l' + Z_0' \cosh \gamma' l'} \right\} \times \dfrac{(Z_0'' \tanh \gamma'' l'' \sinh \gamma' l' + Z_0' \cosh \gamma' l') \cosh \gamma'' l''}{Z_0'}}$$

$$= \frac{V_0}{Z_0 \sinh \gamma l \cosh \gamma' l' \cosh \gamma'' l'' + \dfrac{Z_0 Z_0''}{Z_0'} \sinh \gamma l \sinh \gamma' l' \sinh \gamma'' l'' + Z_0' \sinh \gamma' l' \cosh \gamma l \cosh \gamma'' l'' + Z_0'' \sinh \gamma'' l'' \cosh \gamma l \cosh \gamma' l'}.$$

The same method can be extended to deal with any number of lines, but the formulæ obtained become so complicated as to be of little practical value.

Attenuation and Wave-length Constants.

The values of the attenuation constant β and of the wave-length constant α are derived from the propagation constant γ as follows :—

$$\begin{aligned}\gamma^2 = \beta^2 - \alpha^2 + j2\alpha\beta &= (R + j\omega L)(G + j\omega C)\\ &= GR + j\omega LG + j\omega CR + j^2\omega^2 LC\\ &= GR - \omega^2 LC + j(\omega LG + \omega CR).\end{aligned}$$

$$\therefore\ \beta^2 - \alpha^2 = GR - \omega^2 LC$$

$$2\alpha\beta = \omega LG + \omega CR.$$

$$\therefore\ (\beta^2 - \alpha^2)^2 + (2\alpha\beta)^2 = (GR - \omega^2 LC)^2 + (\omega LG + \omega CR)^2.$$

Whence

$$\beta^2 + \alpha^2 = \sqrt{(G^2 + \omega^2 C^2)(R^2 + \omega^2 L^2)}.$$

Also

$$(\beta^2 + \alpha^2) + (\beta^2 - \alpha^2) = 2\beta^2.$$

Therefore

$$\beta = \sqrt{\tfrac{1}{2}\sqrt{(G^2 + \omega^2 C^2)(R^2 + \omega^2 L^2)} + \tfrac{1}{2}(GR - \omega^2 LC)} \quad (63)$$

$$\alpha = \sqrt{\tfrac{1}{2}\sqrt{(G^2 + \omega^2 C^2)(R^2 + \omega^2 L^2)} - \tfrac{1}{2}(GR - \omega^2 LC)} \quad (64)$$

Conditions for Minimum Attenuation.

The minimum value of β may be found in the usual manner by differentiating it with respect to any of the constants and equating to zero.

$$\beta = \sqrt{\tfrac{1}{2}\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} + \tfrac{1}{2}(RG - \omega^2 CL)}$$

$$\therefore\ 2\beta^2 = \sqrt{R^2(G^2 + \omega^2 C^2) + \omega^2 L^2(G^2 + \omega^2 C^2)} + RG - \omega^2 CL.$$

Differentiating with respect to L (say) and equating to zero,

$$\frac{\omega^2 L(G^2 + \omega^2 C^2)}{\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}} - \omega^2 C = 0,$$

whence

$$LG = CR.$$

Substituting in the expressions for β, Z_o, and α,

$$\beta = \sqrt{RG},$$

$$Z_o = \sqrt{\frac{R}{G}},$$

$$\alpha = \omega\sqrt{LC},$$

and velocity of wave

$$v = \frac{1}{\sqrt{LC}}.$$

Distortionless Lines.

It will be observed that both β and v are independent of ω, that is to say, all waves travel along the line with the same attenuation and the same velocity. Since, therefore, both the fundamental and all the harmonics of any irregular wave impressed upon the line will travel equally, the wave form will be unchanged at the end of the line, and the line is said to be **distortionless.**

The above case is of theoretical rather than of practical importance, since even with the maximum value of L attainable in practice, G would require to be artificially increased in order to satisfy the conditions, with the result that the attenuation would become excessive. Moreover, such a line would possess the additional disadvantage of a very high characteristic impedance.

A somewhat similar case, however, of great practical importance occurs when

$$\omega L >> R$$
$$\omega C >> G.$$

The attenuation constant may then be simplified thus,

$$\beta = \sqrt{\tfrac{1}{2}\sqrt{(R^2 + \omega^2L^2)(G^2 + \omega^2C^2)} + \tfrac{1}{2}(RG - \omega^2CL)}$$
$$= \sqrt{\tfrac{1}{2}\omega^2LC\left(1 + \frac{R^2}{\omega^2L^2}\right)^{\frac{1}{2}}\left(1 + \frac{G^2}{\omega^2C^2}\right)^{\frac{1}{2}} + \tfrac{1}{2}(RG - \omega^2CL)}.$$

Whence, neglecting small quantities above the second order,

$$\beta = \sqrt{\tfrac{1}{2}\omega^2 LC\left(1 + \frac{R^2}{2\omega^2 L^2}\right)\left(1 + \frac{G^2}{2\omega^2 C^2}\right) + \tfrac{1}{2}(RG - \omega^2 CL)}$$

$$= \sqrt{\frac{R^2 C}{4L} + \frac{G^2 L}{4C} + \frac{RG}{2}}$$

$$= \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Similarly,

$$\alpha = \sqrt{\tfrac{1}{2}\omega^2 LC\left(1 + \frac{R^2}{2\omega^2 L^2}\right)\left(1 + \frac{G^2}{2\omega^2 C^2}\right) - \tfrac{1}{2}(RG - \omega^2 CL)}.$$

But, in this case, since the larger terms $\frac{1}{2}\omega^2 CL$ do not cancel out, we may neglect all the terms $\frac{R^2}{2\omega^2 L^2}$, $\frac{G^2}{2\omega^2 C^2}$, and RG, which are of the second order of small quantities, and α becomes

$$\alpha = \omega\sqrt{CL} \quad . \quad . \quad . \quad . \quad (66)$$

If greater accuracy be required, these small terms may be retained, and

$$\alpha = \sqrt{\omega^2 LC + \frac{R^2 C}{4L} + \frac{G^2 L}{4C} - \frac{RG}{2}}$$

$$= \sqrt{\omega^2 CL + \left(\frac{RC - GL}{2\sqrt{LC}}\right)^2}. \quad . \quad . \quad . \quad (67)$$

It will be noticed that the expressions (65) and (66) denote that the line is distortionless as before, and the conditions of this case are actually met with in a heavily loaded telephone line.

CHAPTER III.

APPLICATION TO TELEPHONE TRANSMISSION.

THE sound waves emitted by the human voice and converted by the microphone into electric waves of the same form are all more or less irregular in shape. If analysed into a Fourier series, they consist of a fundamental corresponding to the *pitch* of the voice, and a number of harmonics or overtones of varying prominence which determine the quality or timbre of the voice and the particular vowel sound or consonant which is being pronounced. The range of pitch varies from about 80-500 in the male voice, and from about 150-800 in the female voice. It has, moreover, been found in practice that the suppression of harmonics above a frequency of 2000 p.p.s. does not greatly affect the quality of speech and, except where circuits of special quality are required, it is unnecessary to pay attention to the propagation of harmonics exceeding this frequency. The figure upon which all transmission calculations in telephony are based is 800 p.p.s. ($\omega = 5000$) which, although higher than the average *pitch* of the voice, was decided upon at the Paris conference in 1910 as the rough mean of the frequencies of the important tones and overtones.

In the application of the theory developed in the previous chapter to the transmission of telephonic currents, it should be observed that the theory is strictly applicable only to periodic currents which have reached their steady state; and it might be objected that, since this condition is not accurately attained in speech currents, especially in the case of the sharper consonants such as "p," the theory will not apply. An examination of oscillograms, however, such as those reproduced in

Hill's "Telephonic Transmission," * will show that the change in wave form is not so abrupt as might be expected, and, further, the extremely close agreement which is found in practice to exist between results predicted from the theory and those arrived at by actual experiment, is conclusive proof that the error involved in neglecting the transient effect of changes in wave form is not great.

Attenuation Measurement by Standard Cable.

In determining the efficiency of a circuit for the transmission of speech, account has to be taken of two factors. First, there is the **attenuation** of the wave, which causes the speech to become faint at the distant end. This is measured by the British Post Office in terms of **standard cable** in the following manner. The speech through the line under test is compared with that through a standard artificial non-reactive cable possessing an attenuation equal to the attenuation at 800 p.p.s. of a cable having the following primary and secondary constants :—

R = 88 ohms.	Propagation constant	$= 0{\cdot}15427\angle 46^\circ\ 31'$,
L = ·001 henries.	Attenuation constant	$= 0{\cdot}10616$,
G = 1 micro-mho.	Wave-length constant	$= 0{\cdot}11193$,
C = ·054 mfs.	Characteristic impedance	$= 571{\cdot}4\overline{\diagdown 43^\circ\ 16'}$,

the length of the artificial cable in circuit being adjusted until equality of speech is obtained between the two. If the length of standard cable required to obtain such equality is n miles, then the circuit under test is said to have a **standard cable equivalent or s.c.e.** of n miles. In practice, an artificial standard cable is employed, the equivalent length of which can be immediately adjusted by means of switches. It is useful to observe that, since the attenuation constant of standard cable is ·10616, the volume of speech is approximately halved in traversing $6\frac{1}{2}$ miles of it, for $\epsilon^{6\frac{1}{2}\times\cdot 10616} \approx 2$. The limit of commercial speech is generally assumed to be reached in 35 miles

* "Telephonic Transmission," J. G. Hill (Longmans, Green & Co.).

of standard cable, corresponding to a reduction to one-fortieth in volume, so that, in general, the s.c.e. of a circuit should not exceed 35 miles.*

DISTORTION.

Apart from loss of volume, however, speech may fail because it becomes unintelligible owing to **distortion,** that is to say, because the wave form of the received speech differs too much from that of the sent speech. Such distortion arises from a number of causes, amongst which are the imperfections of the transmitter and receiver, neither of which instruments fulfils perfectly its function of converting sound waves into electric waves and the reverse without change of form. We are concerned here, however, only with the distortion which occurs in the transmission of the electric wave along the line, and the causes of this are as follows.

In the previous chapter it has been shown the attenuation constant and the wave velocity vary, except in certain special cases, with the frequency of the wave. If, therefore, an irregular wave is being transmitted, the harmonics will, in general, travel with an attenuation and velocity differing from one another and from the fundamental, with the result that the wave form at the receiving end will be different from the impressed wave at the sending end. The general tendency is for the attenuation to increase with frequency, so that the higher harmonics are relatively less prominent in the received speech. Distortion is thus seen to arise from two separate causes :—

(*a*) The alteration of the relative amplitudes of the component parts of the wave.

(*b*) The alteration of their phase relation due to their different velocities.

Now, while it is clear that both must cause alteration in the wave form, there is some difference of opinion as to the practical effect of phase change. It has been held by some scientists,

* $\epsilon^{-35 \times \cdot 10616} = \cdot 0243 = \frac{1}{40}$ (approx.).

and experiments have been conducted which lend colour to their theory, that phase alterations are compensated for by the human ear, and that no loss of timbre or distinctness is caused thereby; but the balance of opinion is probably against this view. At the same time, it is almost certainly correct to say that there is *some* compensation, and that distortion of wave form due to the shifting of the phase of the harmonics, provided their amplitude is correctly reproduced, is of less importance than distortion due to alteration in the relative amplitudes of the harmonics.

It is evident, then, that the ideal telephone line should possess not only a low attenuation constant, but that it should also be distortionless. The ordinary line, whether consisting of aerial wires or of underground cable, will be found not to meet these conditions, owing to the fact that the capacity is always relatively too high. The capacity of aerial wires is much the smaller of the two, since the wires are much further apart than in a cable, and, for this reason, an aerial circuit is much better for transmission purposes than an ordinary cable pair; so much so that, for many years, long distance telephony was practically confined to aerial open wires. A comparison between an aerial circuit and a paper cable pair both of 100 lb. gauge will make this clear :—

					$\omega = 5000$		
	R.	L.	G.	C.	Attenuation Constant.	Wave-length Constant.	S.C.E. per mile.
Aerial .	17·60	·0039	10^{-6}	0081	·0120	·0303	·1136
Cable .	17·60	·001	10^{-6}	·065	·0466	·0615	·4388

LOADING.

The relatively poor transmission properties of underground cables have been vastly improved by artificially increasing the inductance of the circuit, the principles underlying which will now be explained.

It was shown in the previous chapter that a line becomes distortionless if $LG = RC$, when the attenuation constant becomes $\sqrt{RG}$. In practice, RC is always greater than LG, so that the condition could be arrived at by artificially increasing the leakance G; but such an operation would be useless, because it would also increase the attenuation constant, with the result that, although the line would be rendered distortionless, it would give extremely heavy losses. But it remains true that, with G kept at its natural low value, the condition $LG = RC$ is the ideal one, since it is the condition for minimum attenuation as well as freedom from distortion, and the nearer we can approach it the better. The discovery of the principles underlying the loading of cables is due to Oliver Heavyside ("Electromagnetic Theory," 1894, Vol. I., Chap. IV.), but for the development of the theory and its application to practice we are indebted to Prof. M. I. Pupin, who expounded the theory in a paper given before the American Institute of Electrical Engineers, March 22nd, 1899, entitled "Propagation of Long Electrical Waves."

Since we cannot, with advantage, increase G, and since it is obviously impossible to decrease R and C in a given line, the only feasible thing to do is to increase L by artificially adding inductance to the line. This procedure, for which the first practical suggestion was made by Prof. Pupin in 1899, is called **loading,** and is now very extensively carried out. It is never possible in practice to add sufficient inductance to reach the point where $LG = RC$, but it is possible to increase L so as to obtain the second condition discussed in Chapter II., viz., $\omega L >> R$ and $\omega C >> G$, when the line again becomes distortionless. One reason why there is a limit, apart even from cost, to the inductance which may usefully be added lies in the consideration that such artificial inductance must increase the resistance losses on account both of the ohmic resistance of the loading coils and of their hysteresis losses, so that, beyond a certain point, the advantage of adding more inductance is neutralised by the increased losses in the line.

Continuous Loading.

There are two methods of loading in general use. The first of these is **continuous** or **krarup** loading, consisting of one or more layers of wire of some magnetic material wound continuously round the conductor. This method is confined to cables, and the inductance of the pair so loaded has been shown by Prof. Breisig * to be

$$L_{(cms.)} = 4l\left\{\log_e \frac{D}{r} + \frac{1}{4} + \frac{\mu\pi t^2}{2(2r+t)(t+a)(1+\alpha)}\right\}$$

where l = length in cms.

D = distance between centres of conductors.
r = radius of conductors.
t = thickness of windings.
μ = permeability of magnetic material.
a = distance between spirals.
α = an empirical factor dependent upon magnetic losses in the winding.

If the spirals of the wire surrounding the conductor are very close together, both a and α become small and may be neglected, when the formula reduces to

$$L = 4l\left\{\log_e \frac{D}{r} + \frac{1}{4} + \frac{\mu\pi t}{2(2r+t)}\right\}$$

which has been found to give results in fairly good agreement with actual measurements. It should be observed that the term $\frac{2l\mu\pi t}{2r+t}$ represents the added inductance, since the part $4l(\log_e \frac{D}{r} + \frac{1}{4})$ is the natural inductance of the circuit unloaded.

This type of loading has certain important advantages, particularly for submarine cables, but is expensive, and, moreover, the amount of inductance which can be added is limited. On the other hand, it seems probable that, with

* "Theoretische Telegraphie," p. 322.

improvements in the magnetic material and method of manufacture, its use will be extended. It is obviously applicable only to new cables.

LUMPED LOADING.

The second method is that of **lumped series** loading, consisting of inductance coils inserted at regular intervals in the line in the manner shown in Fig. 10.

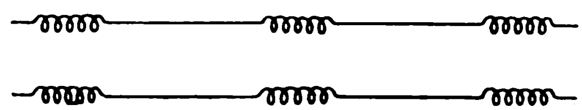


FIG. 10.

In order to preserve the " balance " of the circuit the loading coil at each point consists of a circular core of magnetic material on which are placed two equal windings (Fig. 10), one being inserted in the A line and the other in the B.

The design of these coils is a matter of the greatest importance, since, in addition to fulfilling their primary function of adding inductance, it is very necessary that the harmful effects, which are to some extent unavoidable, should be reduced to a minimum. Thus, their resistance must be as low as possible, and resistance, in this case, includes the losses in the core. The material employed for the core should therefore combine a high permeability with low losses. Iron wire cores were chiefly employed until recently, but the invention of a method of compressing iron in the form of dust into a material of sufficient strength to make the cores of the so-called " dust-core " coils has superseded the use of iron wire in long distance circuits and those in which repeaters are inserted. For shorter circuits, however, iron tape is now employed since coils made from it are both cheaper and smaller than those with dust cores. Further, the windings are bound to possess a certain self-capacity, but since this is exactly the reverse of what is required, the capacity must be kept as low as possible. There are many other requirements to be fulfilled in addition to these, a full description of which, together with the methods employed in practice of connecting the coils to cables and to aerial circuits, will be found in Hill's "Telephonic Transmission," already referred to.

In the same book will be found a description of a third method, known as **leak loading**, which consists in joining loading coils across the circuit at intervals instead of placing them in series in the two wires. This method has not, however, for the present at any rate, the same practical importance as those previously described.

Propagation Constant of Series Loaded Circuit.

The propagation constant of a series loaded circuit may be calculated by Campbell's method * in the following manner:—

Let Z be the impedance of each set of loading coils and d the distance between them, and let I_1 be the current through

Fig. 11.

the first set (considered constant throughout the coils), and I_2 the current through the second set. In order to find the propagation constant, consider a section of line from the middle of the first set of coils to the middle of the second. If the line is long, the average propagation constant of this section is the same as the average propagation constant of the line as a whole, and we may therefore fairly consider the section by itself, disregarding the sections of line both in front of it and following

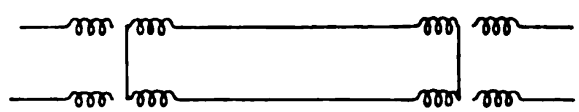


Fig. 12.

it. Hence we can imagine a short-circuit made at the centres of the loading coils as in Fig. 12, and we have, in effect, an

* *Phil. Mag.*, Vol. V., p. 319, March, 1903.

unloaded line of length d closed at the distant end through an impedance Z/2, so that, from (62),

$$\frac{I_1}{I_2} = \cosh \gamma d + \frac{Z}{2Z_0} \sinh \gamma d$$

where γ is the propagation constant of the unloaded line. Again, if γ' is the propagation constant of the loaded line, then we may regard the section alternatively as a length d of loaded line short-circuited at the receiving end, whence, from (58)

$$\frac{I_1}{I_2} = \cosh \gamma' d.$$

So that γ' is given by the relationship

$$\cosh \gamma' d = \cosh \gamma d + \frac{Z}{2Z_0} \sinh \gamma d.$$

If γ' be calculated from the above equation, for different values of d and the real part, which is the attenuation constant of the loaded cable, be plotted, it will be found that it exhibits a sudden and enormous increase as it approaches the value

$$\tfrac{1}{2}\omega d\sqrt{L_1 C} = 1,$$

or

$$d = \frac{2}{\omega\sqrt{L_1 C}},$$

where L_1 is the total inductance of the circuit per mile including the coils. But in a loaded circuit, where $\omega L_1 >> R$ and $\omega C >> G$, we have

wave-length constant $= \omega\sqrt{L_1 C}$,

and wave-length $\lambda = \frac{2\pi}{\omega\sqrt{L_1 C}}$.

Hence, the critical spacing occurs when

$$d = \frac{\lambda}{\pi},$$

or, as the rule is usually stated, **there must be more than π coils per wave-length.**

The increase of the attenuation constant of the loaded line for different spacing of the coils over that of a line in which the inductance is uniformly distributed is given in the following table :—

Coils per Wave-length.	Percentage Increase in Attenuation Constant.
9	——
8	1 per cent.
7	2 ,, ,,
6	3 ,, ,,
5	7 ,, ,,
4	16 ,, ,,
3	200 ,, ,,

from which it will be seen that the divergence begins to be marked when the coils are less than 5 per wave-length.

The practical rule for the spacing of coils is based upon the fact, previously stated, that harmonics above a frequency of 2000 p.p.s. are unimportant. Since the wave-length decreases with increased frequency, it is obvious that a spacing which is sufficiently close for the higher harmonics will be ample for the fundamental and lower harmonics ; so that, if we determine the critical spacing for a wave of 2000 p.p.s., the waves of lower frequency will pass satisfactorily. Neglecting the relatively small natural inductance of the line, the inductance per mile will be $\frac{L}{d}$, where L is the inductance of each coil, so that the critical spacing for 2000 p.p.s. is

$$d = \frac{2}{2\pi \times 2000\sqrt{\frac{L}{d} \times C}},$$

or, if C is in microfarads and L in millihenries, we obtain the practical rule

$$CLd = 25 \text{ (approx.)},$$

which is the spacing rule adhered to in the British Post Office This rule was originally determined by experiment, but the above calculations show that it possesses a sound basis in theory.

Partial Reflection.

In Chapter II. it was shown that the wave is reflected at the end of a line, and that the reflected wave travels back to the sending end. In a similar manner, if at any point in the line there occurs some abrupt change in its characteristics, as, for example, if a length of cable is inserted in an aerial line, then the wave at this point is partially transmitted forwards and partially reflected backwards. Evidence of such partial reflection is supplied as follows. If the sending-end impedance of a finite line is measured over a range of frequencies, the impedance curve exhibits a periodic rise and fall in value, the maximum points occurring at those frequencies at which the impressed and reflected currents and volts are in opposition, and the minimum points at those frequencies at which they are in phase with one another. If the line is so long as to be the equivalent, for transmission purposes, to an infinite line, or if it is closed at the distant end through its own characteristic impedance, then there will be no reflection, and the sending-end impedance curve will be smooth and will not exhibit periodic maxima and minima. Hence, if such a line does show periodic undulations, partial reflection must be occurring at some intermediate point along it.

The distance from the sending end at which the reflection is taking place can be calculated by the following method, due to Mr. Ritter,* of the Post Office Engineering Service.

If α is the wave-length constant of the line at any given frequency, and l the distance of the point of reflection from the sending end, then the change in phase of a wave of that frequency in travelling to the point of reflection and back to the sending end is $2\alpha l$. In addition, its phase will be changed by some indeterminate angle ϕ at the point of reflection itself, so that the total phase change will be $2\alpha l + \phi$. If, therefore, two successive points of maximum impedance occur at fre-

* See I.P.O.E.E. Paper No. 76, by C. Robinson and R. M. Chamney.

quencies f_1 and f_2, it is evident that the total phase change of the second wave must be greater by 2π than the first, or

$$(2\alpha_2 l + \phi_2) - (2\alpha_1 l + \phi_1) = 2\pi$$

where α_1, α_2 and ϕ_1, ϕ_2 are the wave-length constants, and the angles of phase change at the point of reflection of the respective waves

Now, although ϕ_1 and ϕ_2 are indeterminate, it is reasonable to suppose, especially if the reflection is not great, that both are small and that they are approximately equal—an assumption found in practice to be justified. We therefore obtain the simple result that

$$l = \frac{\pi}{\alpha_2 - \alpha_1}.$$

If the line is loaded $\alpha_1 = 2\pi f_1\sqrt{\mathrm{CL}}$ and $\alpha_2 = 2\pi f_2\sqrt{\mathrm{CL}}$ (approx.).

$$\therefore\ l = \frac{\pi}{2\pi\sqrt{\mathrm{CL}}(f_2 - f_1)} = \frac{1}{2\sqrt{\mathrm{CL}}(f_2 - f_1)}.$$

In unloaded lines it may be shown that

$$\alpha_1 = \sqrt{\pi \mathrm{RC} f_1} \text{ and } \alpha_2 = \sqrt{\pi \mathrm{RC} f_2} \text{ (approx.).}$$

$$\therefore\ l = \frac{1{\cdot}772}{\sqrt{\mathrm{RC}(f_2 - f_1)}}.$$

This method is of practical use in locating faults which are not revealed by the ordinary D.C. tests. Thus, in a loaded line, if a loading coil has become short-circuited, it forms a discontinuity in the electrical constants of the line, and causes partial reflection and its position can therefore be located by this means.

CHAPTER IV.

LINE CONSTANTS AND THEIR MEASUREMENT.

(1) RESISTANCE.

THE resistance of a telephone line may be calculated from the size of the wire and the material of which it consists. Thus, the resistance per mile at 68° F. of a single annealed copper wire is

$$R = \frac{\cdot 05475}{d^2},$$

where d is the diameter of the wire in inches.

Or, if the weight of the wire is W lbs. per mile,

$$R = \frac{874\cdot 9}{W}.$$

The wires used for telephone circuits in the British Post Office are gauged by their weight in lbs. per mile. Their resistances are always stated per mile of *loop*, and are, in practice, invariably taken from the table of values as given in Table I.

Copper possesses a considerable temperature coefficient, amounting to

$$\frac{1}{234\cdot 5 + t}$$

where t is in degrees Centigrade. Thus, at 40° C. the coefficient is $\frac{1}{274\cdot 5}$, or ·364 per cent. for a rise in temperature of one degree. The greatest changes in temperature obviously occur in open lines, but calculations based on the constants of such lines are, in any case, only approximate, owing to the great changes which occur in the leakance due to varying weather conditions, so that small variations in resistance can be neglected.

The resistance to alternating currents is also subject to increase owing to the well-known phenomenon called "skin effect," the amount of which may be calculated from Lord Rayleigh's formula,

$$\frac{R'}{R} = 1 + \frac{h^2}{48} - \frac{h^4}{2880} + \frac{h^6}{58647} - \ldots$$

Where R' = high frequency resistance,

R = D.C. resistance,

$$h = \frac{n\pi^2 d^2}{\rho},$$

where n = frequency,

d = diameter in cms.,

ρ = resistivity in absolute C.G.S. units (1600 for copper).

For example, a 400-lb. copper wire has a diameter of approximately 4 cms. Hence, at 800 p.p.s.,

$$h = \frac{800 \times \pi^2 \times \cdot 16}{1600} = \cdot 79$$

$$\frac{R'}{R} = 1 + \frac{(\cdot 79)^2}{48} - \frac{(\cdot 79)^4}{2880} - \ldots = 1 \cdot 013,$$

that is, the resistance offered to an alternating current of this frequency is 1·3 per cent. greater than the resistance offered to a direct current. Aerial wires seldom exceed this gauge, and underground cable wires are usually considerably smaller, so that the increase in resistance due to skin effect may be said to be negligible at telephonic frequency. It becomes marked, however, when dealing with the higher frequencies employed in carrier wave telephony.

(2) Inductance.

The inductance of a metallic circuit may be calculated from the usual formula for a long straight pair, viz.,

$$L = 4l\left(\log_e \frac{D}{r} + \frac{1}{4}\right),$$

where l = length of circuit,
D = distance apart of wires,
r = radius of wires,

and all quantities, including L, are in centimetres. The formula may be written in a more practical form for telephonic calculations, as follows :—

$$L = \cdot 001482 \log_{10} \frac{D}{r} + \cdot 000161,$$

where L is in henries per mile loop.

This formula may be applied with accuracy to the calculation of the inductance of aerial circuits where the distance apart of the wires is considerable compared with their diameters, but it is not very accurate when applied to cable pairs, partly because the formula itself is not strictly accurate, especially at high frequencies, but chiefly because the average distance apart of a pair of wires closely packed in a paper cable is difficult to determine with sufficient exactitude.

The variation of inductance with frequency is negligible over the range employed in telephony.

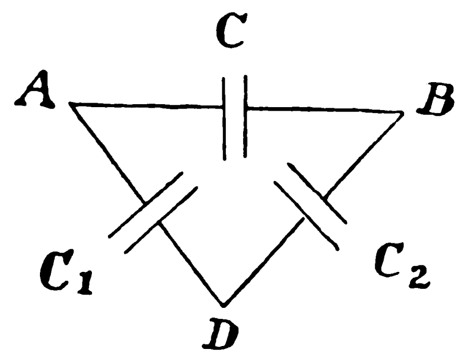
(3) Capacity.

The capacity of an aerial circuit may be accurately calculated from the following formula :—

$$C = \frac{\cdot 0194}{\log_{10} \frac{D}{r}} \text{ mfs. per mile,}$$

where D and r have the same meaning as above.

The capacity of cable circuits, however, depends on so many factors that it can only be determined satisfactorily by experiment The dielectric constant of the insulating material is one factor, and this varies, for example, in the case of paper insulated cables from 1·7 to 1·9. The proximity of other wires and of the sheath are also important considerations.



Thus, in the above diagram, if A and B form the pair and D is a third wire close to them, the actual capacity between A and B is $C + \frac{C_1C_2}{C_1 + C_2}$, indicating the extreme difficulty of calculating the capacity of a pair surrounded by other wires, and in greater or less proximity to the sheath.

(4) Leakance.

The leakance of aerial circuits varies irregularly between wide limits, owing to weather and other conditions, and it is therefore impossible to take accurate account of it in transmission calculations of such circuits. The inaccuracy introduced is not very great, since it is always a comparatively small quantity.

In underground circuits the leakance also varies very greatly indeed, but the variation is regular, and, since it arises from a definite cause, it can be allowed for. In this case, the D.C. leakance is practically nil, being of the order 10^{-9} or less, but with alternating currents a loss of energy occurs in the dielectric due to some form of dielectric hysteresis, which is roughly proportional to the frequency, and which is, at 800 p.p.s.

equivalent to a leakance about 1000 times greater than the normal D.C. leakance. That is to say,

$$\text{Leakance} \approx \text{Constant} \times \omega \approx \mathrm{A}\omega \text{ (say)}$$

where A depends on the type of insulating material.

This being so, it is clear that the formulæ deduced on the theory that G is constant are not strictly accurate, but the leakance is still so small at telephonic frequencies as to be almost negligible, and the inaccuracy is therefore inconsiderable. Moreover, provided the value assigned to G is that which it possesses at 800 p.p.s., its variation over the range of frequencies important in telephony is not such as to matter greatly. At the same time, the author is personally of the opinion that it would be preferable to recognise the variable character of leakance in cables by including it in the form Aω and amending the formulæ accordingly. This is especially necessary when the formulæ are applied, as otherwise they certainly may be, to transmission at the higher frequencies employed in carrier wave telephony, that is, at frequencies from 10,000 p.p.s. upwards, at which leakance becomes not merely an important, but, in some cases, a dominating factor. In this book the usual method has, however, been adhered to as being that, at present, universally employed.

Measurement of Primary Constants.

Since the primary constants of cables, with the exception of resistance, cannot, as a general rule, be calculated with accuracy, it is usual to measure them experimentally. This is done by making open and closed end impedance tests of a measured length of line and calculating the constants from these in the following manner.

From page 30 of Chapter II. we have the relationships,

$$\sqrt{Z_{\text{open}}\, Z_{\text{closed}}} = Z_0 = \sqrt{\frac{\mathrm{R} + j\omega\mathrm{L}}{\mathrm{G} + j\omega\mathrm{C}}}$$

$$\tanh \gamma l = \frac{Z_{\text{closed}}}{Z_0} = \sqrt{\frac{Z_{\text{closed}}}{Z_{\text{open}}}}.$$

Whence γ and Z_0 may be found.

But $$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}.$$

Therefore, $$Z_0\gamma = R + j\omega L$$

and $$\frac{\gamma}{Z_0} = G + j\omega C,$$

whence, by equating real and imaginary parts, all the primary constants may be found.

If the length of the line under test is very short, so that $\gamma l >> 1$, a simpler method of calculating the constants is arrived at in the following manner.

The function tanh γl may be expanded into the series,

$$\gamma l\left\{1 - \frac{(\gamma l)^2}{3} + \frac{2(\gamma l)^4}{15} - \ldots\right\}$$

which is clearly rapidly converging when γl is less than 1, while, if γl is very much less than 1, even the second term becomes negligibly small, and

$$\tanh \gamma l \approx \gamma l.$$

Similarly,

$$\coth \gamma l \approx \frac{1}{\gamma l}.$$

Therefore,

$$\begin{aligned} Z_{\text{closed}} &= Z_0 \tanh \gamma l \\ &= Z_0\gamma l \\ &= l \times \sqrt{\frac{R + j\omega L}{G + j\omega C}} \times \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= l(R + j\omega L), \end{aligned}$$

and

$$\begin{aligned} Z_{\text{open}} &= Z_0 \coth \gamma l \\ &= \frac{Z_0}{\gamma l} \\ &= \frac{1}{l(G + j\omega C)}, \end{aligned}$$

that is to say, the constants are immediately derived from the open and closed end impedances.

This is a convenient method of making tests at audio frequencies on sample lengths of cable of, say, 200 yds. The author has also employed it with great success on lengths as short as 4 yds. when testing cables at carrier wave frequencies with apparatus specially designed for the purpose.

Measurements of line impedance are very important, not only as a means of arriving at the primary constants, but also for other purposes. The first requisite of all methods of measurement is some means of producing alternating currents of pure sine wave form whose frequency can be varied at will over the range required. The Drysdale A.C. potentiometer used in connection with a suitable sine wave alternator and the Franke machine are two good methods when such apparatus is available, but they have the disadvantage of being extremely costly and are not portable. They are, however, extensively used in the British Post Office, and a full description of them will be found in Hill's " Transmission."

The A.C. Bridge.

The advent of the thermionic valve provides a very simple means of producing alternating currents of any desired frequency, and, if proper care is taken, sensibly free from harmonics, and it is therefore particularly suitable for use in connection with these measurements. The means by which a valve can be made to oscillate, and the rate of oscillation controlled, are too well known to require any explanation here. Since, however, the calibration of the frequency of the oscillator is of the greatest importance, it may not be out of place to describe a very accurate method of effecting this.

Let the oscillator be connected across the points AB of the bridge in Fig. 13 (p. 60). The bridge consists of two equal non-inductive resistances r_1, r_2 (usually 1000 ohms), in the arms *a* and *b*, while the two remaining arms *c* and *d* consist of a variable non-inductive resistance box and a variable condenser, these being joined in series in one arm and in parallel in the other. Finally, a telephone receiver is joined across the points C and D.

The bridge is balanced by manipulating the resistance and condenser in either *c* or *d*, the condition of balance being indicated by silence in the telephone receiver. We then have the same relations as in the ordinary Wheatstone bridge, viz.,

$$\frac{a}{b} = \frac{c}{d}$$

That is to say, since *a* and *b* are equal, the impedance of *c* and *d* are equal.

$$\text{Impedance of } c = R_1 - j\frac{1}{\omega C_1}.$$

$$\text{Impedance of } d = \frac{1}{\frac{1}{R_2} - \frac{\omega C_2}{j}}$$

$$= \frac{R_2}{1 + j\omega R_2 C_2}$$

$$= \frac{R_2 - j\omega R_2^2 C_2}{1 + \omega^2 R_2^2 C_2^2}.$$

$$\therefore R_1 - j\frac{1}{\omega C_1} = \frac{R_2}{1 + \omega^2 R_2^2 C_2^2} - \frac{j\omega R_2^2 C_2}{1 + \omega^2 R_2^2 C_2^2}.$$

Whence $$R_1 = \frac{R_2}{1 + \omega^2 R_2^2 C_2^2},$$

or $$1 + \omega^2 R_2^2 C_2^2 = \frac{R_2}{R_1}.$$

And $$\frac{1}{\omega C_1} = \frac{\omega R_2^2 C_2}{1 + \omega^2 R_2^2 C_2^2}$$

$$= \frac{\omega R_2^2 C_2 R_1}{R_2}.$$

Whence $$\omega^2 = \frac{1}{R_1 R_2 C_1 C_2}.$$

The oscillator having been calibrated by this or other means, it may be employed for impedance measurements in connection with any suitable type of A.C. bridge.

In designing the oscillator, however, it should be borne in mind that the frequency is affected to an appreciable extent by the load upon the plate circuit of the valve. Where greater accuracy is required, therefore, it is essential that the oscillator should consist of at least two valves, the first being the oscillating valve with its plate circuit controlling the grid of the second valve. With two valves the error is reduced to under 1 per cent., but if still greater exactitude is required this can be obtained either by using additional valves or by employing the method of a feed back resistance in the oscillating circuit patented by the American Telegraph and Telephone Company of New York.

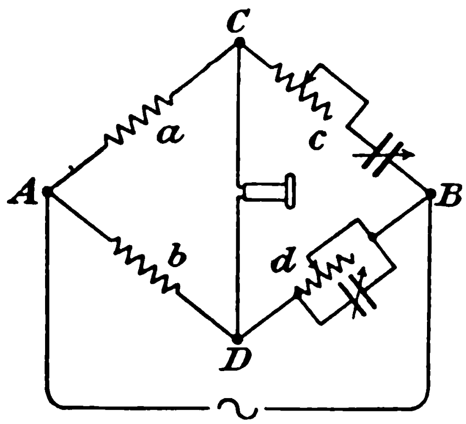


FIG. 13.

Bridges may be designed in several different ways to suit the conditions required, and the theory of each can be easily deduced on the same lines as before.

Thus, in the above arrangement, the impedance of the line is clearly

$$R - j\frac{1}{\omega C}.$$

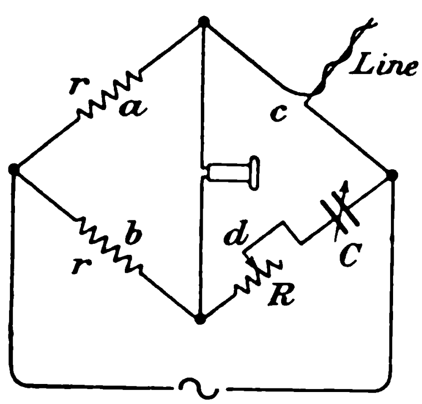


FIG. 14.

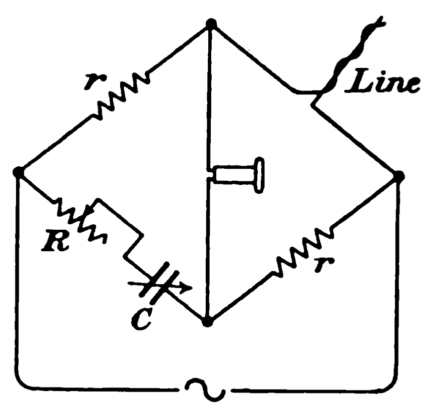


FIG. 15.

Should the line impedance have a positive angle, the arms *b* and *d* are reversed as in Fig. 15.

In this case the line impedance is

$$\frac{r^2}{R - j\frac{1}{\omega C}}.$$

This expression, however, does not lend itself to rapid evaluation, and for ease in calculating results it is simpler to re-arrange the arm b by placing the resistance and condenser in parallel instead of in series, as shown in Fig. 16.

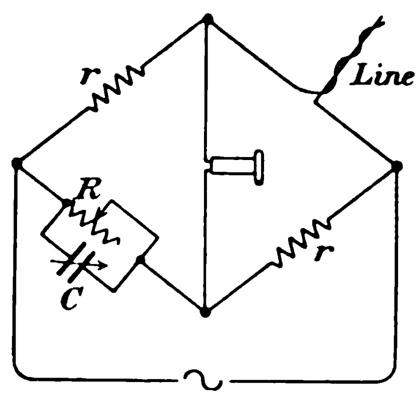


FIG. 16.

The impedance of b is now

$$\frac{1}{\frac{1}{R} + j\omega C},$$

so that the line impedance is

$$r^2\left(\frac{1}{R} + j\omega C\right).$$

Since one does not know whether the line impedance at any given frequency will prove to possess a positive or negative angle, it is convenient in practice to provide a special switch by means of which the connections of the bridge can be quickly changed from those in Fig. 14 to those in Fig. 16.

There are many other methods of making up bridges, not only for measurement of line impedance but also for measuring unknown inductances and capacities. They are very accurate provided proper care is taken in avoiding stray earth capacities and provided the resistances are truly non-inductive, since it is possible with a little practice to obtain an extremely sensitive balance. Their use can be extended to ultra-audio frequencies by introducing a heterodyne arrangement in place of the telephone receiver, but the effect of stray earth capacities becomes more marked at higher frequencies and special precautions are necessary to ensure accuracy. For impedance measurements on lines the method has the great advantage of portability and relative cheapness and is now extensively used.

APPENDIX.

NOTE ON USE OF KENNELLY'S TABLES.

At first sight these tables appear to be incomplete and to present some difficulty in manipulation. The following notes will facilitate their use.

Firstly, it should be noted that q is in *quadrants* and not in radians, so that $q = 2$ is really $q = 2\pi$. Now

$$\sinh(x + iq) = \frac{e^x}{2} \underline{/q} - \frac{e^{-x}}{2} \overline{\diagdown q}.$$

It is therefore evident that $\frac{e^x}{2} \underline{/q}$ and $\frac{e^{-x}}{2} \overline{\diagdown q}$ repeat themselves when $q = q + 2\pi, q + 4\pi$, etc. It follows that

$$\sinh(x + i\overline{2n + q}) = \sinh(x + iq).$$

Since q is in quadrants, the first operation consists in reducing q from radians to quadrants by dividing it by 1·57079. Thus, to find

$$-\sinh(1{\cdot}5 + i8{\cdot}01)$$

$$\text{divide } \frac{8{\cdot}01}{1{\cdot}57079} = 5{\cdot}1$$

subtract $5{\cdot}1 - 4 = 1{\cdot}1,$

then look up $\sinh(1{\cdot}5 + i1{\cdot}1)$

$$= -\underline{\underline{33309 + i\,2{\cdot}32345.}}$$

Similarly,

$$\tanh(1{\cdot}5 + i\,9{\cdot}74)$$

$$\text{quadrant } 9{\cdot}74 = 6{\cdot}2,$$

$$\text{subtract } 6 = {\cdot}2.$$

Look up $\tanh(1{\cdot}5 + i \cdot 2)$,

$$= \underline{\underline{{\cdot}92104 + i\,{\cdot}05404.}}$$

Change of Sign.—If x or iq are negative, change sign of u and iv accordingly.

Thus, $\sinh(x - iq) = u - iv.$

The same applies when the values are given in the form $r \underline{/y.}$ Thus,

$$\sinh(x - iq) = r\overline{\diagdown y.}$$

$$\sinh(-x + iq) = -r\underline{/y.}$$

$$\sinh(-x - iq) = -r\overline{\diagdown y.}$$

VALUES OF q ABOVE 4.

When $x > 4$ the term $\frac{e^{-x}}{2} \overline{\diagdown q}$ becomes very small compared with $\frac{e^{x}}{2} \underline{/q}$, and may be neglected. We have, therefore,

$$\sinh (x + iq) \approx \cosh (x + iq) \approx \frac{e^{x}}{2} \underline{/q},$$

and a range of value is given in Table XIV. For these large values of q, $\tanh (x + iq)$ is obvious ≈ 1.

INTERPOLATION.

Methods of accurate interpolation are given in the explanatory notes appended to the tables. In practice, however, they are cumbersome, and are only of use when extreme accuracy is required in a few calculations. The Chart Atlas, which can be purchased with the tables, is of considerable assistance, but it is the author's experience, when making calculations in practice, that it is often as quick to calculate the values from the identics given in Chapter I., provided good tables of real Circular and Hyperbolic Functions are available. Rough arithmetic interpolation between values given in the tables is, however, both practical and expeditious, and will give results sufficiently accurate for many engineering purposes.

PART II.

POWER TRANSMISSION.

CHAPTER V.

POWER TRANSMISSION.

THE transmission of electric power from the source of supply to the place where it is to be utilised is accomplished by overhead lines or underground cables. In single-phase working the general theory of transmission given in Chapter II. can be applied to the calculation of the voltage and current at any point along the transmission line, and formulæ (41) and (42) used without alteration. Moreover, with but slight modifications they can be applied to the much more important case of 3-phase transmission. The formulæ for propagation, attenuation, and wave-length constants apply equally well to the problems of both telephone and power transmission. It should be noted that the currents used in power transmission have a much lower frequency than those used in telephone transmission, the usual frequencies being 50 and 60 cycles per second.

All modern alternators generate an E.M.F. which is a close approximation to a sine wave. The error introduced by the assumption that all currents and voltages are sinusoidal is therefore small.

Many transmission lines are so short that the effect of the capacity between conductors is negligible, and the current will be constant at all points along the line. This makes it possible to use simple methods of calculation when dealing with such lines. For long transmission lines, however, the capacity effects cannot be ignored, and here the methods of Chapter II. have an important field of application. The trend of development is to transmit power over longer distances, and the methods of Chapter II. as applied to the transmission of power are therefore becoming of increasing practical importance.

In this chapter the simple methods which serve for the solution of short lines will first be given, and then the application of formulæ (41) and (42) of Chapter II. to the solution of long lines will be considered.

Primary Constants of Power Lines.

The primary constants of power lines are the same as those given on pages 52-55 of Chapter IV. for telephone lines, but certain of the formulæ there given can be modified into forms convenient for application to single- and 3-phase transmission lines.

Resistance.—The conductor resistance is the ordinary ohmic resistance of the conductor corrected for skin effect. This correction at the low frequency of power lines will be very small, and can usually be neglected.

Inductance.—The reactance is more frequently required than the inductance, and so the formulæ will be given for the former quantity. The inductance is obtained by dividing the reactance by $2\pi f$ where f is the frequency in cycles per second.

For a single-phase overhead transmission line with conductors of non-magnetic material (e.g. copper or aluminium),

Reactance per mile of conductor

$$= 2\pi f\left(741 \log_{10} \frac{D}{r} + 80\right) \times 10^{-6} \text{ ohms} \quad . \quad (68)$$

r = radius of power conductor.

D = distance separating the centres of the two conductors.

(*N.B.*—D and r must be measured in the same units of length.)

f = frequency in cycles per second.

The reactance of a single-phase concentric cable is very small. When required it may be calculated from the formula :—

Reactance per mile of loop

$$= 2\pi f\left\{741 \log_{10}\frac{d_2}{d_1} + 161\left(\frac{1}{2} + \frac{1}{3}\frac{d_1^2}{d_2^2} - \frac{1}{12}\frac{d_1^4}{d_2^4} + \ldots\right)\right\} \times 10^{-6} \text{ ohms} \quad (69)$$

where d_1 = diameter of inner conductor.

d_2 = inner diameter of outer conductor (see Fig. 20).

For a 3-phase transmission line with conductors of radius r spaced as in Fig. 17,

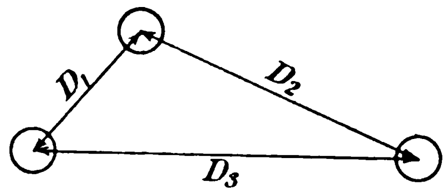


FIG. 17.—3-phase overhead transmission line, conductors unequally spaced.

Reactance per mile of conductor

$$= 2\pi f\left(370\cdot6 \log_{10} \frac{\sqrt[3]{D_1^2 \times D_2^2 \times D_3^2}}{r^2} + 80\right) \times 10^{-6} \text{ ohms} \quad (70)$$

When the conductors are situated at the corners of an equilateral triangle of side D, then $D_1 = D_2 = D_3 = D$.

Reactance per mile of conductor

$$= 2\pi f\left(741 \log_{10} \frac{D}{r} + 80\right) \times 10^{-6} \text{ ohms} . \quad . \quad (71)$$

Thus the reactance per conductor of a 3-phase line with conductors at the corners of an equilateral triangle is equal to the reactance per conductor of a single-phase line of equal length, and with equal spacing between conductors.

Where the spacing is not equilateral (as D_1, D_2, D_3 of Fig. 17) it is often convenient to obtain an equivalent equilateral spacing D which will give the same reactance as the actual spacing.

The equivalent equilateral spacing

$$D = \sqrt[3]{D_1 \times D_2 \times D_3} \quad . \quad . \quad . \quad (72)$$

Typical arrangements of 3-phase conductors with their equivalent spacings are given in Fig. 18.

In Fig. 19 is given the conductor arrangement for a typical 3-phase underground cable with the dimensions D and r indicated.

Capacity.—A distinction must be made between underground cables and overhead lines when giving formulæ for the calculation of capacity. For overhead lines the dielectric separating the conductors is air with a specific inductive

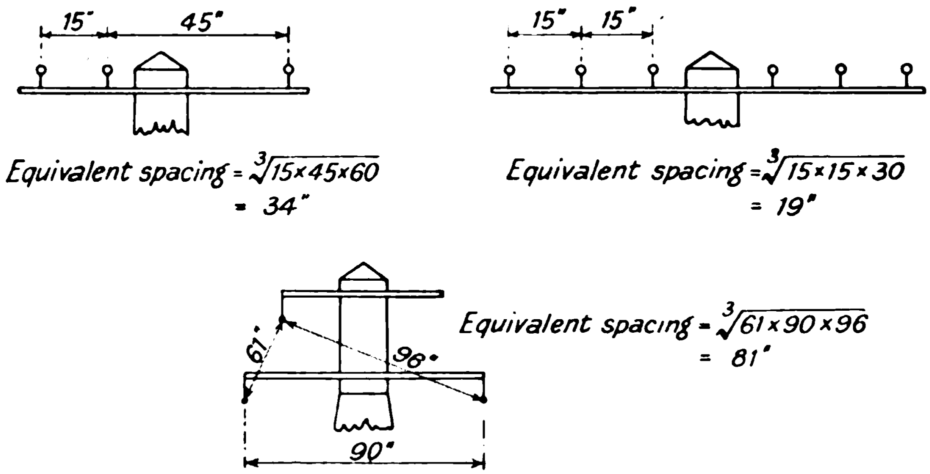


FIG. 18.—3-phase overhead transmission line, equivalent equilateral conductor spacings.

capacity of unity, whilst for underground cables the dielectric is material having a greater specific inductive capacity than

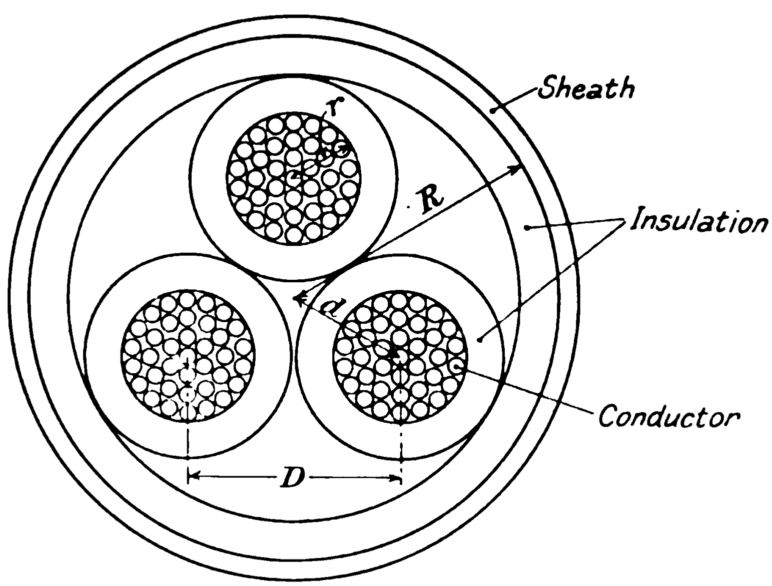


FIG. 19.—3-phase underground cable.

unity. Also, the capacity of cables is affected by other factors which need not be considered for overhead lines, such as the capacity existing between each conductor and earth.

1. Capacity of Overhead Lines.

For a single-phase transmission line,

$$\text{Capacity per mile of loop} = \frac{19{\cdot}4 \times 10^{-3}}{\log_{10} \frac{D}{r}} \text{ microfarads} \quad (73)$$

For a 3-phase transmission line,

$$\text{Capacity per mile of conductor} = \frac{38{\cdot}8 \times 10^{-3}}{\log_{10} \frac{D}{r}} \text{ microfarads,} \quad (74)$$

with the conductors at the corners of an equilateral triangle of side D. When the conductor spacing is D_1, D_2, D_3, as in Fig. 17, the equivalent spacing $D = \sqrt[3]{D_1 \times D_2 \times D_3}$ must be taken.

It should be noted that the above expression for the capacity of a 3-phase line is the capacity of each of three equal condensers which, if connected in "star" across the lines, would take the same charging current as the line actually takes. The capacity as calculated by the above formula will therefore be the "capacity to neutral."

The capacity to neutral of the three-phase line is equal to twice the capacity between conductors of a single-phase line with the same conductor spacing.

2. Capacity of Underground Cables.

Single-phase Concentric Cable.—Referring to Fig. 20, let

d_1 = outside diameter of inner conductor.
d_2 = inside diameter of outer conductor.
K = S.I.C of insulating material.

Capacity in microfarads per mile of loop

$$= \frac{38{\cdot}8 \times 10^{-3} \times K}{\log_{10} \frac{d_2}{d_1}} \quad . \quad . \quad . \quad (75)$$

Three-phase Cables.—Six capacities have to be considered, viz., the three equal capacities between cores C_c and the three equal capacities between each core and the sheath C_s. These are indicated in Figs. 21*a* and 21*b*.

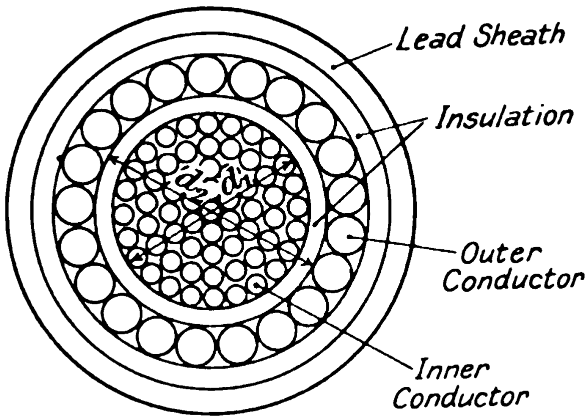


FIG. 20.—Single-phase concentric cable.

As for overhead lines, it is convenient to replace the various capacities by three equal condensers connected in star across the lines, each condenser having a capacity C. The arrangement is indicated in Fig. 21*c*.

C can be calculated from the following formula—

$$C = \frac{0{\cdot}039K}{\log_{10}\left\{\frac{1{\cdot}73d}{r} \times \frac{(R^2 - d^2)}{(R^4 + R^2d^2 + d^4)^{\frac{1}{2}}}\right\}} \text{ microfarads per mile of conductor,} \quad (76)$$

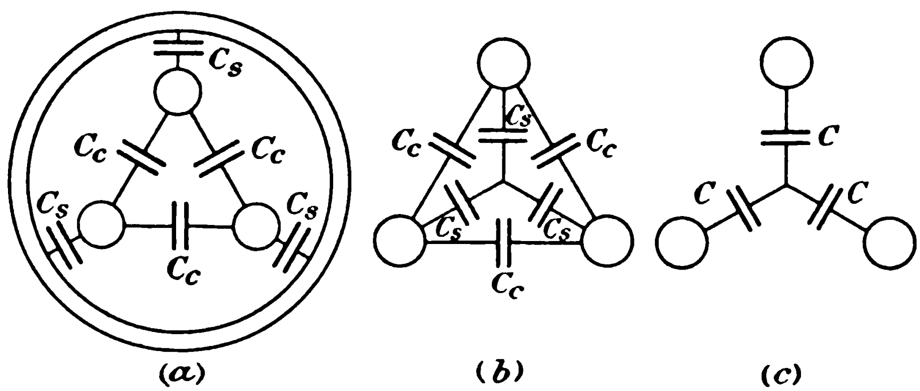


FIG. 21.—Capacity of 3-phase cable.

where R = inside radius of sheath
r = radius of conductor
d = distance between axis of conductor and axis of sheath
} Refer to Fig. 19.

K = S.I.C. of insulating material, and for impregnated paper cables varies from 2·8 to 3·5, according to methods of manufacture.

This formula must be used with caution, since it gives inaccurate results for large conductors closely spaced. In practice it is customary to obtain the value of C by experiment. For detailed information on the capacity of cables, the reader is referred to "Alternating Currents," by Dr. A. Russell.*

Leakance.—The leakance between conductors is very small, and may be neglected. It is interesting to note that the leakance of an overhead line includes not only the leakage over the insulators but also the loss caused by corona discharge.

VOLTAGE DROP ALONG SHORT SINGLE-PHASE TRANSMISSION LINE.

Capacity Effects Neglected.

Neglecting capacity and leakance effects the current will be the same at all points along the line. For purposes of calculation, therefore, it is permissible to consider the resistance

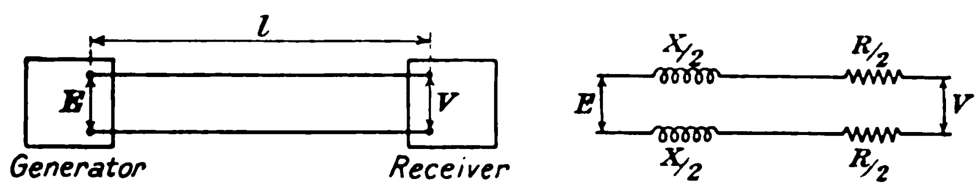


FIG. 22.—Single-phase transmission line having resistance and reactance but no capacity.

and reactance of a single-phase circuit such as that shown on the left of Fig. 22, as if it were lumped as indicated on the right of the same figure.

Let E = voltage between lines at generator.
V = voltage between lines at receiver (or load).
I = current.
$\cos\phi$ = power-factor at receiver.
R/2 = resistance of each conductor.
X/2 = reactance of each conductor.
l = length of line.

It is required to calculate the voltage E at the generator

* "Theory of Alternating Currents," Vol. I., by Alex. Russell. Cambridge University Press.

when the other quantities are specified. Using vector quantities for voltage and current,

$$E = V + I\,(R + jX).$$

The vector diagram for the circuit will be as shown in Fig. 23. Taking I as the reference vector, the voltage V in vector notation will be

$$V = V\cos\phi + jV\sin\phi.$$
$$\therefore E = V\cos\phi + jV\sin\phi + IR + jIX$$
$$= (V\cos\phi + IR) + j(V\sin\phi + IX).$$

The magnitude of E will therefore be

$$E = \sqrt{(V\cos\phi + IR)^2 + (V\sin\phi + IX)^2} \quad . \quad (77)$$

and it will be displaced from the current I by an angle θ where

$$\tan\theta = \frac{V\sin\phi + IX}{V\cos\phi + IR}.$$

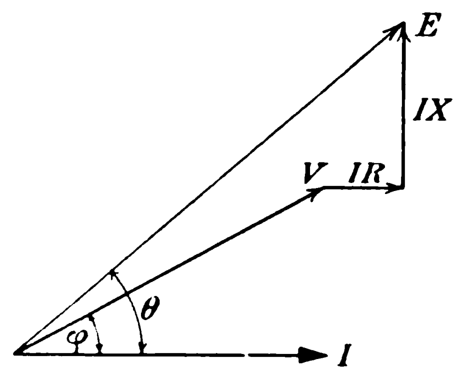


FIG. 23.—Vector diagram for single-phase transmission line.

The voltage drop along the line will therefore be equal to

$$(E - V)$$
$$= \sqrt{(V\cos\phi + IR)^2 + (V\sin\phi + IX)^2} - V.$$

Formula (77) may be simplified into an important practical working formula thus :—

$$E = \sqrt{(V\cos\phi + IR)^2 + (V\sin\phi + IX)^2}$$
$$= \sqrt{V^2 + 2VI(R\cos\phi + X\sin\phi) + I^2(R^2 + X^2)}.$$

When V^2 is large compared with $I^2(R^2 + X^2)$—and this will usually be the case—then $\frac{I^2}{V^2}(R^2 + X^2)$ can be neglected, and

$$E = V\left\{1 + \frac{2I}{V}(R\cos\phi + X\sin\phi)\right\}^{\frac{1}{2}}.$$

Neglecting terms containing V^2 in denominator

$$= V + I(R\cos\phi + X\sin\phi).$$

Voltage drop $= E - V = I\,(R\cos\phi + X\sin\phi)$.

Voltage drop per ampere per mile of conductor $= \frac{(E - V)}{lI}$

$$= 2\left(\frac{R}{2l}\cos\phi + \frac{X}{2l}\sin\phi\right)$$

$$= 2v$$

where $v = \begin{cases} \text{resistance per mile of conductor} \times \cos\phi \\ + \\ \text{reactance per mile of conductor} \times \sin\phi. \end{cases}$

Voltage Drop along Short 3-Phase Transmission Line.

Capacity Effects Neglected.

Let P = power delivered at receiver in watts.
V = voltage between lines at receiver.
$V/\sqrt{3}$ = phase voltage at receiver = V_{ph}.
$\cos\phi$ = power-factor of load at receiver.
E = voltage between lines at generator.
$E/\sqrt{3}$ = phase voltage at generator = E_{ph}.
R = resistance of each conductor.
X = reactance of each conductor.

Line current $I = \frac{P}{\sqrt{3} \times V \times \cos\phi}$ amperes.

The 3-phase transmission scheme is shown in Fig. 24, and the corresponding diagram for line 1, with its resistance and reactance lumped, is given in Fig. 25.

The diagram for lines 2 and 3 will be identical with that for line 1.

No current will flow in the neutral conductor with a balanced 3-phase load, and hence there will be no drop of volts in the neutral.

In practice the neutral conductor is omitted.

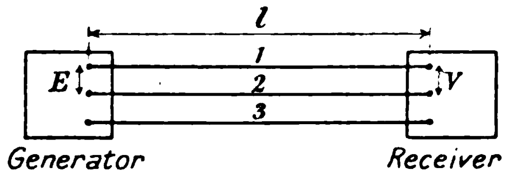


FIG. 24.—3-phase transmission line.

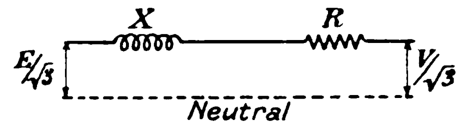


FIG. 25.—3-phase transmission line having resistance and reactance but no capacity.

The vector diagram for the circuit of Fig. 25 is given in Fig. 26.

As for single-phase case,

$$E_{ph} = \sqrt{(V_{ph}\cos\phi + IR)^2 + (V_{ph}\sin\phi + IX)^2}\,. \quad (78)$$

And, since $E = \sqrt{3}E_{ph}$, the voltage at the generator, and hence the voltage drop in the line, can be calculated.

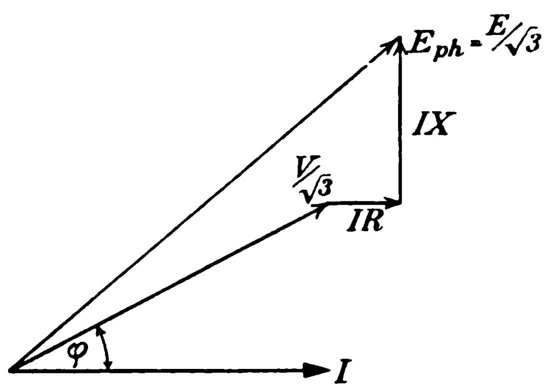


FIG. 26.—Vector diagram for 3-phase transmission line.

In the same manner as for the single-phase case, it can be proved that approximately:—

Phase voltage drop per ampere per mile of conductor $= v$, and therefore the drop as measured between lines $= \sqrt{3}v$

where $v =$ resistance per mile $\times \cos\phi +$ reactance per mile $\times \sin\phi$.

If * R = resistance *per mile* and X = reactance *per mile* of conductor

* This notation is used throughout the remainder of the book.

Then for 3-phase lines the voltage drop

$$= \sqrt{3}\{R \cos \phi + X \sin \phi\}Il \quad . \quad . \quad (79)$$

while for single-phase lines the voltage drop

$$= 2\{R \cos \phi + X \sin \phi\}Il \quad . \quad . \quad . \quad (80)$$

With the new notation (R and X being resistance and reactance *per mile*) formulæ (77) and (78) become

$$E = \sqrt{(V \cos \phi + 2IRl)^2 + (V \sin \phi + 2IXl)^2} \quad . \quad (77a)$$

$$E_{ph} = \sqrt{(V_{ph} \cos \phi + IRl)^2 + (V_{ph} \sin \phi + IXl)^2} \quad (78a)$$

l being the distance of transmission in miles.

These approximate formulæ, (79) and (80), yield results which are sufficiently accurate when applied to the majority of short transmission lines. Their chief use, perhaps, is in obtaining quickly an estimate of what the correct conductor size is to suit a given case, the final calculations being made by using formula (77*a*) or (78*a*).

Table I. gives the voltage drop per ampere-mile for various conductors and spacings commonly used for overhead power transmission, and will be found of considerable assistance when making calculations of size of conductors for overhead transmission lines.

Table II. is similar to Table I., but the conductor sizes and spacings are such as will be met with in underground cables.

Calculation of Conductor Size.

The calculation of voltage drop when the conductor size is known has been dealt with in the preceding sections. The calculation of the conductor size which will give a certain specified voltage drop is more difficult. The most convenient method when tables such as I. and II. are available is to proceed thus :—

1. Calculate the line current I.
2. Knowing the specified voltage drop $E - V$, calculate the permissible value of $v = (E - V)/2Il$ for single-phase and $(E - V)/\sqrt{3}Il$ for 3-phase lines.
3. Using Table I or II., ascertain the conductor size which gives the nearest v to that required.

4. Using this conductor size, calculate the true voltage drop by formula (77*a*) or (78*a*).
5. Adjust conductor size if necessary.

TABLE I.

VOLTAGE DROP PER AMPERE MILE AT POWER-FACTOR 0·8 LAGGING.

Overhead Conductors.

Size.	Area in sq. ins.	Resist. in ohms per mile. R.	12" Spacing.		24" Spacing.		36" Spacing.		48" Spacing.		72" Spacing.	
			Ohms per mile Reactance = X.	*v.*	X.	*v.*	X.	*v.*	X.	*v.*	X.	*v.*
7/064	0·022	1·94	0·514	1·860	0·584	1·900	0·626	1·928	0·654	1·944	0·695	1·969
7/068	0·025	1·72	0·508	1·680	0·578	1·723	0·618	1·747	0·649	1·765	0·689	1·789
7/080	0·035	1·24	0·491	1·287	0·562	1·329	0·603	1·354	0·632	1·371	0·673	1·396
7/097	0·05	0·848	0·472	0·962	0·542	1·004	0·584	1·029	0·613	1·046	0·653	1·070
19/064	0·06	0·717	0·462	0·851	0·532	0·893	0·574	0·918	0·603	0·935	0·644	0·960
19/072	0·075	0·566	0·450	0·723	0·520	0·765	0·562	0·790	0·590	0·807	0·632	0·832
19/083	0·1	0·426	0·436	0·602	0·506	0·644	0·548	0·670	0·576	0·686	0·618	0·712
37/064	0·117	0·368	0·428	0·551	0·498	0·593	0·538	0·617	0·567	0·635	0·609	0·660
37/072	0·15	0·291	0·416	0·482	0·486	0·524	0·528	0·550	0·557	0·567	0·598	0·592
37/083	0·20	0·220	0·402	0·417	0·472	0·459	0·514	0·484	0·543	0·502	0·583	0·526
37/092	0·25	0·179	0·391	0·378	0·462	0·420	0·503	0·445	0·532	0·462	0·573	0·487
37/104	0·30	0·140	0·379	0·339	0·450	0·382	0·492	0·407	0·519	0·423	0·560	0·448

X = reactance per mile of conductor at 50 ~.
R = resistance per mile of conductor.
v = volts drop per ampere-mile of conductor = R cos ϕ + X sin ϕ = 0·8R + 0·6X.

Total voltage drop $\begin{cases} \text{SINGLE-PHASE} = 2vlI \\ \text{THREE-PHASE} = \sqrt{3}vlI \end{cases}$ where l = distance of transmission in miles, I = line current.

This method is illustrated by the following example :—

Example.—It is desired to deliver 1000 kw., cos ϕ = 0·8, 3-phase, 50 cycles, at 10,000 volts to a load at a distance of

TABLE II.

Voltage Drop per Ampere Mile at Power-factor 0·8 Lagging.

Three-phase Underground Cables with Circular Conductors.

Size.	Area in sq. ins.	Resist. in ohms per mile. R.	Working Voltage 660.			Working Voltage 2200.			Working Voltage 3300.			Working Voltage 5500.			Working Voltage 6600.			Working Voltage 11,000.		
			D.	X.	*v.*	D.	X.	*v.*	D.	X.	*v.*	D.	X.	*v.*	D.	X.	*v.*	D.	X.	*v.*
7/064	0·022	1·94	0·272	0·130	1·63	0·312	0·145	1·639	0·332	0·151	1·643	0·372	0·163	1·650	0·392	0·169	1·653	0·492	0·19	1·666
19/052	0·04	1·09	0·34	0·122	0·946	0·38	0·134	0·953	0·40	0·139	0·956	0·44	0·149	0·962	0·46	0·154	0·965	0·56	0·173	0·976
19/064	0·06	0·717	0·40	0·118	0·645	0·44	0·128	0·651	0·46	0·132	0·653	0·50	0·141	0·659	0·52	0·145	0·661	0·62	0·162	0·671
19/072	0·075	0·566	0·44	0·115	0·522	0·48	0·125	0·528	0·50	0·129	0·531	0·54	0·137	0·535	0·56	0·141	0·538	0·66	0·158	0·548
19/083	0·10	0·426	0·495	0·114	0·410	0·535	0·122	0·414	0·555	0·125	0·416	0·595	0·132	0·420	0·615	0·136	0·423	0·715	0·152	0·432
37/072	0·15	0·291	0·584	0·110	0·299	0·624	0·117	0·303	0·644	0·121	0·306	0·684	0·127	0·309	0·704	0·130	0·311	0·804	0·143	0·319
37/083	0·20	0·220	0·661	0·108	0·241	0·701	0·115	0·245	0·721	0·118	0·247	0·761	0·123	0·250	0·781	0·126	0·252	0·881	0·138	0·259
37/092	0·25	0·179	0·734	0·108	0·208	0·761	0·113	0·211	0·784	0·116	0·213	0·824	0·121	0·216	0·844	0·123	0·217	0·944	0·134	0·224

D = distance in inches between centres of conductors.
X = reactance in ohms per mile of conductor.
R = resistance ,, ,, ,,
v = volts drop per ampere-mile of conductor at cos ϕ = 0·8.
= 0·8 R + 0·6 X.
Total voltage drop = $\sqrt{3}vIl$.
I = line current in amperes.
l = distance of transmission in miles.

Note Regarding Value of D.—D = $2r$ + insulation thickness between conductors. Insulation thickness between conductors for the various working voltages is taken from B.E.S.A. Specification No. 7, 1926, for Paper Insulated Cables. Refer to Fig. 19.

10 miles from the power-station. If the permissible voltage drop between load and generator is not to exceed 10 per cent. of the delivered voltage, calculate the necessary size of conductor—

(*a*) If an overhead line with conductors spaced 24 ins. apart is used.

(*b*) If an underground cable is used.

Assume that the possible conductor sizes are limited to those given in Tables I. and II.

$$\text{Line current} = \frac{1000000}{\sqrt{3} \times 10000 \times 0{\cdot}8} = 72 \text{ amperes.}$$

$$\text{Permissible voltage drop} = \frac{1}{10} \times 10000 = 1000 \text{ volts.}$$

Using the approximate formula :—

$$\text{Voltage drop} = \sqrt{3} \times v \times \text{I} \times l.$$
$$\therefore 1000 = \sqrt{3} \times v \times 72 \times 10.$$
$$\therefore v = 0{\cdot}8.$$

I. OVERHEAD LINE.

From Table I., 19/·064 conductor gives $v = 0{\cdot}893$,
19/·072 ,, ,, $v = 0{\cdot}765$.

Work out the correct voltage drops for both conductors thus :—

<u>19/·072.</u>

Resistance per mile = 0·566 ohms = R.
Reactance ,, ,, = 0·52 ,, = X.

$$\text{Generated phase volts} = \sqrt{(\text{V}_{ph}\cos\phi + l\text{IR})^2 + (\text{V}_{ph}\sin\phi + l\text{IX})^2}$$
$$= \sqrt{(4620 + 407)^2 + (3460 + 375)^2}$$
$$= 6320 \text{ volts.}$$

$$\therefore \text{Generated line volts} = \sqrt{3} \times 6320$$
$$= 11{,}000 \text{ volts.}$$

$$\text{Percentage voltage drop} = \frac{1000}{10000} \times 100$$
$$= 10 \text{ per cent.}$$

<u>19/·064.</u>

R = Resistance per mile = 0·717 ohms.
X = Reactance ,, ,, = 0·532 ,,

Generated phase volts $= \sqrt{(V_{ph}\cos\phi + lIR)^2 + (V_{ph}\sin\phi + lIX)^2}$
$= \sqrt{(4620 + 518)^2 + (3460 + 385)^2}$
$= \underline{6420 \text{ volts.}}$

∴ Generated line volts $= \sqrt{3} \times 6420$
$= 11{,}200$ volts.

Percentage voltage drop $= \frac{1200}{10000} \times 100$
$= \underline{\underline{12 \text{ per cent.}}}$

Use 19/·072 conductor.

II. Underground Cable.

As before, $v = 0{\cdot}8$.

From Table II. for 11,000 working voltage—

19/·064 conductor gives $v = 0{\cdot}671$.
19/·052 ,, ,, $v = 0{\cdot}976$.

Calculate correct voltage drops for both conductors thus:—

19/·064.

Resistance per mile = 0·717 ohms = R.
Reactance ,, ,, = 0·162 ,, = X.

Generated phase volts $= \sqrt{(V_{ph}\cos\phi + lIR)^2 + (V_{ph}\sin\phi + lIX)^2}$
$= \sqrt{(4620 + 516)^2 + (3460 + 117)^2}$
$= \underline{6250 \text{ volts.}}$

∴ Generated line volts $= 6250 \times \sqrt{3} = 10{,}820$.

Percentage voltage drop $= \frac{820}{10000} \times 100$
$= \underline{\underline{8{\cdot}2 \text{ per cent.}}}$

19/·052.

Resistance per mile = 1·09 ohms = R.
Reactance ,, ,, = 0·173 ,, = X.

Generated phase volts
$= \sqrt{(V_{ph}\cos\phi + lIR)^2 + (V_{ph}\sin\phi + lIX)^2}$
$= \sqrt{(4620 + 785)^2 + (3460 + 125)^2}$
$= \underline{6480 \text{ volts.}}$

$\therefore$ Generated line volts $= \sqrt{3} \times 6480 = 11,240.$

$$\therefore \text{ Percentage voltage drop } = \frac{1240}{10000} \times 100$$

$$= \underline{\underline{12{\cdot}4 \text{ per cent.}}}$$

Use 19/·064 conductor.

Voltage Drop along Single- or 3-Phase Transmission Line.

Capacity Effects Included.

When capacity effects are included the current varies from point to point along the transmission line, and the simple methods given can no longer be applied.

The general theory of transmission given in Chapter II. includes capacity effects and formula (41) and (42) for the voltage and current at any point along the line can be applied.

It will be convenient, however, to modify the equations slightly so as to make them incorporate the quantities which are usually stated in transmission line problems. Those quantities are :—

(*a*) The load delivered to the receiver in watts.
(*b*) The receiver voltage.
(*c*) The receiver power-factor.

The problem is to calculate the voltage and current at the generator.

If E = voltage at generator.
V = voltage at receiver.
I_G = current at generator.
I_r = current at receiver.
$A = \cosh \gamma l.$
$B = Z_0 \sinh \gamma l.$
* $\boldsymbol{C} = \frac{1}{Z_0} \sinh \gamma l$
γ = propagation constant.

* Do not confuse $\boldsymbol{C}$ with C the symbol used for Capacity.

$\gamma = \sqrt{(R + jX)(G + jwC)}$.

Z_o = characteristic impedance (or surge impedance)

$$= \sqrt{\frac{R + jX}{G + jwC}}.$$

l = distance of transmission in miles.

The equations as modified become—

$$E = VA + I_r B \quad . \quad . \quad . \quad (81)$$

$$I_G = I_r A + VC \quad . \quad . \quad . \quad (82)$$

The formulæ are applicable to both single-phase and 3-phase systems, provided the following distinctions are carefully noted.

SINGLE-PHASE.	3-PHASE.
R = resistance in ohms per mile of loop.	R = resistance in ohms per mile of conductor.
X = reactance in ohms ,, ,,	X = reactance in ohms ,, ,, ,,
C = capacity in farads ,, ,,	C = capacity in farads to neutral per mile of conductor.
G = leakance in ohms ,, ,,	G = leakance in mhos per mile of conductor.
E = voltage between lines at generator.	E = generator volts to neutral $= \frac{\text{generator line volts}}{\sqrt{3}}$.
V = voltage between lines at receiver.	V = receiver volts to neutral $= \frac{\text{receiver line volts}}{\sqrt{3}}$.

If the generator conditions are fixed and it is desired to obtain the conditions at the receiver, the corresponding equations are :—

$$V = EA - I_G B \quad . \quad . \quad . \quad (83)$$

$$I_r = I_G A - EC \quad . \quad . \quad . \quad (84)$$

Example.—A 100-mile, 50-cycle, 3-phase transmission line has its conductors placed at the corners of an equilateral triangle. The conductors have an overall diameter of 0·414 in. and are spaced 10 ft. apart.

Calculate the voltage and current at the power station if 23,000 kilowatts at 110,000 volts, 0·8 power-factor, are delivered to the load at the end of the line.

Resistance of line.—0·426 ohms per mile (from Table I., p. 78).

Primary Constants.—

$$R = 0{\cdot}426 \text{ ohms.}$$
$$G = 0 \text{ (assumed).}$$

$$C = \frac{38{\cdot}8 \times 10^{-3}}{\log_{10}\frac{D}{r}} = \frac{38{\cdot}8 \times 10^{-3}}{\log_{10}\frac{120}{0{\cdot}207}} = 14{\cdot}0 \times 10^{-3} \text{ microfarads per mile.}$$

$$X = 2\pi f\left[741 \log_{10}\frac{120}{0{\cdot}207} + 80\right] \times 10^{-6} = 0{\cdot}67 \text{ ohms per mile.}$$

Characteristic Impedance.—

$$Z_0 = \sqrt{\frac{R + jX}{G + jwC}} = \sqrt{\frac{0{\cdot}426 + j0{\cdot}67}{0 + j2\pi \times 50 \times 14 \times 10^{-9}}}$$
$$= \sqrt{\frac{0{\cdot}794\underline{/57{\cdot}5}}{4{\cdot}4 \times 10^{-6}\underline{/90}}} = \sqrt{180{\cdot}5 \times 10^{3}\underline{/-32{\cdot}5}}$$
$$= \underline{\underline{425\underline{/-16{\cdot}25}}}.$$

$$\gamma = \text{Propagation constant}$$
$$= \sqrt{(R + jX)(G + jwC)}$$
$$= \sqrt{(0{\cdot}426 + j0{\cdot}67)(0 + j4{\cdot}4 \times 10^{-6})}$$
$$= \sqrt{3{\cdot}494 \times 10^{-6}\underline{/147{\cdot}5}}$$
$$= \underline{\underline{1{\cdot}87 \times 10^{-3}\underline{/73{\cdot}75}}}.$$

$$A = \cosh \gamma l$$
$$= \cosh\,[1{\cdot}87 \times 10^{-3}\underline{/73{\cdot}75}] \times 100$$
$$= \cosh\,[0{\cdot}187\underline{/73{\cdot}75}]$$
$$= \cosh\,[0{\cdot}0523 + j0{\cdot}1795]$$
$$= \underline{\underline{0{\cdot}9854\underline{/0{\cdot}55}}}.$$

$$B = Z_0 \sinh \gamma l$$
$$= 425\underline{/-16{\cdot}25}\,\sinh 0{\cdot}187\underline{/73{\cdot}75}$$
$$= 425\underline{/-16{\cdot}25}\,\{0{\cdot}1864\underline{/73{\cdot}9}\}$$
$$= \underline{\underline{79{\cdot}22\underline{/57{\cdot}65}}}.$$

$$C = \frac{1}{Z_0} \sinh \gamma l$$

$$= \frac{1}{425\underline{/-16\cdot25}} 0\cdot1864\underline{/73\cdot9}$$

$$= 0\cdot0004386\underline{/90\cdot15}.$$

$$E = VA + I_rB.$$

$$V = \frac{110000}{\sqrt{3}} = 63{,}510.$$

$$I_r = \frac{23000000}{\sqrt{3} \times 110000 \times 0\cdot8} = 150 \text{ amperes.}$$

Taking V along the axis of reference,

$$V = 63510.$$

$$I_r = 150 \times 0\cdot8 - j150 \times 0\cdot6$$

$$= 150\underline{/-36\cdot9}.$$

$$E = 63510\{0\cdot9854\underline{/0\cdot55}\} + 150\underline{/-36\cdot9}\{79\cdot22\underline{/57\cdot65}\}$$

$$= 73690 + j4806$$

$$= 73850\underline{/3\cdot6} \text{ to neutral.}$$

$$\therefore \text{Generator line volts} = \sqrt{3} \times 73850$$

$$= 128{,}000 \text{ volts.}$$

$$I_G = I_rA + VC.$$

$$= 150\underline{/-36\cdot9}\{0\cdot9854\underline{/0\cdot55}\} + 63510\{0\cdot0004386\underline{/90\cdot15}\}$$

$$= 147\cdot8\underline{/-36\cdot35} + 27\cdot9\underline{/90\cdot15}$$

$$= 133\underline{/-26\cdot6}.$$

Hence the generator line volts = 128,000.
Generator line current = 133 amps.
Voltage drop in line = 128,000 − 110,000
= 18,000 volts.

Open Circuit Load Volts and Charging Current of Line.—Referring to equations (41) and (42),

$$V = V_0 \cosh \gamma x - I_0 Z_0 \sinh \gamma x \quad . \quad . \quad (41)$$

$$I = I_0 \cosh \gamma x - \frac{V_0}{Z_0} \sinh \gamma x \quad . \quad . \quad . \quad (42)$$

These are the general equations for the voltage and current at any point along the line, x being positive when measured from the generator to the load.

In power transmission problems it is occasionally of importance to be able to calculate the two following quantities :—

(*a*) The open-circuit voltage at the load for definite voltage at generator.

(*b*) The current at the generator with zero load current at the receiver. This will be the charging current of the line.

These two quantities can be readily obtained from equations (41) and (42), thus :—

When $x = l, \quad I = 0.$

∴ Substituting in equation (42) and using notation on p. 82—

$$0 = I_G \cosh \gamma l - \frac{E}{Z_0} \sinh \gamma l$$

$$\therefore I_G = \frac{E}{Z_0} \tanh \gamma l, \qquad (85)$$

i.e. charging current $= \dfrac{E}{Z_0} \tanh \gamma l.$

From equation (41),

$$V = E \cosh \gamma l - \left(\frac{E}{Z_0} \frac{\sinh \gamma l}{\cosh \gamma l} \sinh \gamma l\right) Z_0$$
$$= E[\cosh \gamma l - \sinh \gamma l \tanh \gamma l] \qquad (86)$$

where V = open circuit load volts corresponding to a voltage E at the generator.

In previous example the no-load receiver voltage with 128,000 line volts at the generator may be obtained thus—

$$E = 73850 \quad \text{[taking E along reference axis]}$$
$$\cosh \gamma l = 0{\cdot}9854\underline{/0{\cdot}55}$$
$$\sinh \gamma l = 0{\cdot}1864\underline{/73{\cdot}9}.$$

$$\therefore \sinh \gamma l \tanh \gamma l = \frac{(0{\cdot}1864\underline{/73{\cdot}9})(0{\cdot}1864\underline{/73{\cdot}9})}{0{\cdot}9854\underline{/0{\cdot}55}}$$
$$= 0{\cdot}0353\underline{/147{\cdot}25}.$$

$$\therefore \ V = 73850[0{\cdot}9854\underline{/0{\cdot}55} - 0{\cdot}0353\underline{/147{\cdot}25}]$$
$$= \underline{130{,}000 \text{ volts between lines.}}$$

$$\text{Charging current } = \frac{E}{Z_0}\frac{\sinh \gamma l}{\cosh \gamma l}$$
$$= \frac{73850}{425\underline{/-16{\cdot}25}} \times \frac{0{\cdot}1864\underline{/73{\cdot}9}}{0{\cdot}9854\underline{/0{\cdot}55}}$$
$$= \underline{32{\cdot}8\underline{/89{\cdot}6}},$$

i.e. charging current = 32·8 amperes.

Methods of Obtaining cosh γl and sinh γl.

Three methods are available for obtaining the values of cosh γl and sinh γl, viz. :—

(1) From Kennelly's "Tables," interpolation being used when necessary.

(2) From the identities—

$$\cosh (u + jv) = \cosh u \cos v + j \sinh u \sin v$$
and
$$\sinh (u + jv) = \sinh u \cos v + j \cosh u \sin v.$$

The numerical labour involved is very considerably reduced by using tables of log cosh u, log sinh u, log cos v, and log sin v.

(3) From the identities as in (2) above, but using for cosh u the expression $\frac{1}{2}\{\epsilon^u + \epsilon^{-u}\}$, and for sinh u the expression $\frac{1}{2}\{\epsilon^u - \epsilon^{-u}\}$. The values of ϵ^u, ϵ^{-u}, cos v and sin v are obtained from tables.

As an illustration, we will calculate the value of cosh $0{\cdot}187\underline{/73{\cdot}75^\circ}$ (taken from the example on p. 83) by each of the three methods.

I. *From Kennelly's "Tables."*

$$\cosh 0{\cdot}1\underline{/73^\circ} = 0{\cdot}99586\underline{/0{\cdot}161^\circ}$$
$$\cosh 0{\cdot}1\underline{/74^\circ} = \underline{0{\cdot}99576\underline{/0{\cdot}153^\circ}}.$$

$$\text{Difference for } 1^\circ = -\ 0.00010\underline{/0.008}.$$
$$\text{Difference for } 0.75^\circ = -\ 0.000075\underline{/0.006}.$$
$$\therefore \cosh 0.1\underline{/73.75^\circ} = 0.995785\underline{/0.156^\circ}$$
$$\cosh 0.2\underline{/73^\circ} = 0.98350\underline{/0.648^\circ}$$
$$\cosh 0.2\underline{/74^\circ} = 0.98312\underline{/0.614^\circ}.$$

$$\text{Difference for } 1^\circ = -\ 0.00038\underline{/0.034^\circ}.$$
$$\text{Difference for } 0.75^\circ = -\ 0.000285\underline{/0.0255}.$$
$$\therefore \cosh 0.2\underline{/73.75^\circ} = 0.983215\underline{/0.6225}$$
$$\cosh 0.2\underline{/73.75^\circ} = 0.983215\underline{/0.6225}$$
$$\cosh 0.1\underline{/73.75^\circ} = 0.995785\underline{/0.156}.$$

$$\text{Difference for } 0.1 = -\ 0.012570\underline{/0.4665}.$$
$$\therefore \text{Difference for } 0.087 = -\ 0.010936\underline{/0.4059}.$$
$$\therefore \cosh 0.187\underline{/73.75^\circ} = 0.984849\underline{/0.5619^\circ}$$
$$= 0.984849\underline{/0^\circ\ 33.7'}.$$

II. *From cosh (u + jv) = cosh u cos v + j sinh u sin v, and using tables of log cosh u, log sinh u, log cos v, and log sin v.*

$$\cosh 0.187\underline{/73.75^\circ} = \cosh (0.0523 + j0.1795)$$
$$\log \cosh u \cos v = \log \cosh u + \log \cos v.$$

From tables
$$\log \cosh 0.0523 = 0.000594$$
$$\log \cos 0.1795 = \bar{1}.992967.$$
$$\therefore \log \cosh u \cos v = \bar{1}.993561$$
$$\text{Antilog} = 0.98528.$$
$$\therefore \cosh u \cos v = 0.98528.$$

$$\log \sinh u \sin v = \log \sinh u + \log \sin v.$$

Tables of log sinh u may be used, but for small values of u such as will occur in practical problems interpolation is difficult, and the following procedure is advisable :—

$$\sinh u = u \frac{\sinh u}{u}.$$
$$\therefore \log \sinh u = \log u + \log \frac{\sinh u}{u}.$$

Hence $\log \sinh u \sin v = \log u + \log \frac{\sinh u}{u} + \log \sin v.$

Values of $\log \frac{\sinh u}{u}$ are available, and interpolation is much more easily carried out than with $\log \sinh u$.

When $u = 0{\cdot}0523$

$$\log u = \log 0{\cdot}0523 \quad = \bar{2}{\cdot}718502$$

$$\log \frac{\sinh u}{u} = \log \frac{\sinh 0{\cdot}0523}{0{\cdot}0523} = 0{\cdot}000198$$

$$\log \sin v = \log \sin 0{\cdot}1795 \quad = \bar{1}{\cdot}251724.$$

$$\therefore \log \sinh u \sin v = \bar{3}{\cdot}970424.$$

$$\therefore \sinh u \sin v = 0{\cdot}0093417.$$

Hence $\cosh (0{\cdot}0523 + j\,0{\cdot}1795) = 0{\cdot}98528 + j\,0{\cdot}00934$
$= 0{\cdot}98533 \underline{/0^\circ\ 32{\cdot}6'}.$

Tables of $\log \cosh u$, $\log \sinh u$, and $\log \frac{\sinh u}{u}$ are given in "Alternating Current Phenomena in Parallel Conductors," Vol. I., by F. E. Pernot; also in "Smithsonian Mathematical Tables" (*see* Method III.). Suitable tables of logarithmic sines, cosines, and tangents are given in "Chambers' Mathematical Tables," published by W. & R. Chambers.

III. *From Tables giving* ϵ^u, ϵ^{-u}, *cos v, and sin v.*

$$\cosh 0{\cdot}187 \underline{/73{\cdot}75^\circ} = \cosh (0{\cdot}0523 + j\,0{\cdot}1795)$$
$$= \cosh 0{\cdot}0523 \cos 0{\cdot}1795 + j \sinh 0{\cdot}0523 \sin 0{\cdot}1795.$$

Now $\cosh u = \frac{\epsilon^u + \epsilon^{-u}}{2}.$

Tables of ϵ^u and ϵ^{-u} are given in "Smithsonian Mathematical Tables—Hyperbolic Functions," by George F. Becker and C. E. van Orstrand. Interpolation is very simple. Thus, if $u = x + a$ where x is given in the tables and a is a small increment, then

$$\epsilon^u = \epsilon^{x+a} = \epsilon^x \epsilon^a = \epsilon^x \left[1 + a + \frac{a^2}{\lfloor 2} + \ldots \right].$$

Similarly, $\epsilon^{-x-a} = \epsilon^{-x}\epsilon^{-a} = \epsilon^{-x}\left[1 - a + \frac{a^2}{\underline{|2}} + \ldots\right]$.

When $u = 0{\cdot}0523$

$$\epsilon^u = \epsilon^{0{\cdot}0523}.$$

From "Smithsonian Tables," $\epsilon^{0{\cdot}052} = 1{\cdot}053376$.

$$\epsilon^{0{\cdot}0523} = \epsilon^{0{\cdot}052}\,\epsilon^{0{\cdot}0003} = 1{\cdot}053376\,[1 + 0{\cdot}0003 + \ldots]$$
$$= 1{\cdot}053692.$$

Also, $\epsilon^{-0{\cdot}0523} = \epsilon^{-0{\cdot}052}\,\epsilon^{-0{\cdot}0003}$.

From tables $\epsilon^{-0{\cdot}052} = 0{\cdot}9493289$.

$$\therefore\ \epsilon^{-0{\cdot}0523} = 0{\cdot}9493289\,[1 - 0{\cdot}0003]$$
$$= \underline{0{\cdot}9490441}.$$

$$\cosh u = \frac{\epsilon^u + \epsilon^{-u}}{2}.$$

$$\underline{\therefore\ \cosh 0{\cdot}0523 = 1{\cdot}001368.}$$

Similarly, $\sinh 0{\cdot}0523 = 0{\cdot}052324$.

$v = 0{\cdot}1795$ radians $= 10^\circ\ 7{\cdot}04'$.

$\left.\begin{array}{l}\cos v = 0{\cdot}98393\\ \sin v = 0{\cdot}17851\end{array}\right\}$ From suitable tables such as "Chambers' Mathematical Tables."

$$\therefore\ \cosh(0{\cdot}0523 + j\,0{\cdot}1795) = 1{\cdot}001368 \times 0{\cdot}98393 + j\,0{\cdot}052324 \times 0{\cdot}17851$$
$$= 0{\cdot}98528 + j\,0{\cdot}00934$$
$$= \underline{\underline{0{\cdot}98533/0^\circ\ 32{\cdot}6'}}.$$

Example Illustrating Method of Calculation when Great Accuracy is Required.

Consider again the example on page 83. In calculating Z_0 the slide rule was used, and, generally, in the whole of the working there was no striving after extreme accuracy. This procedure is quite satisfactory for most practical cases, but where very accurate results are required a more elaborate method such as that now to be described must be used.

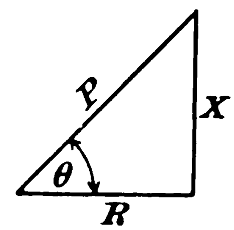
The various steps in the calculation will be illustrated by again working through the example on page 83 and using five-figure logarithms.

$$R = 0{\cdot}426. \qquad G = 0.$$
$$C = 14 \times 10^{-9}. \qquad X = 0{\cdot}67.$$

Calculation of Z_0.

$$Z_0 = \sqrt{\frac{R + jX}{G + j\omega C}}.$$

Express R + *j*X in form P/θ.



$$\tan\theta = \frac{X}{R}.$$

$$\therefore \log\tan\theta = \log X - \log R$$
$$= \bar{1}{\cdot}82607 - \bar{1}{\cdot}62941$$
$$= 0{\cdot}19666. \quad \text{Whence } \theta = 57^\circ\ 33'\ 2''.$$
$$\therefore \log\sin\theta = \bar{1}{\cdot}92627 \text{ from table of log sines, etc.}$$

Also
$$P = \frac{X}{\sin\theta}.$$
$$\therefore \log P = \log X - \log\sin\theta$$
$$= \bar{1}{\cdot}82607 - \bar{1}{\cdot}92627$$
$$= \bar{1}{\cdot}89980.$$
$$\therefore P = 0{\cdot}79396.$$

Hence
$$R + jX = 0{\cdot}79396\underline{/57^\circ\ 33'\ 2''}.$$

Express G + *j*ωC in form P/θ.

$$G + j\omega C = 0 + j\,2\pi f C.$$
$$\log 2\pi = 0{\cdot}79818$$
$$\log f = 1{\cdot}69897$$
$$\log C = \bar{8}{\cdot}14613$$
$$\bar{6}{\cdot}64328.$$
$$\text{Antilog} = 0{\cdot}0000043982$$
$$= 4{\cdot}3982 \times 10^{-6}.$$
$$\therefore G + j\omega C = 4{\cdot}3982 \times 10^{-6}\underline{/90^\circ}.$$

Hence $$Z_0 = \sqrt{\frac{0\cdot79396\underline{/57^\circ\ 33'\ 2''}}{4\cdot3982 \times 10^{-6}\underline{/90^\circ}}}$$

$$\begin{aligned} \log 0\cdot79396 &= \bar{1}\cdot89980 & \quad & 90 \\ \log 4\cdot3982 \times 10^{-6} &= \bar{6}\cdot64328 & & 57^\circ\ 33'\ \ 2'' \\ & \quad 5\cdot25652. & & 32^\circ\ 26'\ 58'' \\ \div\ 2 &= 2\cdot62826. & & -\ 16^\circ\ 13'\ 29'' \\ \text{Antilog} &= 424\cdot87. \\ \therefore Z_0 &= 424\cdot87\underline{/-\ 16^\circ\ 13'\ 29''}. \end{aligned}$$

$$\begin{aligned} \gamma &= \sqrt{(R + jX)(G + j\omega C)} \\ &= \sqrt{(0\cdot79396\underline{/57^\circ\ 33'\ 2''})(4\cdot3982 \times 10^{-6}\underline{/90^\circ})} \\ &= 1\cdot8687 \times 10^{-3}\underline{/73^\circ\ 46'\ 31''}. \end{aligned}$$

Obtain cosh γl in form cosh $(a + jb)$.

Since $l = 100$

$$\begin{aligned} \cosh \gamma l &= \cosh 0\cdot18687\underline{/73^\circ\ 46'\ 31''} \\ &= \cosh [0\cdot18687 \cos 73^\circ\ 46'\ 31'' \\ &\qquad\qquad + j\,0\cdot18687 \sin 73^\circ\ 46'\ 31''] \end{aligned}$$

$$\begin{aligned} \log 0\cdot18687 &= \bar{1}\cdot27154 & \quad \log 0\cdot18687 &= \bar{1}\cdot27154 \\ \log \cos 73^\circ\ 46'\ 31'' &= \bar{1}\cdot44623 & \log \sin 73^\circ\ 46'\ 31'' &= \bar{1}\cdot98235 \\ & \quad \bar{2}\cdot71777 & & \quad \bar{1}\cdot25389 \\ \text{Antilog} &= 0\cdot05221. & \text{Antilog} &= 0\cdot17943. \end{aligned}$$

$$\therefore \cosh \gamma l = \cosh [0\cdot05221 + j\,0\cdot17943].$$

Obtain cosh γl in form P$\underline{/\theta}$.—Using the notation on page 26, viz., $\gamma = \beta + j\alpha$,

$$\begin{aligned} \cosh \gamma l &= \cosh [\beta l + j\alpha l]. \\ \therefore \beta l &= 0\cdot05221 \\ \alpha l &= 0\cdot17943. \end{aligned}$$

Now, $\cosh [\beta l + j\alpha l] = \cosh \beta l \cos \alpha l + j \sinh \beta l \sin \alpha l$.

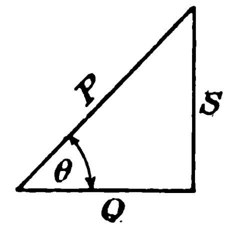
$$\begin{aligned} \log \cosh \beta l &= \log \cosh 0\cdot05221 &&= 0\cdot000592 \text{ (from tables)}. \\ \log \sinh \beta l &= \log \sinh 0\cdot05221 &&= \bar{2}\cdot71795 \text{ (from tables)}. \\ \log \sin \alpha l &= \log \sin 0\cdot17943 &&= \log \sin 10^\circ\ 16'\ 50'' \\ & &&= \bar{1}\cdot25156. \\ \log \cos \alpha l &= \log \cos 10^\circ\ 16'\ 50'' &&= \bar{1}\cdot99297. \end{aligned}$$

Let $Q = \cosh \beta l \cos \alpha l.$

Then $\log Q = \bar{1}\cdot 99356.$

Let $S = \sinh \beta l \sin \alpha l.$

Then $\log S = \bar{3}\cdot 96951$



$$\tan \theta = \frac{S}{Q}.$$

$$\therefore \log \tan \theta = \log S - \log Q$$
$$= \bar{3}\cdot 97595.$$
$$\therefore \theta = 0° \; 32' \; 31''$$

and $\log \cos \theta = \bar{1}\cdot 99998.$

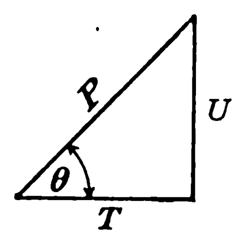
Now $$P = \frac{Q}{\cos \theta}.$$

$$\therefore \log P = \log Q - \log \cos \theta$$
$$= \bar{1}\cdot 99358.$$
$$\therefore P = 0\cdot 98533.$$

Hence $A = \cosh \gamma l$
$$= \underline{0\cdot 98533/\underline{0° \; 32' \; 31''}}.$$

Obtain sinh γl in form $P/\underline{\theta}$.

Let $T = \sinh \beta l \cos \alpha l.$
$U = \cosh \beta l \sin \alpha l.$
$\log T = \bar{2}\cdot 71092.$
$\log U = \bar{1}\cdot 25215.$



Since $$\tan \theta = \frac{U}{T},$$

$$\log \tan \theta = 0\cdot 54123.$$
$$\therefore \theta = 73° \; 57' \; 19''$$
$$\log \sin \theta = \bar{1}\cdot 98275.$$

Since $$P = \frac{U}{\sin \theta},$$

$$\log P = \bar{1}\cdot 26940$$
$$P = 0\cdot 18595.$$
$$\therefore \sinh \gamma l = \underline{0\cdot 18595/\underline{73° \; 57' \; 19''}}.$$

$$B = Z_0 \sinh \gamma l$$
$$= 424{\cdot}87\underline{/-16^\circ\ 13'\ 29''} \times 0{\cdot}18595\underline{/73^\circ\ 57'\ 19''}.$$

$\log 424{\cdot}87 = 2{\cdot}62826$ $\qquad 73^\circ\ 57'\ 19''$

$\log 0{\cdot}18595 = \bar{1}{\cdot}26940$ $\qquad 16^\circ\ 13'\ 29''$

$1{\cdot}89766.$ $\qquad 57^\circ\ 43'\ 50''$

Antilog $= 79{\cdot}006.$

$\therefore B = 79{\cdot}006\underline{/57^\circ\ 43'\ 50''}.$

$$C = \frac{1}{Z_0} \sinh \gamma l$$
$$= \frac{1}{424{\cdot}87\underline{/-16^\circ\ 13'\ 29''}} \times 0{\cdot}18595\underline{/73^\circ\ 57'\ 19''}$$
$$= 0{\cdot}00043767\underline{/90^\circ\ 10'\ 48''}.$$

$$E = VA + I_r B$$
$$= \frac{110000}{\sqrt{3}} \times 0{\cdot}98533\underline{/0^\circ\ 32'\ 31''} + 150\underline{/-36^\circ\ 52'\ 14''} \times 79{\cdot}006\underline{/57^\circ\ 43'\ 50''}.$$

VA in form $P\underline{/\theta}$.

$\log 110000 = 5{\cdot}04139$

$\log 0{\cdot}98533 = \bar{1}{\cdot}99358$

$5{\cdot}03497$

$\log \sqrt{3} = 0{\cdot}23858$

$4{\cdot}79639$

Antilog $= 62573 = P.$

$\therefore VA = 62573\underline{/0^\circ\ 32'\ 31''}.$

$I_r B$ in form $P\underline{/\theta}$.

$\log 150 = 2{\cdot}17609$

$\log 79{\cdot}006 = 1{\cdot}89766$

$4{\cdot}07375$

Antilog $= 11851 = P.$

$-36^\circ\ 52'\ 14'' + 57^\circ\ 43'\ 50''$
$= 20^\circ\ 51'\ 36'' = \theta.$

$\therefore I_r B = 11851\underline{/20^\circ\ 51'\ 36''}.$

VA in form $a + jb$.

$a = 62573 \cos 0^\circ\ 32'\ 31''.$

$\log 62573 = 4{\cdot}79639$

$\log \cos 0^\circ\ 32'\ 31'' = \bar{1}{\cdot}99998$

$4{\cdot}79637$

Antilog $= 62570 = a.$

$I_r B$ in form $a + jb$.

$a = 11851 \cos 20^\circ\ 51'\ 36''.$

$\log 11851 = 4{\cdot}07375$

$\log \cos 20^\circ\ 51'\ 36'' = \bar{1}{\cdot}97056$

$4{\cdot}04431$

Antilog $= 11074 = a.$

$b = 62573 \sin 0^\circ 32' 31''$.

$\log 62573 = 4{\cdot}79639$

$\log \sin 0^\circ 32' 31'' = \bar{3}{\cdot}97583$

$2{\cdot}77222$

Antilog $= 591{\cdot}86 = b$.

$\therefore \mathrm{VA} = 62570 + j\,591{\cdot}86$.

$b = 11851 \sin 20^\circ 51' 36''$.

$\log 11851 = 4{\cdot}07375$

$\log \sin 20^\circ 51' 36'' = \bar{1}{\cdot}55155$

$3{\cdot}62530$

Antilog $= 4219{\cdot}9 = b$.

$\therefore \mathrm{I_rB} = 11074 + j\,4219{\cdot}9$.

$$\mathrm{E} = \mathrm{VA} + \mathrm{I_rB} = 62570 + j\,591{\cdot}86 + 11074 + j\,4219{\cdot}9$$
$$= 73644 + j\,4812.$$

$\mathrm{I_a} = \mathrm{I_rA} + \mathrm{V}C$

$$= 150\angle{-36^\circ 52' 14''} \times 0{\cdot}98533\angle{0^\circ 32' 31''} + \frac{110000}{\sqrt{3}} \times 0{\cdot}00043767\angle{90^\circ 10' 48''}.$$

I_rA in form $P\angle\theta$.

$\log 150 = 2{\cdot}17609$

$\log 0{\cdot}98533 = \bar{1}{\cdot}99358$

$2{\cdot}16967$

Antilog $= 147{\cdot}8 = \mathrm{P}$.

$-36^\circ 52' 14'' + 0^\circ 32' 31''$

$= -36^\circ 18' 43'' = \theta$.

$\therefore \mathrm{I_rA} = 147{\cdot}8\angle{-36^\circ 18' 43''}$.

VC in form $P\angle\theta$.

$\log 110000 = 5{\cdot}04139$

$\log 0{\cdot}00043767 = \bar{4}{\cdot}64115$

$1{\cdot}68254$

$\log \sqrt{3} = 0{\cdot}23858$

$1{\cdot}44396$

Antilog $= 27{\cdot}794 = \mathrm{P}$.

$\therefore \mathrm{V}C = 27{\cdot}794\angle{90^\circ 10' 48''}$.

I_rA in form $a + jb$.

$a = 147{\cdot}8 \cos 36^\circ 18' 43''$.

$\log 147{\cdot}8 = 2{\cdot}16967$

$\log \cos 36^\circ 18' 43'' = \bar{1}{\cdot}90623$

$2{\cdot}07590$

Antilog $= 119{\cdot}1 = a$.

$b = -147{\cdot}8 \sin 36^\circ 18' 43''$.

$\log 147{\cdot}8 = 2{\cdot}16967$

$\log \sin 36^\circ 18' 43'' = \bar{1}{\cdot}77245$

$1{\cdot}94212$

VC in form $a + jb$.

$a = 27{\cdot}794 \cos 90^\circ 10' 48''$

$= -27{\cdot}794 \sin 0^\circ 10' 48''$.

$\log 27{\cdot}794 = 1{\cdot}44396$

$\log \sin 0^\circ 10' 48'' = \bar{3}{\cdot}49715$

$\bar{2}{\cdot}94111$

Antilog $= 0{\cdot}087319 = -a$.

$b = 27{\cdot}794 \sin 90^\circ 10' 48''$

$= 27{\cdot}794 \cos 0^\circ 10' 48''$.

$\log 27{\cdot}794 = 1{\cdot}44396$

$\log \cos 0^\circ 10' 48'' = 0$

THREE-PHASE OVERHEAD TRANSMISSION LINES.

TABLE OF PARTICULARS USED IN EXAMPLES SOLVED BY EQUATIONS (81) AND (82).

Conductor Particulars.	Length of Line.	Frequency.	Primary Constants.	Z_0 and γ.	(1) cosh γl. (2) sinh γl.	Constants A, B, and C in Equations (81) and (82).	Where Used.
Copper, 19/0·083, 0·414 in. diam., Area = 0·1 sq. in., Spacing = 10 ft.	100 miles	50 ~	$R = 0{\cdot}426$ $L = 2130 \times 10^{-6}$ $C = 14 \times 10^{-9}$ $G = 0$	$Z_0 = 425/\underline{-16^\circ 15'}$ $\gamma = 1{\cdot}87 \times 10^{-3}/\underline{73^\circ 45'}$	(1) $0{\cdot}9854/\underline{0^\circ 33'}$ (2) $0{\cdot}1864/\underline{73^\circ 54'}$	$A = 0{\cdot}9854/\underline{0^\circ 33'}$ $B = 79{\cdot}22/\underline{57^\circ 39'}$ $C = 0{\cdot}0004386/\underline{90^\circ 9'}$	Example in text page 83, and Example 12, page 100.
Copper, 19/·083. As above.	50 miles	50 ~	As above.	As above.	(1) $0{\cdot}9963/\underline{0^\circ 8'}$ (2) $0{\cdot}0935/\underline{73^\circ 52'}$	$A = 0{\cdot}9963/\underline{0^\circ 8'}$ $B = 39{\cdot}74/\underline{57^\circ 37'}$ $C = 0{\cdot}00022/\underline{90^\circ 7'}$	Example 12, page 100.
Copper, 37/0·064, 0·448 in. diam., Area=0·117sq.in., Spacing = 15 ft.	200 miles	50 ~	$R = 0{\cdot}368$ $L = 2236 \times 10^{-6}$ $C = 13{\cdot}3 \times 10^{-9}$ $G = 0$	$Z_0 = 437/\underline{-13^\circ 47'}$ $\gamma = 1{\cdot}82 \times 10^{-3}/\underline{76^\circ 13'}$	(1) $0{\cdot}9420/\underline{1^\circ 50'}$ (2) $0{\cdot}3574/\underline{76^\circ 48'}$	$A = 0{\cdot}9420/\underline{1^\circ 50'}$ $B = 156/\underline{63^\circ 1'}$ $C = 0{\cdot}000819/\underline{90^\circ 35'}$	Example 11, page 100.
Copper, 37/0·072, 0·504 in. diam., Area=0·15 sq. in., Spacing = 10 ft.	200 miles	50 ~	$R = 0{\cdot}291$ $L = 2065 \times 10^{-6}$ $C = 14{\cdot}5 \times 10^{-9}$ $G = 0$	$Z_0 = 396/\underline{-12^\circ 4'}$ $\gamma = 1{\cdot}8 \times 10^{-3}/\underline{77^\circ 56'}$	(1) $0{\cdot}9414/\underline{1^\circ 35'}$ (2) $0{\cdot}3534/\underline{78^\circ 27'}$	$A = 0{\cdot}9414/\underline{1^\circ 35'}$ $B = 139{\cdot}95/\underline{66^\circ 23'}$ $C = 0{\cdot}000893/\underline{90^\circ 31'}$	Example 5, page 125.

Antilog $= 87{\cdot}52 = b$.	Antilog $= 27{\cdot}794 = b$.
$\therefore I_rA = 119{\cdot}1 - j\,87{\cdot}52$.	$\therefore VC = -\,0{\cdot}087 + j\,27{\cdot}794$.

$$I_o = I_rA + VC = 119{\cdot}1 - j\,87{\cdot}52 - 0{\cdot}087 + j\,27{\cdot}794$$
$$= 119 - j\,59{\cdot}73.$$

E in form $P/\underline{\theta}$.	I_o *in form* $P/\underline{\theta}$.
$E = 73644 + j\,4812$.	$I_o = 119 - j\,59{\cdot}73$.
$\tan\theta = \dfrac{4812}{73644}$	
$\log\tan\theta = \log 4812 - \log 73644$	
$\log 4812 = 3{\cdot}68233$	$\log 59{\cdot}73 = 1{\cdot}77619$
$\log 73644 = 4{\cdot}86714$	$\log 119 = 2{\cdot}07555$
$\therefore \log\tan\theta = \bar{2}{\cdot}81519$	$\therefore \log\tan\theta = \bar{1}{\cdot}70064$
$\theta = 3^\circ\,44'\,20''$.	$\theta = -\,26^\circ\,39'\,0''$.
$\log\cos\theta = \bar{1}{\cdot}99907$	$\log\cos\theta = \bar{1}{\cdot}95122$
$P = \dfrac{73644}{\cos\theta}$	$P = \dfrac{119}{\cos\theta}$
$\log 73644 = 4{\cdot}86714$	$\log 119 = 2{\cdot}07555$
$\log\cos\theta = \bar{1}{\cdot}99907$	$\log\cos\theta = \bar{1}{\cdot}95122$
$\log P = 4{\cdot}86807$	$\log P = 2{\cdot}12433$
$P = 73802$	$P = 133{\cdot}1$
$\therefore E = 73802/\underline{3^\circ\,44'\,20''}$.	$\therefore I_o = 133{\cdot}1/\underline{-\,26^\circ\,39'\,0''}$.

EXAMPLES ON CHAPTER V.

1. A 3-phase overhead transmission line has conductors spaced with $D_1 = 3$ ft., $D_2 = 4$ ft., $D_3 = 5$ ft., referring to Fig. 17. If the radius of each conductor is 0·096 in. and the frequency is 50 cycles per second, calculate—

   (*a*) The equivalent equilateral spacing.
   (*b*) The reactance per mile of conductor.
   (*c*) The capacity to neutral per mile of conductor.

   Answers { (*a*) 3·914 ft. (*b*) 0·65 ohms. (*c*) 0·0145 microfarads.

2. A 3-phase underground cable has conductors spaced so that, referring to Fig. 19,

$r = 0{\cdot}18$ in. $\quad D = 0{\cdot}5$ in. $\quad R = 0{\cdot}61$ in. $\quad K = 3$.

Calculate—

(*a*) The reactance per mile of conductor at a frequency of 50 cycles per second.

(*b*) The capacity to neutral per mile of conductor.

Answers { (*a*) 0·129 ohms.
(*b*) 0·417 microfarads.

3. Show that when the fall of voltage due to resistance and reactance is small compared with the line voltage, the fall of voltage along a 3-phase transmission line per ampere per mile is given by the expression $\sqrt{3}\{R \cos \phi + X \sin \phi\}$ where R is the resistance per mile of conductor, X is the reactance per mile of conductor, and $\cos \phi$ the power-factor of the load.

Find the fall of voltage along a 3-phase transmission line, the line pressure at the load being 30,000 volts, and the length of the line being 30 miles, when 5000 k.v.a. are delivered at a power-factor of 0·8, the current lagging. The resistance and reactance per mile are 0·72 *w* and 0·6 *w* respectively.

Answer : 4680 by formula (80), 4700 by formula (79). [London University, 1925.]

4. Prove that in a symmetrically arranged 3-phase transmission line the inductive drop and resistance drop between any two wires is the same as that which would occur if half the total power were transmitted at the same voltage and frequency along two of the wires.

An overhead 3-phase line consists of three wires each 0·8 in. in diameter spaced 4 ft. apart. The current flowing per wire is 300 amps. at 50 cycles. Calculate the resistance and inductive drops per mile of line ; the specific resistance of copper is 0·67 microhms per inch cube.

Answer : RI = 43·9 volts, XI = 264·5 volts. [London University, 1921.]

5. 1000 kw. of 3-phase power are to be delivered over a distance of 20 miles to a star-connected receiving circuit, the power-factor of which is 0·85. The voltage between lines at the receiving end is 20,000 and the frequency is 50. What must be the voltage at the transmitting end if the resistance per mile of each line conductor is 1·5 ohms and its inductance per mile is 0·0015 henry.

Answer : 21,810 volts. [London University, 1918.]

6. It is required to deliver 250 k.w. at 6600 volts 0·8 power-factor, 50 cycles per second, to a load which is situated 10 miles from the source of supply. The permissible voltage drop being 5 per cent. of the delivered voltage, calculate the necessary conductor size—

   (*a*) If an overhead line is used with conductors spaced 2 ft. apart.

   (*b*) If an underground cable is used.

   Assume that the choice of conductor size is limited to those given in Tables I. and II.

   Answers { (*a*) *Overhead Line.*—19/·083 conductor gives 4·1 per cent. drop.
   19/·072 conductor gives 5·3 per cent. drop.
   Hence 19/·083 conductor must be used.
   (*b*) *Cable.*—19/·064 conductor gives 4·6 per cent. drop and should be used.

7. An overhead transmission line has primary constants per mile of $R = 0{\cdot}5$ ohms, $C = 10 \times 10^{-9}$ farads, $G = 0$, and an inductance such that the reactance X at 50 cycles per second is 0·6 ohms. Calculate for this frequency :—

   (*a*) The characteristic impedance Z_0.

   (*b*) The propagation constant γ.

   Slide rule accuracy only is required.

   Answers { (*a*) $Z_0 = 498\underline{/- 19{\cdot}4^\circ}$.
   (*b*) $\gamma = 1{\cdot}57 \times 10^{-3}\underline{/70{\cdot}1^\circ}$.

8. An overhead transmission line has the following primary constants per mile: $R = 0{\cdot}291$, $C = 14{\cdot}5 \times 10^{-9}$, $G = 0$, and inductance such that the reactance X at a frequency of 50 cycles per second is 0·649 ohms. Calculate for this frequency :—

   (*a*) The characteristic impedance Z_0.

   (*b*) The propagation constant γ.

   Employ method given on pages 91 and 92, and use five-figure logarithms.

   Answers { (*a*) $Z_0 = 395{\cdot}14\underline{/- 12^\circ\ 4'\ 30''}$.
   (*b*) $\gamma = 1{\cdot}8 \times 10^{-3}\underline{/77^\circ\ 55'\ 30''}$.

   Note.—Compare answers with values given in table on page 96.

9. An overhead transmission line has a propagation constant

$$\gamma = 2 \times 10^{-3}\underline{/60^\circ}.$$

If the length of the line l is 100 miles, calculate cosh γl and sinh γl, having given $\epsilon^{0\cdot1} = 1\cdot105$ and $\epsilon^{-0\cdot1} = 0\cdot905$. Slide rule accuracy only is required.

$$\textit{Answer}\begin{cases}\cosh \gamma l = 0\cdot9899 + j\,0\cdot0172 = 0\cdot99/\underline{1^\circ} \\ \sinh \gamma l = 0\cdot0985 + j\,0\cdot1733 = 0\cdot2/\underline{60\cdot37^\circ.}\end{cases}$$

10. An overhead transmission line has a propagation constant

$$\gamma = 1\cdot8 \times 10^{-3}/\underline{77^\circ\ 56'.}$$

If the length of the line l is 200 miles, calculate cosh γl and sinh γl.

Employ method 2 on page 87, and use five-figure logarithms and tables of log cosh x and log sinh x.

$$\textit{Answer}\begin{cases}\cosh \gamma l = 0\cdot94178/\underline{1^\circ\ 34'\ 48''} \\ \sinh \gamma l = 0\cdot35293/\underline{78^\circ\ 26'\ 20''.}\end{cases}$$

Note.—Compare answers with values given in table on page 96.

11. A 3-phase overhead transmission line supplies 100 amperes at 200 kilovolts, 50 cycles to a load, the power-factor being 0·8 lagging. The line is 200 miles long and has a resistance per mile of 0·368 ohms, reactance 0·705 ohms, capacity of $13\cdot3 \times 10^{-9}$ farads and negligible leakance. Calculate the voltage and current at the generator.

$$\textit{Answers}\begin{cases}\text{Voltage at generator} = 123{,}150/\underline{4^\circ\ 50'.} \\ \text{Current} \quad ,, \qquad ,, \quad = 86\cdot3/\underline{27^\circ\ 59'.}\end{cases}$$

12. Taking transmission line particulars as given in the example on page 83, calculate :—

(*a*) The voltage and current at the generator when delivering 28,000 k.w. to the load at 0·8 power-factor, 100,000 volts, 50 cycles.

(*b*) The voltage and current halfway along the line when delivering this load of 28,000 k.w.

$$\textit{Answers :}\ (a)\begin{cases}\text{Voltage} = 71{,}950/\underline{4^\circ\ 55'.} \\ \text{Current} = 183/\underline{-\ 29^\circ\ 57'}\ \text{amperes.}\end{cases}$$

$$(b)\begin{cases}\text{Voltage} = 64{,}990/\underline{2^\circ\ 36'.} \\ \text{Current} = 192/\underline{-\ 33^\circ\ 44'}\ \text{amperes.}\end{cases}$$

Note.—Voltage and current are given in the answers as phase quantities, the load conditions being :—

Voltage = 57,700.
Current = $200/\underline{-\ 36^\circ\ 52'.}$

CHAPTER VI.

EFFECT OF TRANSFORMERS ON VOLTAGE DROP IN TRANSMISSION CIRCUIT.

MOST transmission schemes include step-up and step-down transformers, and it is of considerable practical importance to be able to calculate the voltage drop with the transformers included in the line.

Consider the following simple single-phase transmission scheme as illustrated in Fig. 27.

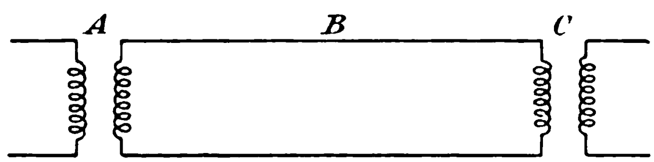


FIG. 27.—Single-phase transmission scheme consisting of transmission line and step-up and step-down transformer.

A is a step-up transformer installed, say, in a power-station.

B is the transmission line.

C is the step-down transformer installed, say, in a sub-station.

The problem is to calculate the total drop in volts from the low-tension terminals at A to the low-tension terminals at C The drop as calculated will include the drop in volts in the transformers A and C as well as in the line B.

Equivalent Network of Transformer.—The calculation of the voltage drop in the transformer is most readily performed by using an equivalent network to replace each transformer. It is proved in text-books on the subject that any transformer can be replaced by a network of suitably arranged resistances and reactances.

Thus a single-phase transformer with a voltage V_1 applied to the primary, and a voltage V_2 at the secondary terminals, can be replaced by either of the simple networks shown in Figs. 28 and 29. As will be seen later, these simple networks do not exactly reproduce the transformer conditions, but they are sufficiently accurate for most purposes.

The following symbols are employed in Figs. 28 and 29 :—

r_1 = primary resistance. r_2 = secondary resistance.
x_1 = primary reactance. x_2 = secondary reactance.
I_1 = primary current. V_1 = primary applied volts.
I_2 = secondary current. V_2 = secondary voltage.
a = ratio of transformation = primary turns/secondary turns.

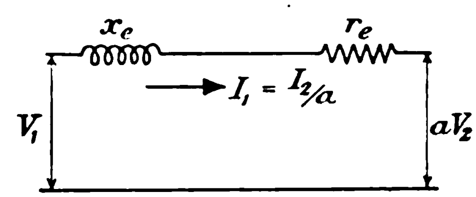


FIG. 28.—Equivalent approximate network of transformer. All quantities referred to primary side.

$x_e = x_1 + a^2x_2 \qquad r_e = r_1 + a^2r_2$

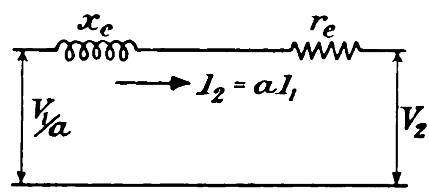


FIG. 29.—Equivalent approximate network of transformer. All quantities referred to secondary side.

$x_e = \frac{x_1}{a^2} + x_2. \qquad r_e = \frac{r_1}{a^2} + r_2.$

The primary winding is, of course, that winding to which power is supplied, and it should be noted that the ratio of transformation a will be a whole number or a fraction, depending upon whether the primary or the secondary is the high-tension winding.

In Fig. 28 everything is referred to the primary winding, and when using this network the current flowing is the actual primary current.

x_e = equivalent reactance of transformer referred to primary winding.

r_e = equivalent resistance of transformer referred to primary winding.

In Fig. 29 everything is referred to the secondary winding, and when using this network the current flowing is the actual secondary current.

x_e = equivalent reactance of transformer referred to secondary winding.

r_e = equivalent resistance of transformer referred to secondary winding.

The calculation of the total voltage drop in A, B, and C is now simple. It is convenient to express all the transformer quantities in terms of the high-tension winding, hence, since at the power-station A the primary winding is the low-tension

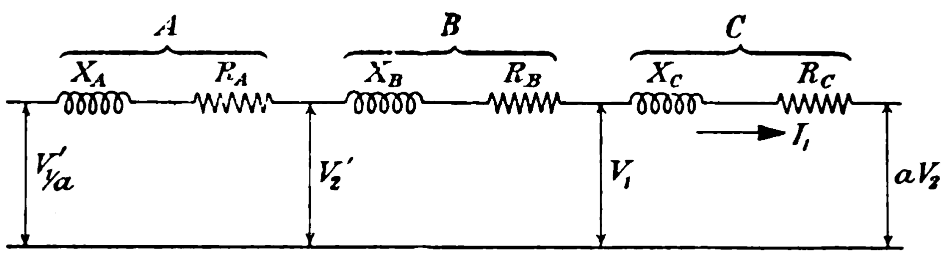


FIG. 30.—Equivalent circuit for single-phase transmission scheme.
X_A = equivalent reactance of transformer A referred to secondary.
R_A = equivalent resistance of transformer A referred to secondary.
X_B = reactance of transmission line B.
R_B = resistance of transmission line B.
X_C = equivalent reactance of transformer C referred to primary.
R_C = equivalent resistance of transformer C referred to primary.

winding, we replace the transformer at A by a network where everything is referred to the secondary side (Fig. 29), while at the sub-station C, the primary winding being the high-tension winding, we replace transformer C by a network where everything is referred to the primary side (Fig. 28). The equivalent circuit for the transmission scheme of Fig. 27 is therefore as indicated in Fig. 30.

The vector diagram for this circuit is given in Fig. 31.

It is clear that in order to calculate the voltage drop, including the transformers, all that is necessary is to add the equivalent resistances and reactances of transformers at A and C expressed in terms of the high-tension winding, to the resistance and reactance of the transmission line, and the problem can be solved from formula (77) where for R is substituted ($R_A + R_B + R_C$), and for X is substituted ($X_A + X_B + X_C$). The current to be used is the high-tension current throughout, and the resultant vector OA is the voltage at the low-tension side of transformer at A divided by the ratio of transformation a. Since at A the primary winding is the low-tension winding, this ratio will be a fraction.

Example.—A single-phase transmission scheme consists of an overhead line and step-up and step-down transformers at A and C as in Fig. 27. The load delivered at the low-tension side at C is 300 k.v.a. at 2300 volt 0·8 power-factor lagging. The high-voltage side at C is directly connected to the overhead

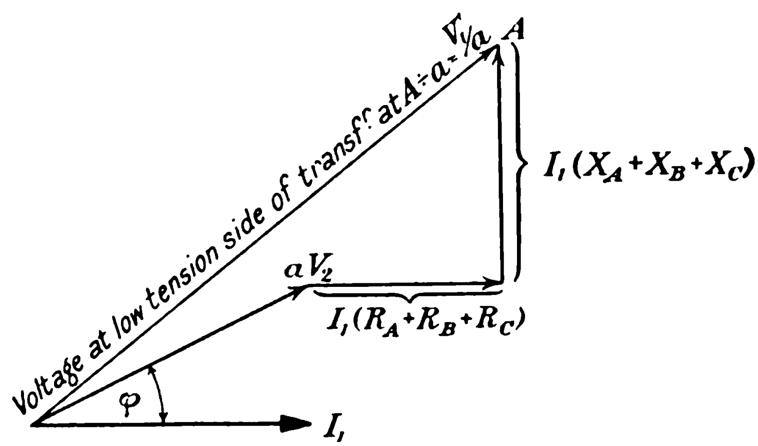


FIG. 31.—Vector diagram for single-phase transmission scheme.

line, the ratio of transformation being 4·79. The step-up transformer at A is identical with that at C. The total resistance of the overhead line is 10 ohms, and its reactance 30 ohms. The resistance of the low-voltage winding of each transformer is 0·05 ohms, and its reactance 0·162 ohms. The resistance of the high-tension winding is 1·28 ohms, and its reactance 4·28 ohms. Calculate the voltage on the low-tension side of the power-station transformer at A.

Current delivered at low-tension side of transformer at C

$$= 300000/2300 = 130 \text{ amperes.}$$

At C the high-tension side is the primary and the low-tension side is the secondary. Referring transformer to high-tension (i.e. primary winding) the network is

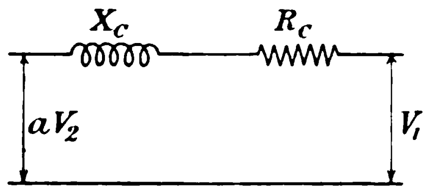


FIG. 32.

$$
\begin{aligned}
X_c &= x_1 + a^2 x_2 \\
&= 4{\cdot}28 + (4{\cdot}79)^2 \times 0{\cdot}162 \\
&= 4{\cdot}28 + 3{\cdot}73 \\
&= \underline{8{\cdot}01 \text{ ohms.}}
\end{aligned}
$$

$$R_c = r_1 + a^2 r_2 = 1{\cdot}28 + (4{\cdot}79)^2 \times 0{\cdot}05 = 2{\cdot}43 \text{ ohms.}$$

At A the low-tension side is the primary and the high-tension side is the secondary.

Referring everything to the high-tension, i.e. to the secondary winding, the network is

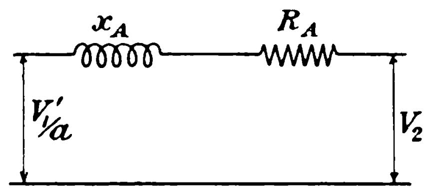


FIG. 33.

$$x_A = \frac{x_1}{a^2} + x_2 = \frac{0{\cdot}162}{\left(\frac{1}{4{\cdot}79}\right)^2} + 4{\cdot}28 = 8{\cdot}01 \text{ ohms.}$$

$$R_A = 2{\cdot}43 \text{ ohms.}$$

Referring to Fig. 31,

$$aV_2 = 4{\cdot}79 \times 2300 = 11{,}000,$$

$$I_1 = \frac{I_2}{a} = \frac{130}{4{\cdot}79} = 27 \text{ amperes.}$$

$$I_1\{R_A + R_B + R_C\} = 27\{2{\cdot}4 + 10 + 2{\cdot}4\} = 400.$$

$$I_1\{X_A + X_B + X_C\} = 27\{8 + 30 + 8\} = 1240.$$

$$
\begin{aligned}
OA &= \sqrt{(11000 \times 0{\cdot}6 + 1240)^2 + (11000 \times 0{\cdot}8 + 400)^2} \\
&= \underline{12{,}100 \text{ volts.}}
\end{aligned}
$$

Voltage at primary (i.e. low-tension side) of transformer at

$$A = V_1 \text{ and } \frac{V_1}{a} = 12{,}100.$$

$$a = \frac{1}{4{\cdot}79} \quad \therefore\ 4{\cdot}79V_1 = 12{,}100.$$

$$\underline{V_1 = 2530 \text{ volts.}}$$

Effect of Transformers when Magnetising Current is Included.

As indicated in a previous section, the circuits of Figs. 28 and 29 do not represent with complete accuracy the equivalent network of a transformer. The error arises from the fact that they do not allow for the magnetising current flowing in the primary winding. This magnetising current may be divided into two components, viz., a purely wattless current which supplies the magnetising ampere turns for the core, and a purely wattful current required to supply the iron losses in the core.

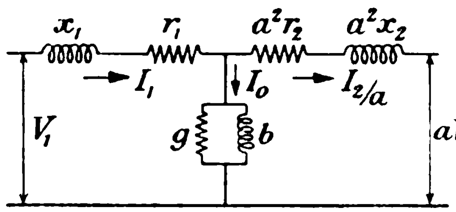


FIG. 34.—Accurate equivalent network of transformer.

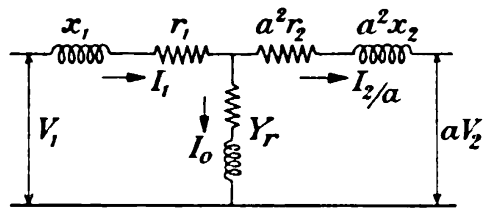


FIG. 35.—Accurate equivalent network of transformer.

An accurate network is represented in Fig. 34, where everything is expressed in terms of the primary winding. In this network g is a pure non-inductive resistance which takes a current equal to the wattful component of the magnetising current, while b is a pure inductance which takes a current equal to the wattless component of the magnetising current. Fig. 35 reproduces Fig. 34, excepting that g and b in parallel have been replaced by a single equivalent admittance Y_r.

Whilst the approximate methods already given are sufficiently accurate for most purposes, it is interesting to calculate the voltage drop in a transmission scheme, including transformers represented by the accurate network of Fig. 34, instead of the approximate ones of Figs. 28 and 29. The method which will be given is due to Messrs. R. D. Evans and H. K. Sels,* and is instructive as an application of the principles of Chapter I. for the solution of alternating current problems by the methods of vector algebra, as well as important in the theory of the transmission of electric power. In this method the transformer primary and secondary impedances are assumed to be equal when expressed in terms of either the high-tension or low-tension winding. The error introduced by this assumption is small, and the resulting equations embody transformer quantities which are readily obtained by simple tests, and are therefore usually stated by transformer manufacturers.

When currents and voltages are expressed in vector quantities in the manner described in Chapter I., the equations for the solution of an alternating current network can be written as if the network were traversed by direct current. Thus, in Fig. 36, where everything is expressed in terms of the high-tension winding:

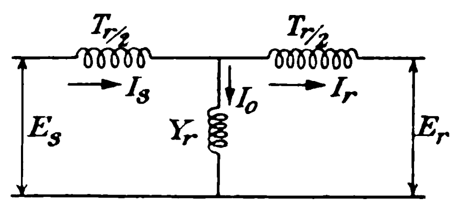


FIG. 36.—Equivalent network of transformer when primary and secondary impedances are equal.

Let T_r = equivalent impedance of transformer.

Y_r = admittance which takes current equal to magnetising current.

E_s = voltage at sending end.

E_r = voltage at receiver.

Then
$$E_s = E_r + I_s\frac{T_r}{2} + I_r\frac{T_r}{2}$$
$$= E_r + \frac{T_r}{2}[I_s + I_r].$$

* *Electric Journal*, Vol. 18, p. 306.

Also
$$I_o = \left\{E_s - I_s\frac{T_r}{2}\right\}Y_r$$
$$= \left\{E_r + \frac{T_r}{2}(I_s + I_r) - I_s\frac{T_r}{2}\right\}Y_r$$
$$= \left(E_r + I_r\frac{T_r}{2}\right)Y_r.$$

Applying Kirchoff's laws—
$$I_s = I_r + I_o$$
$$= I_r + E_rY_r + \frac{T_rI_rY_r}{2}$$
$$= I_r\left[1 + \frac{Y_rT_r}{2}\right] + E_rY_r$$
$$= I_rD + E_rC \quad . \quad . \quad . \quad . \quad (87)$$

Where $C = Y_r$ and $D = \left[1 + \frac{T_rY_r}{2}\right]$

$$E_s = E_r + \frac{T_r}{2}[I_s + I_r]$$
$$= E_r + \frac{T_r}{2}\left[I_r\left\{1 + \frac{T_rY_r}{2}\right\} + E_rY_r + I_r\right]$$
$$= E_r + \frac{T_r}{2}\left[I_r\left\{1 + \frac{T_rY_r}{2} + 1\right\} + E_rY_r\right]$$
$$= E_r + \frac{I_rT_r}{2}\left[2 + \frac{T_rY_r}{2}\right] + E_rT_r\frac{Y_r}{2}$$
$$= E_r\left[1 + \frac{Y_rT_r}{2}\right] + I_rT_r\left[1 + \frac{Y_rT_r}{4}\right]$$
$$= E_rA + I_rB \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Where $A = 1 + \frac{Y_rT_r}{2}$ $\quad B = T_r\left\{1 + \frac{Y_rT_r}{4}\right\}.$

MEASUREMENT OF T_r, Y_r, r_e, AND x_e.

These values can be readily determined for any transformer by two simple tests, viz. :—

1. An open-circuit test.
2. A short-circuit test.

The open-circuit test will give the value of Y_r.

The short-circuit test will give the values of T_r, x_e, and r_e.

Open-circuit Test.—The diagram of connections for this is given in Fig. 37.

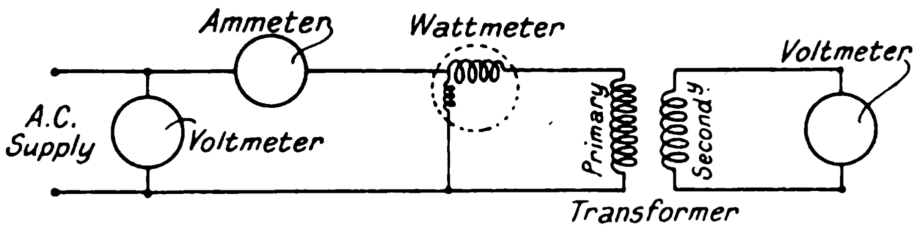


FIG. 37.—Open-circuit test of transformer.

The normal voltage at normal frequency is applied to the primary circuit. The secondary is on open-circuit. The input amperes and input watts are measured.

On open-circuit the transformer will, since it is working at full voltage, have the normal flux in its core. The full iron loss will therefore be present and will be measured by the wattmeter. The current flowing will be the no-load current of the transformer, and this will always be very small compared with the normal full load current. Under these circumstances the

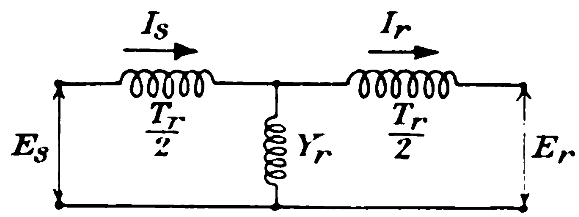


FIG. 38.—Equivalent network of transformer in open-circuit.

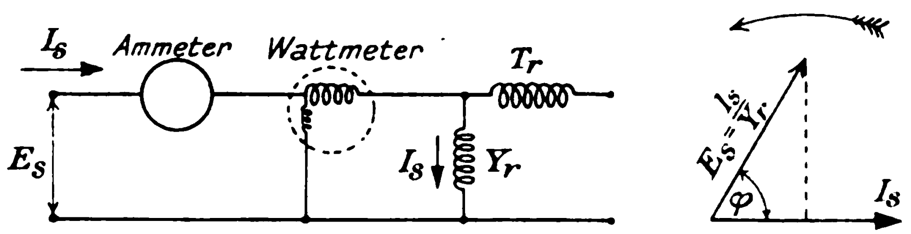


FIG. 39.—Equivalent network and vector diagram of single-phase transformer on open-circuit.

voltage drop in the primary impedance $\frac{T_r}{2}$ (Fig. 38) will be very small, and the current input I_s will be practically unaltered if

Y_r is represented as being joined across the primary terminals as in Fig. 39.

Obviously $I_s = E_s Y_r$, i.e. $Y_r = \frac{I_s}{E_s}$ . . . . . (89)

If the wattmeter reading is W, then the power-factor of the input current I_s is

$$\cos \phi = W/E_s I_s \qquad (90)$$

In this way the vector Y_r is determined both in magnitude and direction.

For *three-phase transformers* E_s must be "voltage to neutral," and thus is equal to the line voltage during the test divided by $\sqrt{3}$, and Y_r will be the admittance per phase.

Transformer manufacturers give the information necessary for the determination of Y_r in various ways. Sometimes it is given in terms of "magnetising watts" and "magnetising K.V.A." Sometimes the magnetising watts are called "no-load watts" or "core loss," while the magnetising K.V.A. may be called "no-load K.V.A." The magnetising (or no-load) watts are the watts input during the open-circuit test, i.e. they are the core loss at normal primary voltage. The magnetising (or no-load) K.V.A. is the K.V.A. input to the transformer during the open-circuit test. The manner in which Y_r can be determined when these quantities are given will be explained by an example.

Example.—A 3-phase, 1000 K.V.A., 6600 volt transformer has a magnetising K.V.A. of 10 per cent. and magnetising watts of 20,000. Calculate Y_r.

Under open-circuit conditions input to transformer = 100 K.V.A. The input to any 3-phase transformer is $\sqrt{3}E_L I_L$ volt-amperes,

where E_L = line volts,
I_L = line amperes.

$$\therefore \sqrt{3} \times 6600 \times I_L = 100{,}000.$$

$$\underline{I_L = 8{\cdot}7 \text{ amperes.}}$$

[NOTE.—When in 3-phase problems a voltage is specified (as 6600 in this problem) it always means *line* voltage unless the contrary is definitely indicated.]

Watts input to any 3-phase transformer

$$= \sqrt{3}E_L I_L \cos\phi = (\text{K.V.A.}) \times \cos\phi,$$

where $\cos\phi$ is the power-factor.

Watts input under open-circuit conditions = 20,000.

$$\therefore 20{,}000 = 100{,}000 \cos\phi$$
$$\cos\phi = 0{\cdot}2.$$

Thus the no-load current is 8·7 amperes lagging behind the voltage by an angle ϕ, where $\cos\phi = 0{\cdot}2$.

$$\therefore Y_r = \frac{I_s}{E_s} = \frac{8{\cdot}7\underline{/-\phi}}{6600/\sqrt{3}} = 0{\cdot}00227\underline{/-78^\circ\ 28'} \text{ mhos per phase.}$$

Short-circuit Test.—The diagram of connections for this test is given in Fig. 40.

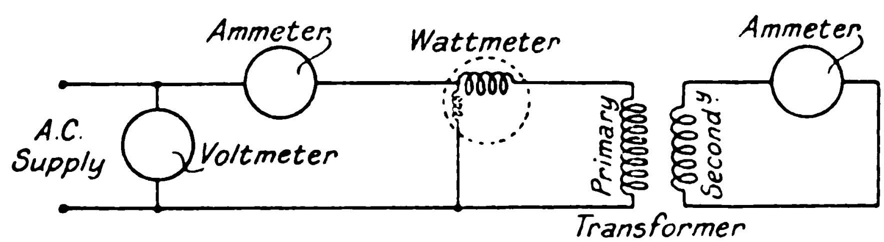


FIG. 40.—Short-circuit test of transformer.

The secondary is short-circuited on itself, and it is therefore necessary for the primary volts to be reduced to a small fraction of their normal value, otherwise the currents flowing in primary and secondary windings would be excessive. Under these circumstances the flux in the core is also a small fraction of its normal value, and the iron loss and magnetising current will therefore be very small. The circuit of the transformer may now be represented with a high degree of accuracy by Fig. 41.

The shunt admittance Y_r is omitted. It was only included in the general diagram to take account of the magnetising current and the iron loss, and these are so small during short-circuit as to be negligible.

Referring to Fig. 41,

$$\text{Input current} = I_s = I_r.$$

$$\therefore E_s = I_s T_r$$

$$T_r = \frac{E_s}{I_s} \quad . \quad . \quad . \quad (91)$$

i.e. for a single-phase transformer the equivalent impedance is equal to the input voltage divided by the input current during short-circuit. This impedance is referred to that side of the transformer which was used as primary during the short-circuit test.

If P is the power input as measured by the wattmeter W,

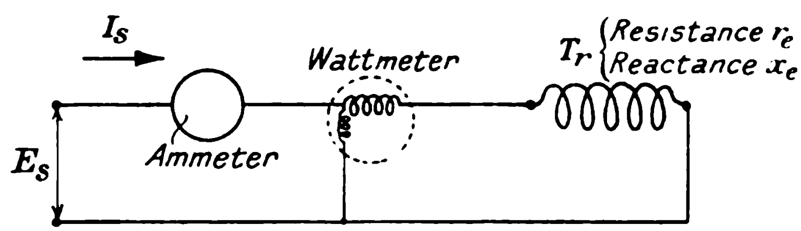


FIG. 41.—Equivalent network of single-phase transformer on short-circuit.

then this input is all expended as resistance losses, i.e. "copper losses" in the transformer winding.

$$\therefore P = I_s^2 r_e \text{ and } r_e = \frac{P}{I_s^2} \quad . \quad . \quad (92)$$

where r_e is the equivalent resistance of the transformer referred to the side used as primary during the test.

The equivalent reactance x_e is obtained from

$$x_e = \sqrt{T_r^2 - r_e^2} \quad . \quad . \quad . \quad (93)$$

Thus by measuring the input volts, amperes, and watts during the short-circuit test, we obtain T_r in the form

$$T_r = r_e + jx_e.$$

For *three-phase* transformers the following modifications are necessary.

During short-circuit—

let P = total input power in watts.

E_L = voltage between any two lines.

I_L = line current.

Then since line voltages must be made " voltages to neutral,"

$$E_s = \frac{E_L}{\sqrt{3}}$$

$$I_s = I_L.$$

$$\therefore T_r = \frac{E_L}{\sqrt{3} I_L} \text{ ohms per phase} \qquad (94)$$

Also, since the total copper loss is divided equally between the three phases, the loss per phase is $\frac{P}{3}$ watts.

$$\therefore \frac{P}{3} = I_L^2 r_e$$

$$r_e = \frac{P}{3 I_L^2} \text{ ohms per phase} \qquad (95)$$

Transformer manufacturers usually give the information necessary for the determination of T_r in the form of percentage impedance drop and full load copper loss.

Percentage Impedance Voltage Drop at any specified output is the ratio of the internal impedance voltage drop at that output to the no-load secondary voltage with the rated voltage on the primary, the result being expressed as a percentage.

The copper loss is also frequently given as a percentage thus :—

Full load copper loss per cent.

$$= \frac{\text{Copper loss in K.W. at full output}}{\text{Full load output in K.V.A.}} \times 100.$$

It is obvious that both the full load impedance drop and the full load copper loss can be obtained from the short-circuit test by adjusting I_s to be the normal full load current of the transformer.

Example.—A 100 K.V.A., 440 volt, 3-phase transformer has a no-load transformation ratio of 3300 to 440 volts. The full load impedance drop is 6 per cent., and the copper loss 2 per cent. Calculate r_e and x_e.

8

Referred to 440 volt side,

$$\text{output current} = \frac{100000}{\sqrt{3} \times 440} = 130 \text{ amperes.}$$

This calculation assumes that the voltage on the secondary side is the same at full load as at no load. This is incorrect, since the voltage changes with the load, but the error introduced is usually very small, and it is customary to neglect it in commercial transformer practice.

$$\text{Impedance drop at full load} = 0{\cdot}06 \times \frac{440}{\sqrt{3}} = 15{\cdot}2 \text{ volts.}$$

∴ Equivalent impedance T_r referred to low-tension side

$$= \frac{15{\cdot}2}{130} = \underline{0{\cdot}117 \text{ ohms per phase.}}$$

Copper loss at full load $= 0{\cdot}02 \times 100{,}000 = 2000$ watts.

∴ $3 \times 130^2 \times r_e = 2000.$

$$\underline{r_e = 0{\cdot}04 \text{ ohms per phase.}}$$

$$\therefore x_e = \sqrt{0{\cdot}117^2 - 0{\cdot}04^2} = \underline{0{\cdot}109 \text{ ohms per phase.}}$$

∴ Referred to low-tension side,

$$T_r = 0{\cdot}04 + j\,0{\cdot}109 \text{ ohms per phase.}$$

Referred to high-tension side,

$$T_r = (0{\cdot}04 + j\,0{\cdot}109) \times \left(\frac{3300}{440}\right)^2$$

$$= \underline{2{\cdot}21 + j\,6{\cdot}1 \text{ ohms per phase.}}$$

Example.—A 3-phase transformer having an open-circuited transformation ratio of 11,000 to 110,000 is operated from 11,000 volt mains. It gave the following results on open-circuit and short-circuit tests :—

Open-circuit.—Primary volts = 11,000 ; primary amperes = 75 ; watts = 285,000.

Short-circuit.—Primary volts = 170 ; primary amperes = 500 ; watts = 33,000.

Calculate the values of Y_r and T_r for this transformer.

Refer all quantities to primary side.

From *Open-circuit* test, $I_s = E_s Y_r$.

$$\therefore Y_r = \frac{75 \times \sqrt{3}}{11000} = 0{\cdot}0118 \text{ mhos.}$$

$$\cos\alpha = \frac{285000}{\sqrt{3} \times 11000 \times 75} = 0{\cdot}2,$$

where $\cos\alpha$ = open-circuit power-factor.

$$\therefore \alpha = -78{\cdot}5^\circ.$$

$$\therefore Y_r = 0{\cdot}0118 \underline{/-78{\cdot}5^\circ}$$

$$= \underline{0{\cdot}00236 - j\,0{\cdot}01158.}$$

From *Short-circuit* test, $I_s = \frac{E_s}{T_r}$.

$$\therefore T_r = \frac{170}{\sqrt{3} \times 500} = 0{\cdot}197 \text{ ohms.}$$

$$\text{Resistance loss} = 33{,}000 = 3 \times 500^2 \times r_e.$$

$$\therefore \underline{r_e = 0{\cdot}044 \text{ ohms.}}$$

$$x_e = \sqrt{0{\cdot}197^2 - 0{\cdot}044^2}$$

$$= \underline{0{\cdot}192 \text{ ohms.}}$$

$\therefore$ Referred to low-tension side, $\begin{cases} Y_r = 0{\cdot}00236 - j\,0{\cdot}01158. \\ T_r = 0{\cdot}044 + j\,0{\cdot}192. \end{cases}$

Referred to high-tension side, $\begin{cases} Y_r = 0{\cdot}0000236 - j\,0{\cdot}0001158. \\ T_r = 4{\cdot}4 + j\,19{\cdot}2. \end{cases}$

Calculation of Voltage Drop along Short Transmission Line Including Transformers Represented by Accurate Network.

Applying equation (88), the voltage E_s at the high-tension side of the transformer at sub-station C (Fig. 27) can be obtained for a given value of E_r, the voltage on the low-tension side and I_r the low-voltage current.

Also, the current in the line I_s can be calculated from equation (87). Knowing the line current and the voltage at the receiver end of the line, the voltage at the generator end of the line can be obtained from equation (77).

This gives the voltage and current at the high-tension side of the transformer at A, and an application of equations (88) and (87) will now give the voltage and current at the low-tension terminals of the transformer at A.

Example—A 3-phase transmission scheme consists of an overhead line with step-up and step-down transformers at A and C (see Fig. below). The load delivered to the low-tension side at C is 1000 K.W. at 6000 volts, 0·8 power-factor lagging. The high-tension side at C is connected to the overhead line. Ratio of transformation, 10. The step-up transformer at A is identical with that at C. Test data for both transformers are as follows:—

Open-circuit.—Primary volts = 6000; primary amps. = 9; power = 9300 watts.

Short-circuit.—Primary volts = 360; primary amps. = 120; power = 8150 watts.

The total resistance of the overhead line B is 10 ohms per conductor, and its reactance is 30 ohms per conductor.

Calculate the voltage and current on the low-tension side of the transformer at A. Represent the transformers by the accurate network of Fig. 36.

Current delivered on low-tension side of transformer at C

$$= \frac{1000000}{\sqrt{3} \times 6000 \times 0{\cdot}8} = 120 \text{ amps.}$$

STEP-DOWN TRANSFORMER AT C.

Referring all transformer quantities to low-tension side.

From *Open-circuit* test—

$$Y_r = \frac{9\sqrt{3}}{6000} = 0{\cdot}0026 \text{ mhos.}$$

$$\cos \alpha = \frac{9300}{\sqrt{3} \times 6000 \times 9} = 0{\cdot}1. \qquad \therefore \alpha = 84{\cdot}2^\circ.$$

$$\therefore Y_r = 0{\cdot}0026 \,\underline{/ - 84{\cdot}2^\circ.}$$

$$= \underline{0{\cdot}00026 - j\,0{\cdot}00258.}$$

From *Short-circuit* test—

$$T_r = \frac{360}{\sqrt{3} \times 120} = 1{\cdot}732 \text{ ohms.}$$

$$\text{Resistance loss} = 3 \times 120^2 \times r_e = 8150.$$

$$\therefore\ \underline{r_e = 0{\cdot}189 \text{ ohms.}}$$

$$x_e = \sqrt{1{\cdot}732^2 - 0{\cdot}189^2}$$
$$= \underline{1{\cdot}72 \text{ ohms.}}$$

$$\therefore\ \underline{T_r = 0{\cdot}189 + j1{\cdot}72.}$$

Referred to high-tension side,

$$\begin{cases} Y_r = 0{\cdot}0000026 - j0{\cdot}0000258. \\ T_r = 18{\cdot}9 + j172. \end{cases}$$

Referring all quantities to high-tension side,

$$E_r = \frac{60000}{\sqrt{3}} = 34700.$$

$$I_r = 12\,\underline{/-\phi} = 12\,\underline{/-36{\cdot}9^\circ}$$
$$= \underline{9{\cdot}6 - j7{\cdot}2.}$$

$$Y_rT_r = (0{\cdot}0000026 - j0{\cdot}0000258)(18{\cdot}9 + j172)$$
$$= \underline{0{\cdot}00449 - j0{\cdot}000041.}$$

$$\therefore A = 1 + \frac{Y_rT_r}{2} = 1{\cdot}00224 - j0{\cdot}00002.$$

$$B = T_r\left\{1 + \frac{Y_rT_r}{4}\right\} = (18{\cdot}9 + j172)(1{\cdot}00112 - j0{\cdot}00001)$$
$$= \underline{18{\cdot}92 + j172{\cdot}19.}$$

$$C = Y_r = 0{\cdot}0000026 - j0{\cdot}0000258.$$

$$D = A = 1{\cdot}00224 - j0{\cdot}00002.$$

$$I_s = I_rD + E_rC.$$
$$= (9{\cdot}6 - j7{\cdot}2)(1{\cdot}00224 - j0{\cdot}00002) + 34700(0{\cdot}0000026 - j0{\cdot}0000258)$$
$$= 9{\cdot}71 - j8{\cdot}12$$
$$= \underline{12{\cdot}7\,/-39{\cdot}9^\circ.}$$

$$
\begin{aligned}
E_s &= E_r A + I_r B \\
&= 34700\,[1{\cdot}00224 - j0{\cdot}00002] + (9{\cdot}6 - j7{\cdot}2)(18{\cdot}92 + j172{\cdot}19) \\
&= 36200 + j1516 \\
&= \underline{36220 \angle 2{\cdot}4^\circ.}
\end{aligned}
$$

The vector diagram for the circuit is given in Fig. 42.

Transmission Line.

A B C

R = 10 X = 30

$E_s = 36220 \angle 2{\cdot}4^\circ$
$I_s = 12{\cdot}7 \angle -39{\cdot}9^\circ$

Applying the formula for a short transmission line,

Voltage at H.T. transformer terminals at A = E_A.

$$
\begin{aligned}
E_A &= \sqrt{(E_s \cos 42{\cdot}3 + IR)^2 + (E_s \sin 42{\cdot}3 + IX)^2} \\
&= \sqrt{26920^2 + 24760^2} \\
&= \underline{36{,}570 \text{ volts.}}
\end{aligned}
$$

$$
\tan \alpha = \frac{24760}{26920}. \qquad \alpha = 42{\cdot}6^\circ.
$$

So that, referred to E_r as reference axis,

$$
\begin{aligned}
E_A &= 36570 \angle 2{\cdot}7^\circ. \\
I_A &= 12{\cdot}7 \angle -39{\cdot}9^\circ.
\end{aligned}
$$

Step-up Transformer at A.

For the transformer at A, referring everything to high-tension side,

$$
\begin{aligned}
I_r &= 12{\cdot}7 \angle -39{\cdot}9^\circ = 9{\cdot}74 - j8{\cdot}15. \\
E_r &= 36570 \angle 2{\cdot}7^\circ = 36530 + j1720. \\
A &= 1{\cdot}00224 - j0{\cdot}00002. \\
B &= 18{\cdot}92 + j172{\cdot}19. \\
C &= 0{\cdot}0000026 - j0{\cdot}0000258. \\
D &= 1{\cdot}00224 - j0{\cdot}00002.
\end{aligned}
$$

$$
\begin{aligned}
E_s &= E_r A + I_r B \\
&= (36530 + j1720)(1{\cdot}00224 - j0{\cdot}00002) \\
&\qquad + (9{\cdot}74 - j8{\cdot}15)(18{\cdot}92 + j172{\cdot}19) \\
&= 38200 + j3240 \\
&= 38340 \angle 4{\cdot}8^\circ.
\end{aligned}
$$

$$\begin{aligned} I_s &= I_r D + E_r C \\ &= (9\cdot74 - j8\cdot15)(1\cdot00224 - j0\cdot00002) \\ &\qquad + (36530 + j1720)(0\cdot0000026 - j0\cdot0000258) \\ &= 9\cdot86 - j9\cdot11 \\ &= 13\cdot4 \underline{/- 42\cdot7^\circ}. \end{aligned}$$

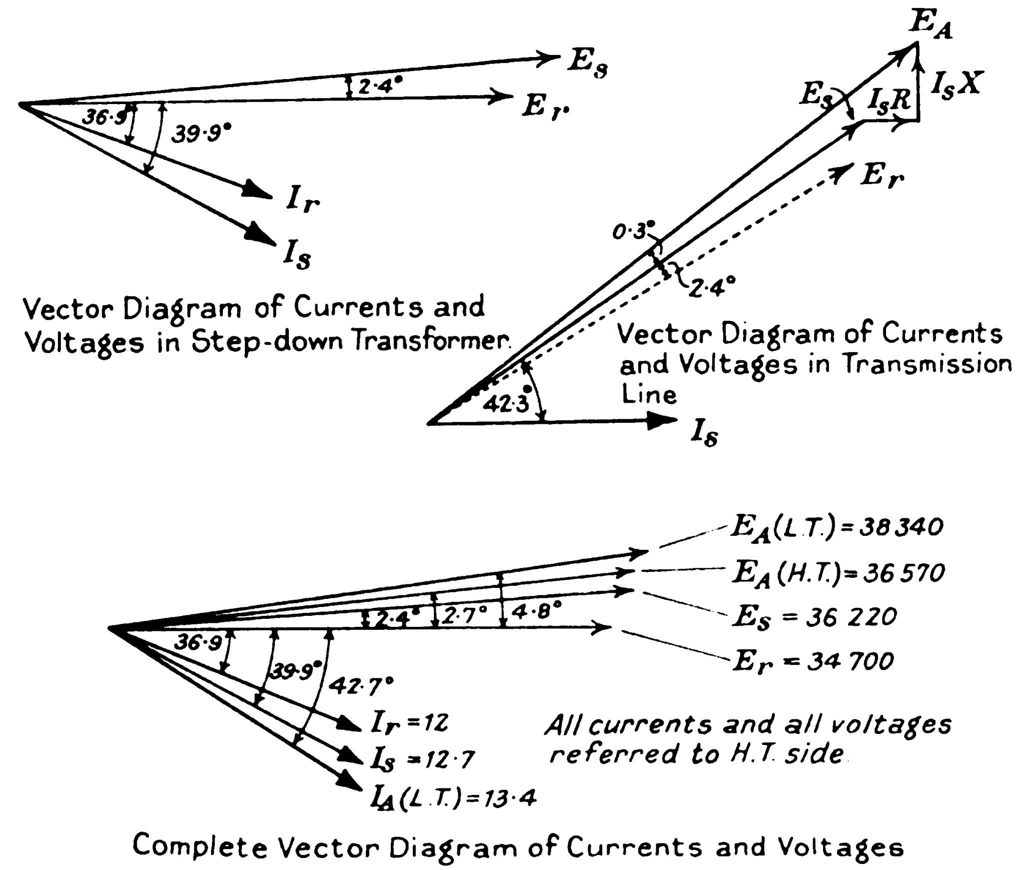


FIG. 42.—Vector diagrams for transmission scheme, consisting of short overhead line with step-up and step-down transformers.

Therefore { voltage on low-tension side of transformer at A $= 3834\sqrt{3} = 6630.$
current on low-tension side of transformer at A $= 134$ amperes.

Vector diagrams are given in Fig. 42.

Calculation of Voltage Drop in Long Transmission Line and Including Transformers Represented by Accurate Network.

The important result follows from equations (87) and (88) that the primary and secondary current and voltages of a transformer are connected by the following equations :—

$$E_s = E_r A + I_r B$$
$$I_s = I_r D + E_r C.$$

These equations are of the same form as equations (81) and (82), which connect the generator and receiver voltages and currents of a long transmission line.

A general solution to the problem of a long transmission circuit including step-up and step-down transformers can now be made.

Starting at the low-tension terminals of the transformer at C (Fig. 27), the voltage and current at the high-tension terminals B can be obtained from the equations,

$$\left.\begin{aligned} E_2 &= A_1 E_r + B_1 I_r \\ I_2 &= C_1 E_r + D_1 I_r \end{aligned}\right\}$$ where A_1, B_1, C_1, and D_1 are the circuit constants for the transformer at C.

The high-tension terminals are connected to the transmission line, and the voltage and current at the generator end of the transmission line can be obtained from—

$$\left.\begin{aligned} E_3 &= A_2 E_2 + B_2 I_2 \\ I_3 &= C_2 E_2 + D_2 I_2 \end{aligned}\right\}$$ where A_2, B_2, C_2, and D_2 are the circuit constants for the transmission line,

and D_2 will equal A_2.

This brings us to the high-tension side of the transformer at A, and the voltage and current at the low-tension terminals can be obtained from—

$$\left.\begin{aligned} E_s &= A_3 E_3 + B_3 I_3 \\ I_s &= C_3 E_3 + D_3 I_3 \end{aligned}\right\}$$ where A_3 B_3 C_3, and D_3 are the circuit constants for the transformer at A.

The networks representing the transformer C at the load and the transmission circuits may be replaced by a single equivalent network with constants A_4, B_4, C_4, and D_4 by eliminating E_2 and I_2 from the first four equations given above. We can then pass direct from the low-tension transformer terminals at C to the high-tension transformer terminals at A by means of equations,

$$E_3 = E_r A_4 + I_r B_4,$$
$$I_3 = E_r C_4 + I_r D_4.$$

Similarly, if we wish to pass direct from the low-tension terminals of the transformer at C to the low-tension terminals of the transformer at A, we can do so by obtaining constants from equations above such that

$$E_s = E_r A_o + I_r B_o \quad . \quad . \quad . \quad (96)$$
$$I_s = E_r C_o + I_r D_o \quad . \quad . \quad . \quad (97)$$

It can be readily proved that the constants A_0, B_0, C_0, and D_0 have the following values—

$$A_o = A_3(A_1A_2 + C_1B_2) + B_3(A_1C_2 + C_1D_2).$$
$$B_o = A_3(B_1A_2 + D_1B_2) + B_3(B_1C_2 + D_1D_2).$$
$$C_o = C_3(A_1A_2 + B_2C_1) + D_3(A_1C_2 + C_1D_2).$$
$$D_o = C_3(B_1A_2 + D_1B_2) + D_3(B_1C_2 + D_1D_2).$$

It is important to remember that all of the above circuit constants are complex quantities and must be dealt with by the methods described in Chapter I.

It is clear, then, that knowing the load conditions on one end of such a transmission system as that indicated in Fig. 27 we can calculate the conditions at the generator end by means of the general circuit constants A_o, B_o, C_o, and D_o, and equations (96) and (97) above.

If the generator conditions are fixed, the load conditions may be determined by means of the general circuit constants A_o, B_o, C_o, and D_o and the following equation :—

$$E_r = A_o E_s - B_o I_r \quad . \quad . \quad . \quad (98)$$
$$I_r = - C_o E_s + D_o I_r \quad . \quad . \quad . \quad (99)$$

Application to Three-phase Circuits.—The theory has been worked out for the case of single-phase transmission circuits, but the results apply equally well to 3-phase circuits, provided that the following modifications are made.

All voltages must be voltages to neutral, and all transformer resistances and reactances must be resistances and reactances to neutral. The following example illustrates the application of the principles discussed to a 3-phase system.

Example.—A 3-phase, 50-cycle transmission circuit consists of a transmission line with a step-up transformer at one end and a step-down transformer at the other. The two transformers are identical 3-phase 11,000 to 110,000 volt units, and their particulars are given in the example on page 115.

The transmission line is as given on page 83, i.e. 100 miles of 19/0·083 conductor.

If 10,000 kw. at 0·8 power-factor are delivered to the load on the low tension of the step-down transformer at a voltage of 11,000, calculate the voltage and current on the low-tension side of the step-up transformer.

Load Particulars.—

$$\text{Load current} = \frac{10\,000\,000}{\sqrt{3} \times 11000 \times 0{\cdot}8} = 656 \text{ amperes.}$$

Referred to high-tension side, load current = 65·6 amperes.

$$\therefore E_r = 110000/\sqrt{3} = 63510.$$

$$I_r = 65{\cdot}6\underline{/-\phi} = 52{\cdot}48 - j39{\cdot}36.$$

Transformer Particulars (from p. 115).—

$$Y_r = 0{\cdot}0000236 - j0{\cdot}0001158.$$

$$T_r = 4{\cdot}4 + j19{\cdot}2.$$

$$\therefore Y_rT_r = 0{\cdot}002327 - j0{\cdot}0000564.$$

$$A = 1 + \frac{Y_rT_r}{2} = 1{\cdot}00116 - j0{\cdot}0000282.$$

$$B = T_r\left\{1 + \frac{Y_rT_r}{4}\right\} = 4{\cdot}4028 + j19{\cdot}2111.$$

$$C = Y_r = 0{\cdot}0000236 - j0{\cdot}0001158.$$

$$D = A = 1{\cdot}00116 - j0{\cdot}0000282.$$

I. Step-down Transformer.

$$\begin{aligned} I_2 &= I_rD + E_rC \\ &= (52{\cdot}48 - j39{\cdot}36)(1{\cdot}00116 - j0{\cdot}0000282) \\ &\qquad + 63{,}510\,(1{\cdot}00116 - j0{\cdot}0000282) \\ &= 54{\cdot}04 - j46{\cdot}76 \\ &= 71{\cdot}5\underline{/-40^\circ\,52'}. \end{aligned}$$

$$\begin{aligned} E_2 &= E_rA + I_rB \\ &= 63{,}510\,(1{\cdot}00116 - j0{\cdot}0000282) \\ &\qquad + (52{\cdot}48 - j39{\cdot}36)(4{\cdot}4028 + j19{\cdot}2111) \\ &= 64571 + j833 \\ &= 64580\underline{/0^\circ\,44'}. \quad \text{(See Fig. 43.)} \end{aligned}$$

II. Transmission Line.

Particulars from pages 91 to 94.

$$V = 64571 + j833 = 64580\underline{/0^\circ\,44'}.$$

$$I_r = 54{\cdot}04 - j46{\cdot}76 = 71{\cdot}5\underline{/-40^\circ\,52'}.$$

$$\begin{aligned} E_3 &= VA + I_rB. \quad \text{From equation (81), page 83.} \\ &= 64580\underline{/0^\circ\,44'}\{0{\cdot}9853\underline{/0^\circ\,33'}\} + 71{\cdot}5\underline{/-40^\circ\,52'}\{79\underline{/57^\circ\,44'}\} \\ &= 63636\underline{/1^\circ\,17'} + 5649\underline{/16^\circ\,52'} \\ &= 69030 + j3060 \\ &= \underline{\underline{69090/2^\circ\,32'}}. \end{aligned}$$

$$\begin{aligned} I_3 &= I_rA + VC. \quad \text{From equation (82), page 83.} \\ &= 71{\cdot}5\underline{/-40^\circ\,52'}\{0{\cdot}9853\underline{/0^\circ\,33'}\} \\ &\qquad + 64580\underline{/0^\circ\,44'}\{0{\cdot}0004377\underline{/90^\circ\,11'}\} \\ &= 70{\cdot}45\,\underline{/-40^\circ\,19'} + 28{\cdot}26\underline{/90^\circ\,55'} \\ &= 53{\cdot}27 - j17{\cdot}32 \\ &= 56\underline{/-18^\circ}. \quad \text{(See Fig. 43.)} \end{aligned}$$

III. Step-up Transformer.

$$\begin{aligned} I_s &= I_3D + E_3C \\ &= (53{\cdot}27 - j17{\cdot}32)(1{\cdot}00116 - j0{\cdot}0000282) \\ &\qquad + (69030 + j3060)(0{\cdot}0000236 - j0{\cdot}0001158) \\ &= 55{\cdot}3 - j25{\cdot}3 \\ &= 60{\cdot}8\underline{/-24^\circ\,32'}. \end{aligned}$$

$$E_s = E_3A + I_3B$$
$$= (69030 + j3060)(1\cdot00116 - j0\cdot0000282)$$
$$+ (53\cdot27 - j17\cdot32)(4\cdot4028 + j19\cdot2111)$$
$$= 69670 + j4013.$$
$$= \underline{\underline{69780/3^\circ\ 18'.}}$$

The vector diagram is given in Fig. 43.

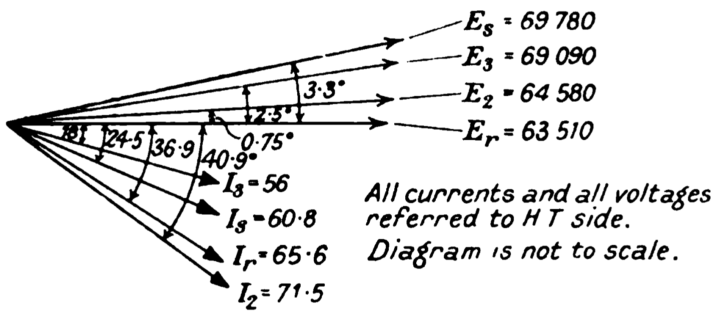


FIG. 43.—Vector diagram for transmission scheme, consisting of long overhead line and step-up and step-down transformers.

EXAMPLES ON CHAPTER VI.

1. A 1000 K.V.A., 3-phase transformer is operated from 11,000-volt mains. It has a no-load transformation ratio of 11,000 to 440 volts. At full load the impedance drop is 5 per cent. and the copper loss is 1 per cent. Calculate the equivalent resistance and reactance referred to both high-tension and low-tension sides.

Answer
- Referred to high-tension side: $r_e = 1\cdot21$ ohms; $x_e = 5\cdot9$,,
- ,, ,, low-tension side: $r_e = 0\cdot00193$ ohms; $x_e = 0\cdot0094$,,

2. A 3-phase, 60 K.V.A., 5000-volt transformer has a no-load transformation ratio of 5000 to 220 volts. It gave the following results on open-circuit and short-circuit tests :—

Open-circuit.—Primary volts = 5000, primary amps. = 0·4, watts = 600.

Short-circuit.—Primary volts = 200, primary amps. = 10, watts = 1000.

Calculate the equivalent impedance referred to high-tension side in the form $T_r = r_e + jx_e$, where r_e = equivalent resistance, x_e = equivalent reactance.

Calculate also the value of Y_r.

Answer $\begin{cases} T_r = 3{\cdot}3 + j11. \\ Y_r = 0{\cdot}000139\underline{/-\alpha} \text{ where } \cos\alpha = 0{\cdot}174 \text{ referred} \\ \qquad \text{to high-tension side.} \end{cases}$

3. A 3-phase overhead line has a resistance per mile of 0·35 ohms and a reactance per mile of 0·6 ohms. At the power-station the line is connected directly to the generator, and after proceeding for 20 miles it is connected to a step-down transformer which delivers a load of 600 kw. at 2200 volts 0·8 p.f. on its secondary side. Referred to its primary side the equivalent resistance of the transformer is 2·5 ohms, and its equivalent reactance is 15 ohms. The ratio of primary to secondary turns is 6 to 1. Calculate the voltage at the high-tension side of the transformer, and also at the generator. Neglect capacity of line and no-load current of transformer.

[London University, 1927.]

Answer $\begin{cases} \text{Voltage at transformer} = 13{,}800. \\ \quad ,, \quad \text{generator} = 13{,}920. \end{cases}$

4. If a 3-phase load is supplied through a step-up transformer at the power-station, a 10 mile overhead line, and a step-down transformer at the load, calculate the voltage at the power-station end, under the following conditions :—

Load delivered = 2000 kw. at 6000 volts, 0·85 power-factor.

The transformers are identical, having a ratio of transformation of 3 and resistances and reactances per phase as follows :—

Resistance of low-tension side = 0·1 ohms.
Reactance of low-tension side = 0·5 ,,
Resistance of high-tension side = 1 ,,
Reactance of high-tension side = 5 ,,

No-load current may be neglected.

The overhead line has a resistance per mile of conductor of 0·4 ohms, a reactance per mile of 0·45 ohms, and negligible capacity.

Answer : 6880 volts.

5. A 3-phase 50 cycle transmission scheme consists of an overhead line 200 miles long with a step-up transformer at the generating station and a step-down transformer at the load. The overhead line consists of copper conductors 0·504 ins. diameter spaced 10 ft. apart. The resistance per mile of conductor is 0·291 ohms. The two transformers are identical, 12,000 to 120,000 volt units. The equivalent impedance of each referred to the high-tension side is $T_r = 5 + j\,20$ ohms, whilst the admittance

$$Y_r = 0{\cdot}00002 - j0{\cdot}0002 \text{ mhos.}$$

If 10,000 kw. are delivered to the load at 12,000 volts, 0·8 power-factor (lagging), calculate :—

(*a*) The voltage and current at the high-tension terminals of the step-down transformer.

(*b*) The voltage and current at the high-tension terminals of the step-up transformer.

(*c*) The voltage and current at the low-tension terminals of the step-up transformer.

Answers :—

[*Note*.—Voltage and current are given as phase quantities, and are referred to high-tension side. Refer to Table, p. 96.]

Step-down transformer—

Low-tension side : Voltage = 69280.

Current = $60 \underline{/ -36^\circ\ 52'}$ amperes.

(*a*) High-tension side : Voltage = $70400 \underline{/ 0^\circ\ 37'}$.

Current = $70{\cdot}3 \underline{/ -45^\circ\ 16'}$.

Step-up transformer—

(*b*) High-tension side : Voltage = $75650 \underline{/ 4^\circ\ 36'}$.

Current = $49{\cdot}7 \underline{/ 20^\circ\ 11'}$.

(*c*) Low-tension side : Voltage = $75760 \underline{/ 5^\circ\ 21'}$.

Current = $49{\cdot}5 \underline{/ 2^\circ\ 34'}$.

CHAPTER VII.

TRAVELLING WAVES IN TRANSMISSION LINES.

SWITCH-IN PHENOMENA.

THE equations for the voltage and current at any point along a transmission line have been developed in Chapter II, and are given in equations 41 and 42. It is now necessary to point out that these equations hold only when the line is in what is referred to as a "steady state." They give correct values of current and voltage only when some time has elapsed since the circuit conditions were last altered. When circuit conditions are altered certain transient phenomena appear and equations 41 and 42 take no account of these. As the name implies, such phenomena are transitory in character and very quickly vanish, leaving equations 41 and 42 to express correctly voltage and current along the line. Short though the life of these transients is, during the time of their occurrence they produce important results. In this chapter and the following one, the theory of transient phenomena will be discussed. Our study is concerned with what happens in the exceedingly short interval of time after circuit conditions are altered before equations 41 and 42 hold.

Circuit conditions are most obviously altered by switching, and switching a transmission line into circuit or switching it out causes transient phenomena. Any alteration in the load on the system (and included in this we consider an earth on the system) produces like effects. Also, atmospheric conditions such as occur during thunderstorms may produce transients.

In this chapter we will consider the phenomena accompanying the switching into circuit of a transmission line.

SWITCHING IN AN OPEN-CIRCUITED TRANSMISSION LINE.

Consider a direct current generator* which is switched on to an open-circuited transmission line as in Fig. 44. The commonly held idea is that with the closing of the switch, the voltage E of the generator is established instantaneously over the whole length of the line. This is incorrect. The voltage E takes an exceedingly short, but none the less definite time to establish itself at all points along the line. At the instant of closing the switch, the voltage E is applied to the extreme end of the line, and a certain current i flows from the generator into the line.

Assume that the line is of infinite length, and that the resistance and leakage conductance of the line are negligible. Consider an exceedingly short length Δx of the line immediately adjacent to the switch. A moment before the closing of the switch there is no current, and consequently there is no magnetic flux linked with the portion Δx. After a short interval of time Δt, there is a current i over the length Δx and a flux $\mathrm{L}i\Delta x$ linked with it; L being the inductance of the line per unit length in henries. The creation of these flux linkages generates in the length Δx a voltage $= \mathrm{L}i\frac{\Delta x}{\Delta t}$, which must be equal and opposite to the applied voltage E.

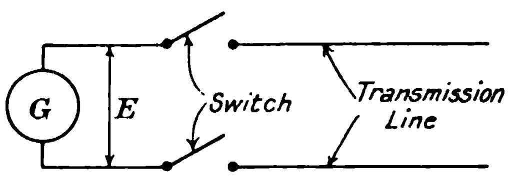


FIG. 44.—Switching-in an open-circuited transmission line.

$\therefore\ \mathrm{E} = \mathrm{L}i\frac{\Delta x}{\Delta t} = \mathrm{L}iv$ in the limit, where v is the velocity of propagation of the current along the wire. Hence it is seen that the current must be passing along the wire with sufficient velocity to generate a voltage E in the extreme end portion adjacent to the switch.

* For note regarding alternating current case see end of Chapter VIII.

Another very important change has accompanied the creation of the current i. The length Δx is now at a definite voltage E, and therefore between conductors there must be a quantity of electricity $C\Delta xE$ coulombs where C is the capacity between lines in farads per unit length.

This quantity of electricity has been supplied by the generator in time Δt, and since current is rate of change of quantity,

$$\therefore\ i = \text{CE}\frac{\Delta x}{\Delta t}\ \text{amps.}$$

Hence, in the limit

$$i = \text{CE}v\ \text{amps.} \qquad (100)$$

But

$$\text{E} = \text{L}iv\ \text{volts} \qquad (101)$$

$$\therefore \frac{i}{\text{E}} = \frac{\text{C}}{\text{L}}\frac{\text{E}}{i} \qquad i = \text{E}\sqrt{\frac{\text{C}}{\text{L}}} \qquad (102)$$

$$\text{Also} \quad i\text{E} = \text{CL}i\text{E}v^2, \quad v = \sqrt{\frac{1}{\text{CL}}} \qquad (103)$$

i.e. the requirements of simultaneous establishment of flux (magnetic and dielectric) along the line cause, at the instant of closing the switch, a voltage E and a current i moving with velocity v to be established at the extreme end of the line. The voltage E of the generator is now transferred to an exceedingly short length Δx immediately adjacent to the switch. The portion of the line immediately adjacent to the length Δx is now in exactly the same state as the length Δx was when the switch was closed, viz. there is a voltage E applied to it. By exactly the same reasoning as before, the current i and voltage E will be established along the length of line adjacent to Δx, and so on all the way along the line.

We see, then, that the voltage E is established progressively along the line as by the passage of a wave which has a voltage distribution of magnitude E volts associated with it. The front of the wave is quite definitely located at any time. The wave also has a current distribution associated with it, and this is connected with the voltage distribution by the relation

$$i = E\sqrt{\frac{C}{L}}, \text{ i.e. } i = \frac{E}{\sqrt{\frac{L}{C}}} = \frac{E}{Z},$$

where $Z = \sqrt{\frac{L}{C}}$, and this (see p. 83) is the NATURAL IMPEDANCE of the line. It is also called the SURGE IMPEDANCE or the WAVE RESISTANCE Such a wave as we are here considering is called a COMPLETE or PURE TRAVELLING WAVE.

Complete or Pure Travelling Wave.—A complete or pure travelling wave in a conductor consists of a voltage distribution and a current distribution, which travel along together. The current i and the voltage E at any point in the wave are connected by the relation

$$i = \frac{E}{Z}.$$

The travelling voltage distribution is frequently called the VOLTAGE WAVE. The travelling current distribution is frequently called the CURRENT WAVE.

If we imagine a voltmeter and ammeter whose moving parts had no inertia, installed at any point A, along the line, the ammeter and voltmeter would read zero until the front of the wave reached A. They would then suddenly record readings of E and i, and remain constant at these values.

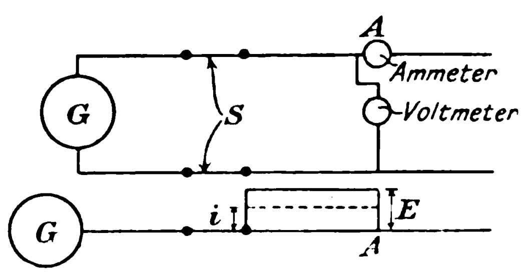


FIG. 45.—Voltage and current waves along transmission line. Switch S closed.

As long as the switch S remains closed, the back of the wave will remain established at the generator. The front of the wave moves forward as discussed in order to generate a voltage in the line equal and opposite to the generator voltage E. Suppose that the switch S is opened. There is now no applied voltage to the line, and therefore there must be no resultant generated voltage in the line. What happens is that the back of the wave leaves the generator and moves along the line with velocity v. Flux linkages due to the travelling current distribution are now lost at the back of the wave as quickly as they are established at the front, and no voltage will be generated in the line. We now have a rectangular wave with its voltage and current distributions moving along the line with velocity v as indicated in Fig. 46. Instruments at A will continue to read E and i only until the back of the wave reaches them. They will then suddenly drop to zero as the back of the wave passes. The pressure E and current i persist over the whole length of the wave. They vanish only at the end of the last element, and are produced anew at the beginning of the first.

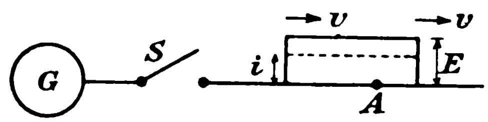


FIG. 46.—Voltage and current waves along transmission line. Switch S opened.

The pressure of the wave mounts steeply up to the value E at its commencement, and falls rapidly to zero at its end.

Inductance of Two-conductor Line having Zero Resistance.—It has been assumed in the above theory that the conductors have zero resistance. The current in such conductors would be entirely confined to their surfaces and there would be no internal flux linkages. The inductance as derived from equation (68), page 68, viz. :—

Inductance per mile of conductor

$$= L = 0{\cdot}00008 + 0{\cdot}000741 \log \frac{D}{r} \text{ henries,}$$

assumes that the current is uniformly distributed over the section of the conductors. The formula must be modified for a resistanceless line by omitting the first term, which represents the portion of the inductance due to internal flux linkages. The formula thus modified is :—

Inductance per mile of conductor

$$= L = 0{\cdot}000741 \log \frac{D}{r} \text{ henries} \quad . \quad . \quad (104)$$

The absence of internal flux linkages does not affect the capacity which remains correctly expressed by formula (73), page 71.

Reflection of Travelling Waves.—Consider now what happens if instead of the line being of infinite length it is of a certain definite length. As before, let it be open-circuited at its far end. Consider the state of affairs indicated in Fig. 47, when a wave of voltage E and current i has just reached the open end of the line. As soon as the front of the wave reaches

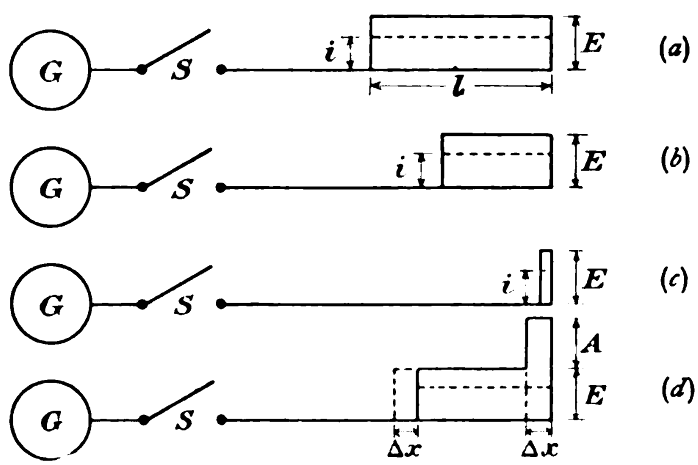


FIG. 47.—Reflection at end of open-circuited line.

the end of the line, the current i at the end vanishes. All conditions elsewhere along the line are unaltered, and therefore need for their satisfaction the continuance of the wave. Is it correct to argue that what happens is this: As the back of the wave gets nearer and nearer to the end of the line, the wave gets shorter and shorter as indicated in Figs. 47*b* and 47*c*, until when the back of the wave reaches the end of the line it will have completely vanished? This statement cannot be correct. In Fig. 47*a* the wave has a magnetic field and an electric field associated with it over a length l. This represents an amount of energy $= (\frac{1}{2}Li^2 + \frac{1}{2}CE^2)l$. What has become of this energy? It cannot disappear. What happens is that the vanishing of current from a certain length of the line must be accompanied by the formation of an amount of electrostatic energy equal to that released from the electromagnetic form. Referring to Fig. 47*d*, let us assume that a length Δx of the wave has vanished, and that an *extra* voltage A has been generated over the length Δx by the formation of electrostatic energy equal to the released electromagnetic energy.

Then $$\tfrac{1}{2}Li^2\Delta x = \tfrac{1}{2}CA^2\Delta x,$$

$$A = \sqrt{\frac{L}{C}}\, i = E,$$

i.e. the *extra* voltage A created must equal E. Hence the voltage at the end of the line rises to 2E. There is thus an

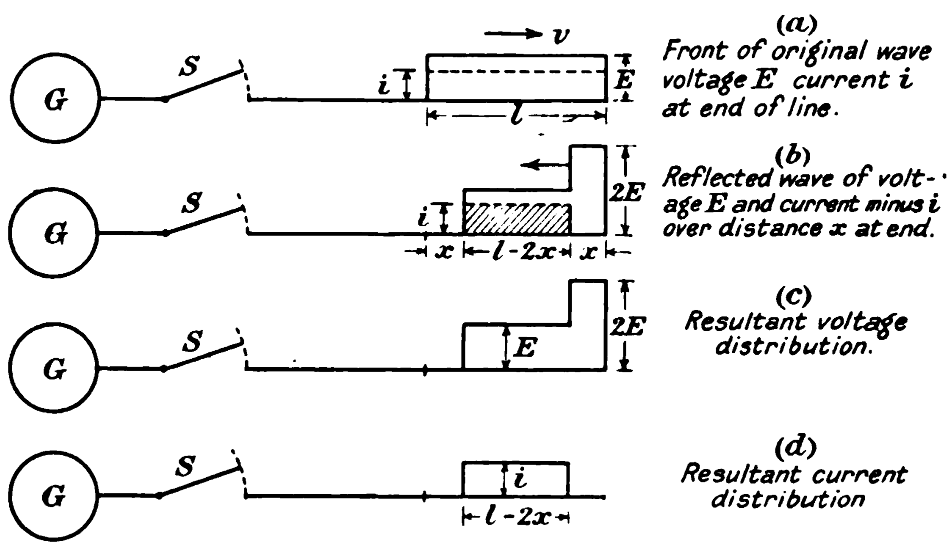


FIG. 48.—Reflection of travelling wave at open end of line.

unbalanced voltage E at the end of the line which calls for the formation of a *reflected* wave of current i and voltage E. What happens then when the wave strikes the open end of the line is not that the wave disappears, but that a reflected wave of voltage E and current i is created which travels back along the line with velocity v. Confirmation of this statement can be obtained by examining Fig. 48, where a portion x of the original wave has vanished.

Electromagnetic energy in original wave $= \frac{1}{2}Li^2 \,.\, l.$
Electrostatic energy in original wave $= \frac{1}{2}CE^2 l.$
Total energy $= (\frac{1}{2}Li^2 + \frac{1}{2}CE^2)l = \frac{1}{2}CE^2 2l,$

since
$$i = E\sqrt{\frac{C}{L}}.$$

Electromagnetic energy when reflected wave exists over length x in Figs. 48 or 49

$$= \frac{1}{2}Li^2(l - 2x).$$
$$\text{Electrostatic energy} = \frac{1}{2}CE^2(l - 2x) + \frac{1}{2}C \,.\, 4E^2 \,.\, x$$
$$= \frac{1}{2}CE^2(l + 2x).$$
$$\text{Total energy} = \frac{1}{2}Li^2(l - 2x) + \frac{1}{2}CE^2(l + 2x)$$
$$= \frac{1}{2}CE^2 2l \text{ since } i = E\sqrt{\frac{C}{L}},$$

which equals the energy in the original wave.

Hence, when the wave of voltage E and current i strikes the open end of the line, at that instant there is created a reflected wave of voltage E having the same sign as the original voltage and current i having the opposite sign. The length of the reflected wave is equal to that of the incident wave. The two sets of waves combine where they exist simultaneously to give resultant waves, thus the voltage at end of line will be 2E and the current zero (Figs. 48 and 49). If the incident wave is of finite length and the line is a long one, then the reflected

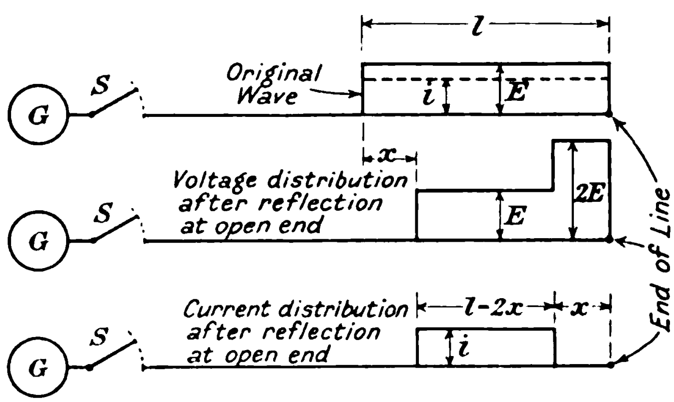


FIG. 49.—Reflection of travelling wave at open end of line.

wave will ultimately pass through the incident wave and will be the only one left in the circuit.

It is instructive to carry this consideration further by considering the case illustrated in Figs. 50 to 55, where it is assumed that the connection to the supply is maintained throughout.

When the wave reaches the open end of the line, it is reflected. The result of this reflection is, as we have seen, that the voltage is doubled and the current becomes zero (Fig. 51).

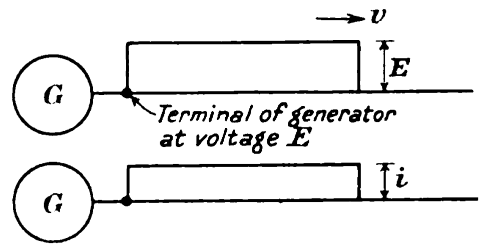


FIG. 50.—Voltage and current waves travelling along line with velocity v.

When the reflected wave reaches the generator, the voltage everywhere along the line is 2E and the current is zero (Fig. 52).

The state of affairs is now exactly reversed from what it is at the instant when the switch was first closed. Then the generator was at a voltage E above that of the line, but now the line is at a voltage E above that of the generator. The voltage at the end of the line *must* be E, and what happens there is that the excess pressure (= E volts) is used for the formation

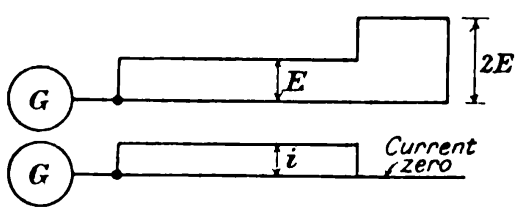


FIG. 51.

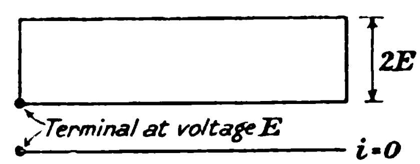


FIG. 52.

of a new current by the electrostatic energy changing into electromagnetic. If i is the current thus formed,

$$\tfrac{1}{2}\,Li^2 = \tfrac{1}{2}\,CE^2. \qquad \therefore\ i = \pm\, E\sqrt{\frac{C}{L}}.$$

The reduction of the voltage from 2E to E may be considered as being due to a reducing wave of voltage − E and current $-i$. Hence there is now created a new wave of voltage and current which gives conditions thus (Fig. 53) :—

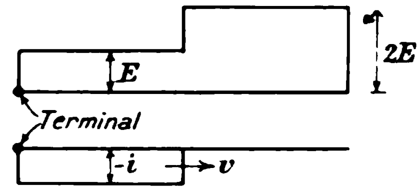


FIG. 53.

When this new wave reaches the end of the line, there will be a voltage E over the whole length and a current $-i$ (Fig. 54).

As soon as the wave of current $-i$ reaches the end of the line a voltage $-$ E is created, giving a reflected wave of voltage $-$ E and of current $+i$ (Fig. 55).

When the wave reaches the generator zero voltage and zero current exist throughout the line, which is now once again in its original state, and the cycle of operations is repeated. Thus, when an open-circuited transmission line is switched into

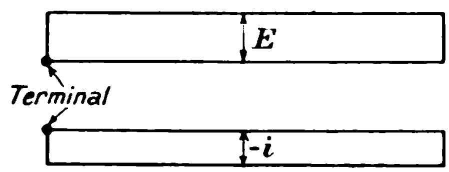


FIG. 54.

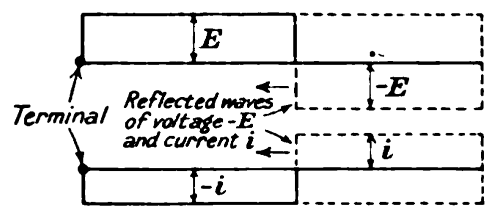


FIG. 55.

circuit, a switch-in wave is created. The greatest voltage associated with this wave is twice the voltage of the supply.

The frequency of the switch-in wave is the number of such complete cycles as have just been described which are performed per second.

If v is the velocity of the wave, then the time for a cycle is the time taken to travel four times the length of the line $= \frac{4l}{v}$

$$\text{frequency} = \frac{v}{4l} = \frac{1}{4l\sqrt{LC}}.$$

Switch-in Phenomena when Line is Short-circuited at the End.—A very similar sequence of events to that discussed in the previous section occurs if the line is short-circuited at its end, only here the necessary requirement is that as soon as the original wave front reaches the end the voltage E and its stored electrostatic energy must vanish and be converted into electromagnetic. This necessity produces a reflected wave of current i of the same sense as the incident wave, and a reflected wave of voltage E of opposite sense. These incident

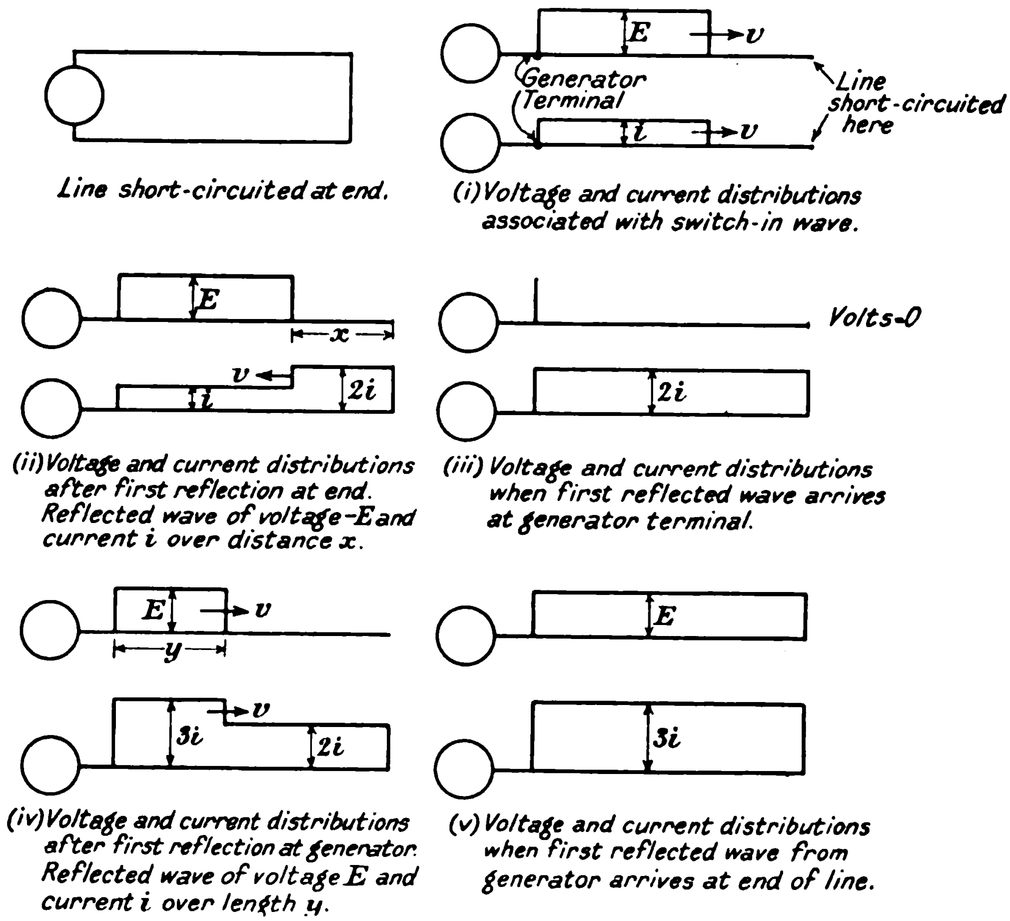


FIG. 56.—Reflection of travelling wave—short-circuited line.

and reflected waves combine as shown in the above diagrams, which are self-explanatory (Fig. 56).

Switch-in Phenomena when Line is Closed at End through Resistance.—Now consider a general case of which these two examples of short-circuited and open-circuited are only particular cases. Suppose the wave strikes a resistance of R ohms. Assume the wave has associated with it a voltage

E and current I. When it strikes the resistance assume that a wave E′ and I′ is reflected, leaving a wave E_{II} and I_{II} to pass into the resistance (Fig. 57).

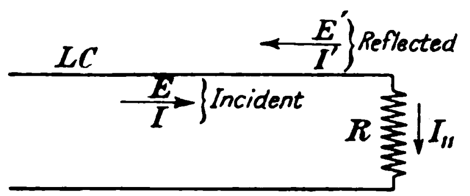


FIG. 57.—Reflection of travelling wave at resistance connected to end of line.

Then, at the moment of incidence, the voltage across the resistance is $E + E' = I_{II}R$.

The current which has been abstracted from the incident wave is $I - I_{II}$, and this created the voltage E_I in the reflected wave where $E_I = (I - I_{II})\sqrt{\frac{L}{C}}$. This pressure augments the pressure E, and the resistance has therefore the voltage $E + (I - I_{II})\sqrt{\frac{L}{C}}$ across it.

$$\therefore\ E + (I - I_{II})\sqrt{\frac{L}{C}} = I_{II}R.$$

$$\therefore\ E + I\sqrt{\frac{L}{C}} - I_{II}\sqrt{\frac{L}{C}} = I_{II}R \text{ and } I\sqrt{\frac{L}{C}} = E.$$

$$\therefore\ 2E - I_{II}\sqrt{\frac{L}{C}} = I_{II}R.$$

$$\therefore\ I_{II} = \frac{2E}{R + \sqrt{\frac{L}{C}}}.$$

$$\therefore\ E + E' = I_{II}R = \frac{2ER}{R + \sqrt{\frac{L}{C}}} = 2E\left\{\frac{R}{\sqrt{\frac{L}{C}} + R}\right\}.$$

$$\text{If } R = \sqrt{\frac{L}{C}} \qquad E + E_I = E \qquad \text{i.e. } E_I = 0 \qquad I_{II} = \frac{E}{R},$$

i.e. there is no reflected wave, all of the current associated with the incident wave passes into the resistance.

The magnitude of reflected wave of voltage

$$= E_I = (I - I_{II})\sqrt{\frac{L}{C}}$$

$$\therefore\ E_{\text{I}} = I_{\text{II}}R - E$$

$$= \frac{2ER}{R + \sqrt{\frac{L}{C}}} - E = E\left[\frac{2R}{R + \sqrt{\frac{L}{C}}} - 1\right]$$

$$= E\left\{\frac{R - \sqrt{\frac{L}{C}}}{R + \sqrt{\frac{L}{C}}}\right\}.$$

Similarly, magnitude of reflected wave of current $= I_{\text{I}}$.

$$I_{\text{I}} = I_{\text{II}} - I$$

$$= \frac{2E}{R + \sqrt{\frac{L}{C}}} - \frac{E}{\sqrt{\frac{L}{C}}}$$

$$= \frac{E}{\sqrt{\frac{L}{C}}}\left\{\frac{\sqrt{\frac{L}{C}} - R}{\sqrt{\frac{L}{C}} + R}\right\}$$

$$= -I\left\{\frac{R - \sqrt{\frac{L}{C}}}{R + \sqrt{\frac{L}{C}}}\right\}.$$

Reflected wave of voltage $E_{\text{I}} = E\left\{\frac{R - \sqrt{\frac{L}{C}}}{R + \sqrt{\frac{L}{C}}}\right\}$.

Reflected wave of current $I_{\text{I}} = -I\left\{\frac{R - \sqrt{\frac{L}{C}}}{R + \sqrt{\frac{L}{C}}}\right\}$.

If $\sqrt{\frac{L}{C}}$ is called the wave resistance and written $= w$,

then $$E_{\text{I}} = E\left(\frac{R - w}{R + w}\right) \quad . \quad . \quad . \quad (105)$$

$$I_{\text{I}} = -I\left(\frac{R - w}{R + w}\right) \quad . \quad . \quad . \quad (106)$$

The following are particular cases :—

(*a*) If $R = \infty$, i.e. receiving circuit is open,

$$I_1 = -I \text{ and } E_1 = E.$$

(*b*) If $R = O$, i.e. receiving circuit is short-circuited,

$$E_1 = -E \text{ and } I_1 = I.$$

(*c*) If $R = \sqrt{\frac{L}{C}}$,

$$E_1 = O \text{ and } I_{11} = O.$$

No reflection, the entire wave being absorbed.

Signs of Voltage and Current in Incident and Reflected Waves.

In Fig. 58 the two conductors of a transmission system are shown. Assume that the conductors are initially at the same potential and that at one end they are suddenly connected to the terminals of a generator. This impressed voltage will cause current to flow in the conductors and will give rise to waves of voltage and current as discussed on page 128. If the polarity of the generator is as indicated in the figure, then the direction of current flow in the *positive* conductor will be from

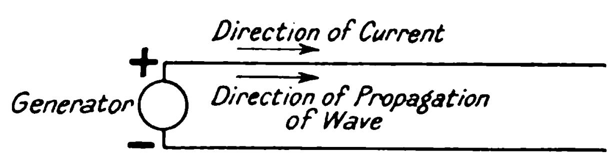


Fig. 58.—Connection between direction of current, polarity of conductor and direction of wave propagation.

left to right. The direction of propagation of the wave is also from left to right. It follows from this that the direction of current flow in the *positive* conductor is the direction of propagation for the waves of voltage and current.

For reflected waves the direction of propagation is in the opposite direction to that of the incident waves. Reversal of the direction of wave propagation can mean only one of two things, viz. :—

Either the conductor which is positive for the incident wave is negative for the reflected ; or

The conductors have the same polarity for both incident and reflected waves, but the direction of the current is reversed for the two sets of waves.

The truth of these statements is made evident by an examination of Figs. 59 and 60.

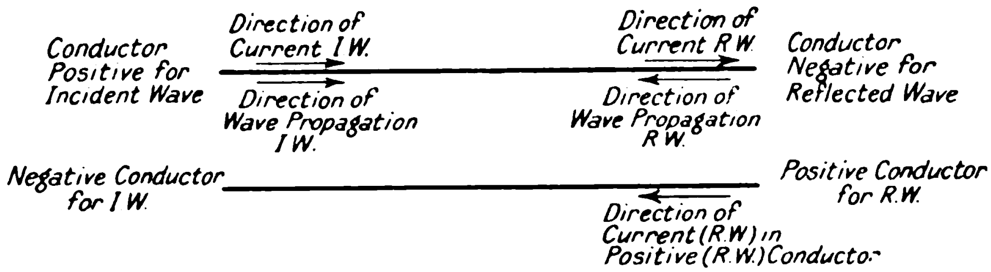


FIG. 59.—Incident and reflected waves. Currents in same direction. Voltages in opposite directions.

It follows that if two sets of waves are travelling in opposite directions (as in incident and reflected waves) they must have either currents or voltages in opposite senses.

For waves travelling to the right in Figs. 59 and 60, i.e. for incident waves, the relation has been proved to be $I = \frac{E}{w}$.

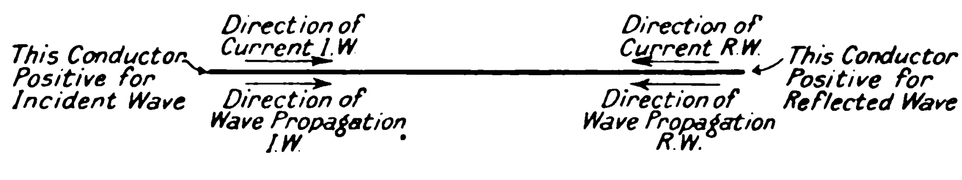


FIG. 60.—Incident and reflected waves. Currents in opposite directions. Voltages in same direction.

Therefore, for waves travelling to the left, i.e. for reflected waves, since if E is positive I is negative, and *vice versa*, $I = -\frac{E}{w}$. The work already done in this chapter has proved that for an open-circuited line the incident and reflected waves have voltages in the same sense, but currents in the opposite sense. For short-circuited lines the incident and reflected waves have currents in the same sense, but voltages in the opposite sense.

Cable and Overhead Line in Series.

It frequently happens in practice that it is convenient to insert short lengths of underground cable into an overhead line. For example, if an overhead line passes across railways or roads, strict regulations as to guarding against possibilities of broken wires falling to the ground have to be complied with. Under such circumstances it is common practice for the overhead line to be discontinued and the line continued under the road or railway by cable. Also where the line has to be brought into a sub-station or a power-station, the overhead line frequently ends a short distance from the building, and the circuit is concluded by a length of underground cable. It is, therefore, of considerable practical importance to consider what happens if travelling waves are set up in a circuit which contains overhead line and underground cable in series.

Three typical cases will be considered :—

(1) A long (assumed infinitely long) overhead line in series with a short cable open at its end.

(2) A long (assumed infinitely long) cable in series with a short overhead line open at its end.

(3) A short cable joining two long (assumed infinitely long) overhead lines.

(1) Long Overhead Line in Series with Short Cable. Cable Open at End. Waves Originate in Overhead Line.

Assume that there is a pure travelling wave of voltage E and current I in the overhead line. When the wave reaches the junction between overhead line and cable, a portion of the wave will be transmitted into the cable, and a portion reflected back into the overhead line.

Let the pressure of the wave entering the cable be e_K and the current i_K. Then

$$e_K = i_K w_K \quad . \quad . \quad . \quad . \quad (107)$$

where w_K is the wave resistance of the cable.

In Fig. 61*a*, *b* and *c* the electrical state of the system is represented at the instants just before and just after the incident wave reaches the junction.

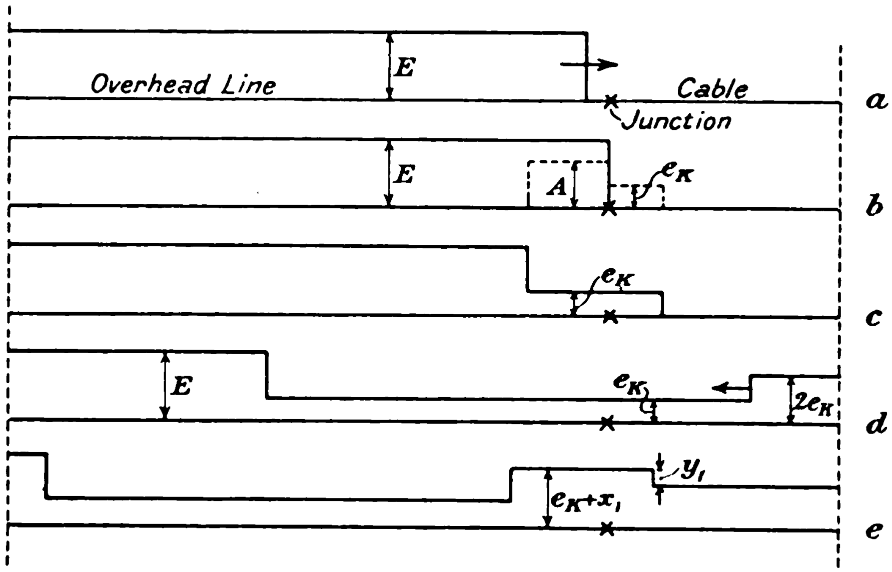


FIG. 61.—Long overhead line in series with short cable open at end.

Let A be the voltage of the wave reflected back into overhead line.

Then

$$E + A = e_K, \text{ i.e. } A = e_K - E \quad . \quad . \quad (108)$$

Let B be the current in the wave reflected back into overhead line.

Then

$$I + B = i_K, \text{ i.e. } B = i_K - I \quad . \quad . \quad (109)$$

If w_L is the wave resistance of the overhead line, then

$$A = - Bw_L \quad . \quad . \quad . \quad (110)$$

the negative sign appearing because A and B are associated with a reflected wave. It follows that $e_K - E = - (i_K - I)w_L$, substituting (108) and (109) in (110). Also, since $Iw_L = E$ and $i_K w_K = e_K$,

then

$$e_K - E = -i_K w_L + I w_L$$
$$= -i_K w_L + E.$$
$$\therefore 2E = e_K + i_K w_L$$
$$= e_K + \frac{e_K}{w_K} w_L$$
$$= e_K \left[\frac{w_K + w_L}{w_K}\right].$$
$$\therefore e_K = 2E \frac{w_K}{w_L + w_K} \quad . \quad . \quad . \quad (111)$$

Also

$$i_K = \frac{2E}{w_L + w_K} \quad . \quad . \quad . \quad . \quad (112)$$

The wave which enters the cable travels to the open end, where it is reflected, the pressure being doubled and the current falling to zero. These effects are produced (as discussed on p. 133) by a pure wave of voltage e_K and current $-i_K$, reflected from the open end. As the reflected wave (voltage $= e_K$) travels towards the junction it will raise the voltage of the cable to $2e_K$. Fig. 61*d* shows the conditions after reflection at the open end. When the reflected wave (voltage $= e_K$) reaches the junction part of it is transmitted into the overhead line and part is reflected back into the cable.

Let x_1 be the voltage of the wave which penetrates into the overhead line, x_1 is obtained from equation (111), but since the wave is now passing from cable to overhead line instead of *vice versa*, w_K and w_L must be interchanged. Also, the incident wave has a voltage e_K, not E.

$$\therefore x_1 = 2e_K \cdot \frac{w_L}{w_L + w_K}.$$

Fig. 61*e* represents conditions at a time just after the reflection from the open end of the cable has got back to the junction.

At the junction there is now a voltage of $E + A + x_1$.

$$\therefore \text{Voltage at junction} = E + e_K - E + 2e_K \cdot \frac{w_L}{w_L + w_K}$$
$$= e_K + 2e_K \frac{w_L}{w_L + w_K}.$$

The voltage of the cable immediately before the reflected wave arrived at the junction was $2e_K$. There must be a wave having a voltage y_1 reflected back into the cable, where

$$2e_K + y_1 = e_K + 2e_K \cdot \frac{w_L}{w_L + w_K}.$$

$$\therefore y_1 = e_K \frac{w_L - w_K}{w_L + w_K}.$$

This wave (voltage $= y_1$) is reflected at the open end of the cable, a reflected wave of equal voltage y_1, starting at the open end and travelling back to the junction. When this, the second reflected wave from the open end, reaches the junction a wave having a voltage x_2 is transmitted into the overhead line, and a wave having voltage y_2 is reflected back into the cable.

It has been proved that when a travelling wave in the cable has a voltage e_K and reaches the junction, a wave having a voltage x_1 is transmitted, and a wave having a voltage y_1 is reflected where

$$x_1 = 2e_K \frac{w_L}{w_L + w_K},$$

$$y_1 = e_K \cdot \frac{w_L - w_K}{w_L + w_K}.$$

Therefore, for a travelling wave in the cable having a voltage y_1,

$$x_2 = 2y_1 \frac{w_L}{w_L + w_K},$$

$$y_2 = y_1 \frac{w_L - w_K}{w_L + w_K} = e_K \left(\frac{w_L - w_K}{w_L + w_K}\right)^2.$$

The process repeats itself, and

$$y_3 = e_K \left(\frac{w_L - w_K}{w_L + w_K}\right)^3.$$

In general, after n reflections,

$$y_n = e_K \left(\frac{w_L - w_K}{w_L + w_K}\right)^n$$

$$= 2E \frac{w_K}{w_L + w_K} \left(\frac{w_L - w_K}{w_L + w_K}\right)^n.$$

If E_n is the accumulated pressure in the cable after the nth reflection,

$$\begin{aligned}
E_n &= e_K + e_K + y_1 + y_1 + y_2 + y_2 + \ldots + y_n + y_n \\
&= 2e_K + 2y_1 + 2y_2 + \ldots + 2y_n \\
&= 4E\frac{w_K}{w_L + w_K} + \sum_{n=n}^{n-1} 2y \\
&= 4E\frac{w_K}{w_L + w_K}\left[1 + \frac{w_L - w_K}{w_L + w_K} + \left(\frac{w_L - w_K}{w_L + w_K}\right)^2 + \ldots\right] \\
&= 2E\left[1 - \left(\frac{w_L - w_K}{w_L + w_K}\right)^{n+1}\right] \quad . \quad . \quad . \quad . \quad (113)
\end{aligned}$$

For $n = \infty$, $E_n = 2E$.

The charging of the cable, therefore, takes place by increments. The charging time for each increment is obtained from the length of the cable and the wave velocity. For each increment the wave has to travel a distance equal to the length of the cable. If the length of the cable is l and the wave velocity is v, then the time between voltage increments is $t = \frac{l}{v}$.

The problem dealt with has been that of an overhead line and cable, but it is obvious that a general statement of the results obtained may be given thus:—

If a pure travelling wave has a voltage Z and is travelling in a conductor of wave resistance w_A, then when it strikes a junction where the conductor wave resistance changes to w_B, the original incident wave is partly reflected back into the conductor of resistance w_A, and partly transmitted into the conductor of resistance w_B. The relations between the voltages of incident reflected and transmitted waves are:—

$$\text{Incident wave} = Z.$$

$$\text{Transmitted wave} = 2Z\frac{w_B}{w_A + w_B} \quad . \quad . \quad (114)$$

$$\text{Reflected wave} = Z\frac{w_B - w_A}{w_A + w_B} \quad . \quad . \quad (115)$$

This general statement is illustrated in Fig. 62, which also shows the two particular cases of a travelling wave in an overhead line striking a junction whence the line is continued by cable, and the case of a travelling wave in a cable striking a junction whence the line is continued overhead.

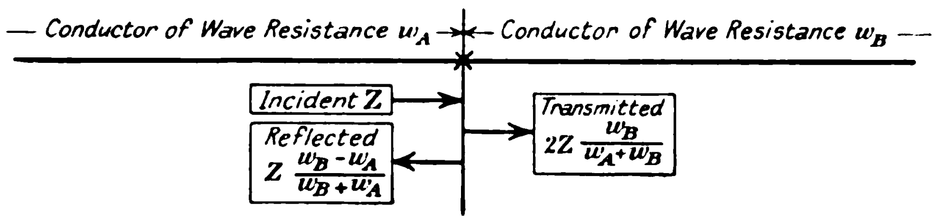


FIG. 62*a*.—General case of conductors of wave resistances w_A and w_B in series Wave originating in conductor of resistance w_A.

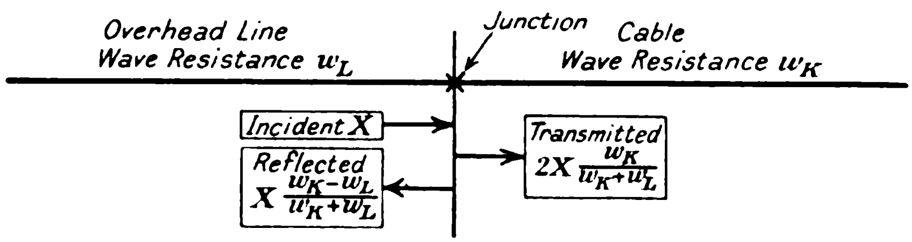


FIG. 62*b*.—Overhead line (wave resistance w_L) and cable (w_K) in series. Wave originating in overhead line.

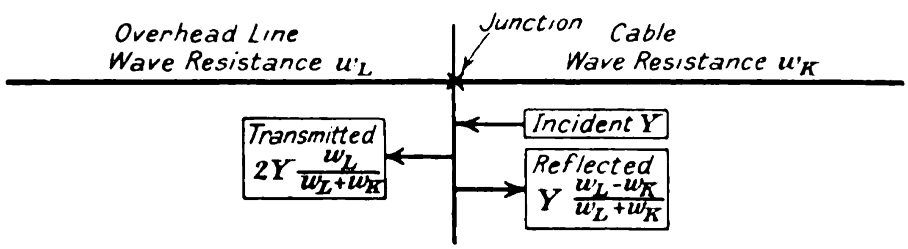


FIG. 62*c*.—Overhead line (wave resistance w_L) and cable (w_K) in series. Wave originating in cable.

The problem just dealt with, i.e. a travelling wave in a long overhead line striking a junction between overhead line and cable, can now be summarised with the aid of Fig. 62 thus :—

Voltage of original wave transmitted into cable from overhead line $= e_K$.

Reflection at open end produces reflected wave having voltage $= e_K =$ Incident Y (Fig. 62).

Voltage of cable $= 2e_K$.

Incident Y (Fig. 62) produces reflected wave $e_K \frac{w_L - w_K}{w_L + w_K}$.

Voltage of cable $= 2e_K + e_K \frac{w_L - w_K}{w_L + w_K}$.

Reflection at open end produces reflected wave $e_K \frac{w_L - w_K}{w_L + w_K}$ = Incident Y (Fig. 62).

Voltage of cable $= 2e_K + 2e_K \frac{w_L - w_K}{w_L + w_K}$.

Incident Y (Fig. 62) produces reflected wave $e_K \left(\frac{w_L - w_K}{w_L + w_K}\right)^2$.

Voltage of cable $= 2e_K + 2e_K \frac{w_L - w_K}{w_L + w_K} + e_K \left(\frac{w_L - w_K}{w_L + w_K}\right)^2$.

Reflection at open end produces reflected wave $e_K \left(\frac{w_L - w_K}{w_L + w_K}\right)^2$ = Incident Y (Fig. 62).

Voltage of cable $= 2e_K + 2e_K \frac{w_L - w_K}{w_L + w_K} + 2e_K \left(\frac{w_L - w_K}{w_L + w_K}\right)^2$.

After n reflections,

Voltage of cable

$$= 2e_K + 2e_K \frac{w_L - w_K}{w_L + w_K} + \ldots + 2e_K \left(\frac{w_L - w_K}{w_L + w_K}\right)^n,$$

which agrees with the expression on page 146.

Example.—If $w_L = 600$ and $w_K = 60$, then $e_K = 2E \frac{w_K}{w_L + w_K}$

$$= 2E \frac{60}{660}$$

$$= \frac{2}{11}E.$$

Voltage of cable builds up thus :—

Time 1. When first reflection from open end reaches junction, voltage $= 2e_K = \frac{4}{11}E = 0{\cdot}36E$.

Time 2. When second reflection from open end reaches junction, voltage $= 2e_K + 2e_K \cdot \frac{w_L - w_K}{w_L + w_K}$

$$= \frac{4}{11}E + \frac{4}{11}E\frac{540}{660} = 0{\cdot}66E.$$

Time 3. When third reflection from open end reaches junction, voltage $= \frac{4}{11}E + \frac{4}{11}E\frac{540}{660} + \frac{4}{11}E\left(\frac{540}{660}\right)^2$
$= 0{\cdot}90E.$

Time 4. When fourth reflection from open end reaches junction, voltage = 1·11E.

Time 5. When fifth reflection from open end reaches junction, voltage = 1·26E.

After ∞ reflections, voltage = 2E.

The way in which the voltage builds up is shown in Fig. 63. The voltage of the cable finally reaches a value of 2E after an

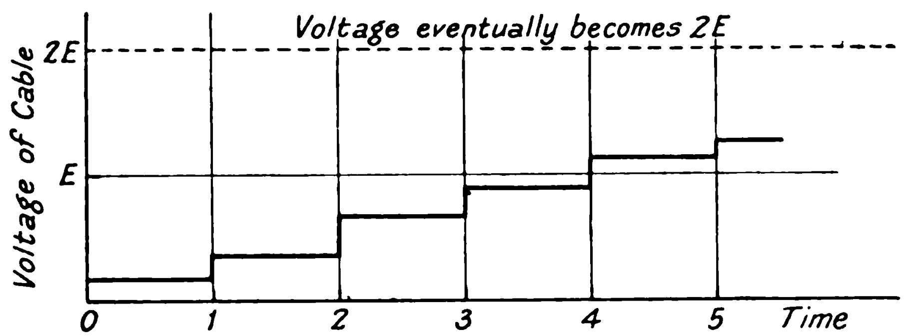


FIG. 63.—Long overhead line and short cable in series. Cable open at end. Figure shows voltage increments in cable.

infinite number of oscillations of the switch-in wave in the cable.

Example.—A transmission line is 33 miles long. It runs overhead for 25 miles and then underground in cables for 8 miles.

The overhead line is connected to a continuous voltage of 10,000, the distant end of the line being open.

The inductance and capacity of the line are :—

	Overhead.	Cable.
Millihenries per mile of loop .	2·87	0·6
Microfarads ,, ,, .	0·01	0·072

Calculate, and illustrate by sketches, the voltage distribution at the following instants :—

(*a*) Immediately before the wave front of the switch-in wave reaches the junction.
(*b*) Immediately after the wave front reaches the junction.
(*c*) Immediately after first reflection from open end of cable has travelled back to junction.

$$\sqrt{\frac{L}{C}} = w = \text{wave resistance,}$$

$$w_K = \sqrt{\frac{0 \cdot 6 \times 10^{-3}}{0 \cdot 072 \times 10^{-6}}} = \sqrt{8 \cdot 3 \times 10^3} = 91 \text{ ohms,}$$

$$w_L = \sqrt{\frac{2 \cdot 87 \times 10^{-3}}{0 \cdot 01 \times 10^{-6}}} = \sqrt{2870 \times 10^3} = 535 \text{ ohms.}$$

$$\text{Velocity of wave propagation} = v = \frac{1}{\sqrt{LC}}.$$

$$v \text{ for overhead} = \frac{1}{\sqrt{2 \cdot 87 \times 10^{-3} \times 0 \cdot 01 \times 10^{-6}}} = 186{,}000 \text{ miles per second.}$$

$$v \text{ for cable} = \frac{1}{\sqrt{0 \cdot 6 \times 10^{-3} \times 0 \cdot 072 \times 10^{-6}}} = 152{,}000 \text{ miles per second.}$$

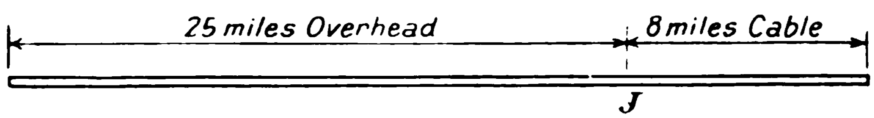


FIG. 64*a*.

(*a*) Immediately before front of switch-in wave reaches junction J, there is a voltage of 10,000 along the whole length of overhead line (see Fig. 64*b*).

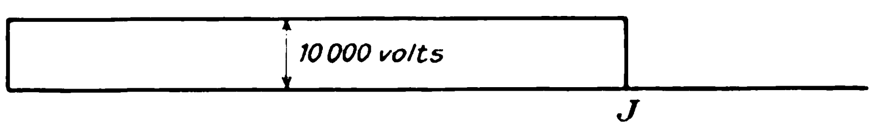


FIG. 64*b*.

(*b*) When wave-front reaches J, voltage of wave transmitted into cable $= 20000 \times \frac{91}{626} = \underline{2900 \text{ volts.}}$

∴ Voltage of wave reflected back into overhead line = − 7100 (Fig. 64*c*).

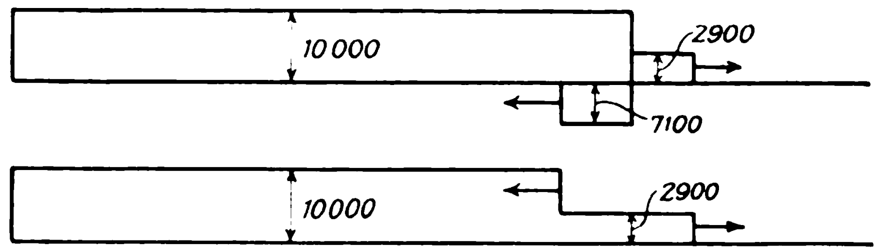


FIG. 64*c*.

(*c*) When wave in cable reaches end, a reflected wave having a voltage of 2900 is reflected back and travels in cable towards junction J.

When this reflected wave reaches J, voltage of wave transmitted into overhead line

$$= 2900 \times \frac{2 \times 535}{626} = \underline{4960 \text{ volts.}}$$

Voltage of wave reflected back into cable = 2060 volts.

During time taken for original transmitted wave into cable to travel 16 miles, reflected wave into overhead will have travelled $16 \times \frac{186}{152} = 19 \cdot 6$ miles.

∴ Voltage distribution will be as shown (Fig. 64*d*).

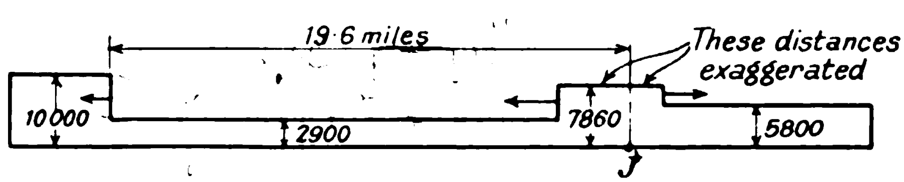


FIG. 64*d*.

(2) LONG CABLE IN SERIES WITH SHORT OVERHEAD LINE. OVERHEAD LINE OPEN AT END. WAVES ORIGINATE IN CABLE.

The theory already obtained for Case I. holds in its entirety, only changing w_L and w_K in the equations thus :—

After arrival of the travelling wave at the junction the voltage of the wave transmitted into the overhead line (see Fig. 62) is

$$e_L = 2E \frac{w_L}{w_K + w_L}.$$

The pressure reached in the overhead line after a series of n reflections is

$$E_n = 2E\left[1 - \left(\frac{w_K - w_L}{w_K + w_L}\right)^{n+1}\right] \quad . \quad . \quad (116)$$

In the case of the example previously considered with $w_L = 600$ and $w_K = 60$, the voltage of the overhead line builds up thus :—

Time 1. When first reflection from open end reaches junction, voltage = 3·63E.

Time 2. When second reflection from open end reaches junction, voltage = 0·66E.

Time 3. When third reflection from open end reaches junction, voltage = 3·10E.

Time 4. When fourth reflection from open end reaches junction, voltage = 1·12E.

Time 5. When fifth reflection from open end reaches junction, voltage = 2·74E.

After ∞ reflections, voltage = 2E.

The way in which the voltage builds up is shown in Fig. 65.

In general terms (referring to Fig. 62), the problem just considered is one where the wave passes from a conductor of low wave resistance (w_A small) to a conductor of high wave resistance (w_B large). In such cases the voltage jumps are always considerably greater than where the wave passes from conductors of large to conductors of small wave resistance, as from overhead line to cable.

Example.—A long transmission line starts from a generating station and is carried underground by cable, excepting for a short length spanning a river. The line is carried overhead across the river, and terminates in a sub-station on the river bank. The junction between cable and overhead line is made at a junction box, the insulators of which flash over at a voltage of 100,000. If 20,000 volts is switched on to the cable, the switches in the sub-station being open, calculate the maximum

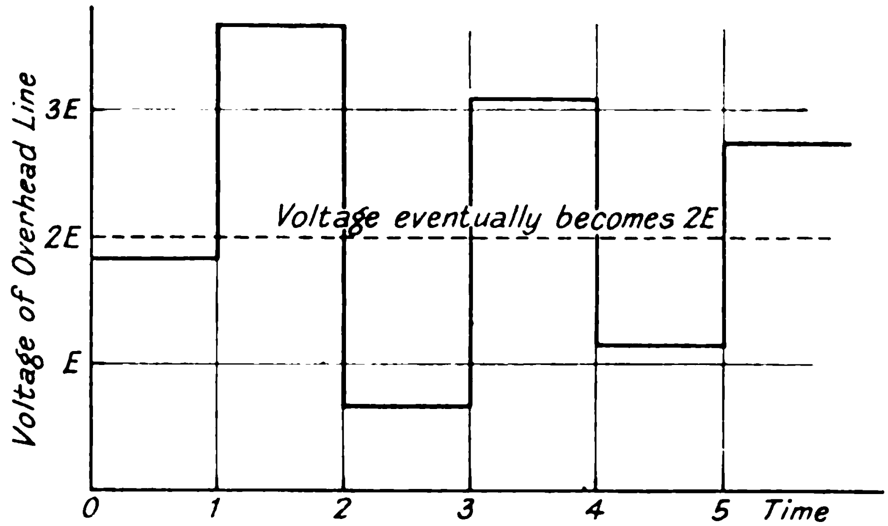


FIG. 65.—Long cable and short overhead line in series. Overhead line open at end. Figure shows voltage increments in overhead line.

voltage which the insulators of the junction box will be called upon to withstand.

Wave resistance of overhead line = 600 ohms.
Wave resistance of cable = 60 ,,

From formula on page 152 maximum voltage at junction of overhead line and cable = 3·63 E

$$= 3{\cdot}63 \times 20{,}000$$
$$= \underline{72{,}600 \text{ volts.}}$$

The flash-over voltage of the insulators being 100,000, they will not flash-over. The question serves to illustrate the high voltage (several times the working voltage) which insulation at the junction of cables and overhead lines may be called upon to withstand.

(3) SHORT CABLE JOINING TWO LONG OVERHEAD LINES. OVERHEAD LINES IDENTICAL ON EACH SIDE OF CABLE. WAVES ORIGINATE IN OVERHEAD LINE.

Assume that an overhead line is cut at some point along its length, and that the two portions are then connected by a short length of cable. If a pure travelling wave of vertical wave-front and voltage E is set up in the left-hand portion of the overhead line (see Fig. 66), the voltage characteristics of the wave which passes through the cable and into the right-hand portion will now be considered. It will be proved that the presence of the cable materially alters the original wave. In particular, it changes its front from a vertical to a sloping one, and this reduces the possibility of damage resulting to machinery installed at the end of the second length of overhead line.

In Fig. 62, Z, X, and Y are the voltages associated with any incident waves, and the relations between the voltages of incident, reflected, and transmitted waves are as indicated.

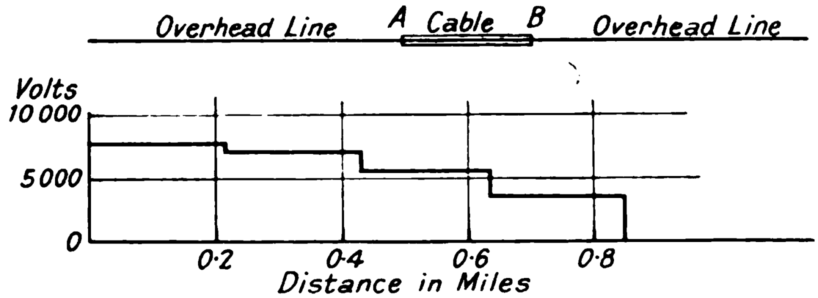


FIG. 66.—Short length of cable inserted into overhead line. Figure shows voltage distribution along line when 5th wave emerges from cable.

When an incident wave having voltage E strikes the junction of overhead line and cable A, then a wave having voltage $2E \frac{w_K}{w_L + w_K}$ is transmitted into the cable. This is now the voltage Y of the incident wave at the junction B, and a wave having voltage $2E \frac{w_K}{w_L + w_K} \frac{2w_L}{w_L + w_K} = E \frac{4w_K w_L}{(w_L + w_K)^2}$ is transmitted into overhead line. Also, a wave is reflected back into the cable, the voltage of it being

$$2E \left(\frac{w_K}{w_L + w_K}\right) \left(\frac{w_L - w_K}{w_L + w_K}\right) = 2E \frac{w_K(w_L - w_K)}{(w_L + w_K)^2}.$$

This reflected wave travels back towards B, and when it arrives the conditions are those holding for the incident wave having voltage Y in Fig. 62, since the wave is travelling in the cable towards the overhead line.

The voltage of the incident wave corresponding to Y in Fig. 62 is $2E\,\frac{w_K(w_L - w_K)}{(w_L + w_K)^2}$.

Of this incident wave a portion passes into the overhead line, but a wave having a voltage

$$2E\,\frac{w_K(w_L - w_K)}{(w_L + w_K)^2}\left(\frac{w_L - w_K}{w_L + w_K}\right) = 2E\,\frac{w_K(w_L - w_K)^2}{(w_L + w_K)^3}$$

is reflected back into the cable. This is now the voltage of the new incident wave (i.e. Y in Fig. 62), and the overhead line beyond B receives a second transmitted wave having a voltage

$$2E\,\frac{w_K(w_L - w_K)^2}{(w_L + w_K)^3} \times 2\frac{w_L}{w_L + w_K} = E\,\frac{4w_Kw_L}{(w_L + w_K)^2}\left(\frac{w_L - w_K}{w_L + w_K}\right)^2.$$

This process repeats itself, and at intervals, separated by the time taken for a wave to travel twice the length of the cable, the overhead line to the right of B receives a new travelling wave. The voltages of these successive waves are—

$$\text{1st transmitted wave} = E \times \frac{4w_Kw_L}{(w_L + w_K)^2}.$$

$$\text{2nd transmitted wave} = E \times \frac{4w_Kw_L}{(w_L + w_K)^2} \times \left(\frac{w_L - w_K}{w_L + w_K}\right)^2.$$

$$\text{3rd transmitted wave} = E \times \frac{4w_Kw_L}{(w_L + w_K)^2} \times \left(\frac{w_L - w_K}{w_L + w_K}\right)^4.$$

$$\text{4th transmitted wave} = E \times \frac{4w_Kw_L}{(w_L + w_K)^2} \times \left(\frac{w_L - w_K}{w_L + w_K}\right)^6.$$

$$n\text{th transmitted wave} = E \times \frac{4w_Kw_L}{(w_L + w_K)^2} \times \left(\frac{w_L - w_K}{w_L + w_K}\right)^{2n-2}.$$

The voltage at B after the nth wave has entered the overhead line

$$= E \times \frac{4w_Kw_L}{(w_L + w_K)^2} + E\,\frac{4w_Kw_L}{(w_L + w_K)^2}\left(\frac{w_L - w_K}{w_L + w_K}\right)^2 + \dots$$

$$+ E\,\frac{4w_Kw_L}{(w_L + w_K)^2}\left(\frac{w_L - w_K}{w_L + w_K}\right)^{2n-2}$$

$$= E\,.\,\frac{4w_Kw_L}{(w_L + w_K)^2}\left[1 + \left(\frac{w_L - w_K}{w_L + w_K}\right)^2 + \dots + \left(\frac{w_L - w_K}{w_L + w_K}\right)^{2n-2}\right]$$

$$= E \cdot \frac{4w_K w_L}{(w_L + w_K)^2}\left[\frac{1 - \left(\frac{w_L - w_K}{w_L + w_K}\right)^{2n}}{1 - \left(\frac{w_L - w_K}{w_L + w_K}\right)^{2}}\right]$$

$$= E\left[1 - \left(\frac{w_L - w_K}{w_L + w_K}\right)^{2n}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

If $n = \infty$, $E_n = E$.

After a large number of reflections and transmissions the voltage of the wave in the overhead line to the right of B approaches a maximum value of E, which is the voltage of the original incident wave. The insertion of the cable has therefore not reduced the maximum value of the voltage associated with the travelling wave, but it has smoothed out its front in the manner indicated in Fig. 66. This smoothing out of the front is very desirable, and the damage to apparatus is thereby reduced.

Example.—A cable 100 yds. long, having a wave-resistance of 60 ohms, is inserted between two long lengths of overhead line of wave-resistance 600 ohms. Calculate and show by means of a diagram the voltage wave along one section of the overhead line due to a travelling wave of 10,000 volts magnitude in the other section. Assume this wave is of infinite length, and of vertical wave front. Neglect resistance and leakance. Velocity of wave in cable = 100,000 miles per second.

Time between successive transmissions into second part of overhead line, $t = \frac{2l}{v}$ seconds,

where $l = 300/5280$ miles.

v = velocity of wave in cable = 100,000 m./sec.

$$\therefore t = \frac{2 \times 300}{5280 \times 100000} = 0{\cdot}00000114 \text{ seconds}$$

$$= 1{\cdot}14 \text{ micro-seconds.}$$

Voltage of overhead line after nth transmitted wave enters

$$= E\left[1 - \left(\frac{w_L - w_K}{w_L + w_K}\right)^{2n}\right].$$

Hence voltage due to first transmitted wave

$$= E\left[1 - \left(\frac{w_L - w_K}{w_L + w_K}\right)^2\right].$$

Now, $$\frac{w_L - w_K}{w_L + w_K} = \frac{600 - 60}{600 + 60} = 0{\cdot}82.$$

Voltage due to first transmitted wave

$$= 10000[1 - (0{\cdot}82)^2] = 3300 \text{ volts.}$$

After an interval of 1·14 micro-seconds this is followed by the second transmitted wave. In this time the first wave will have travelled

$$1{\cdot}14 \times 10^{-6} \times 186{,}000 = 0{\cdot}212 \text{ mile.}$$

So that when the second transmitted wave emerges the 0·212 mile next to the junction is at a voltage of 3300.

The second transmitted wave raises the voltage to

$$E[1 - (0{\cdot}82)^4] = 5500 \text{ volts.}$$

The third transmitted wave raises the voltage to

$$E[1 - (0{\cdot}82)^6] = 7000 \text{ volts.}$$

The distance between fronts of second and third waves = 0·212 mile.

Further voltage increments come thus :—

After 4th wave $E[1 - (0{\cdot}82)^8] = E \times 0{\cdot}797 = 7970$ volts.
„ 5th „ $E[1 - (0{\cdot}82)^{10}] = E \times 0{\cdot}862 = 8620$ „

This process continues, the voltage eventually reaching 10,000.

EXAMPLES ON CHAPTER VII.

1. A single-phase transmission line has copper wires 0·25 in. diameter, spaced 3 ft. apart.

Calculate—

(*a*) The inductance of the line per mile of loop.
(*b*) The capacity of the line per mile of loop.
(*c*) The velocity of electric wave propagation in the conductors.
(*d*) The wave resistance of the conductors.

Answers : (*a*) 0·00365 henry ; (*b*) 0·0079 mfds. ; (*c*) 186,000 miles per sec. ; (*d*) 680 ohms.

2. A single-phase concentric cable has the following particulars :—
$d_1 = 0{\cdot}25$ in., $d_2 = 0{\cdot}85$ in. (Fig. 20). The inductance per mile of loop is 0·0012 henry. K = 3.
Calculate (*b*), (*c*), and (*d*) as in Question 1.

Answers : (*b*) 0·219 mfds. ; (*c*) 62,000 miles per sec. ; (*d*) 74 ohms.

[*Note.*—In the following examples the overhead line has the particulars calculated in Question 1, and the underground cable has particulars calculated in Question 2. Resistance and leakance to be neglected.]

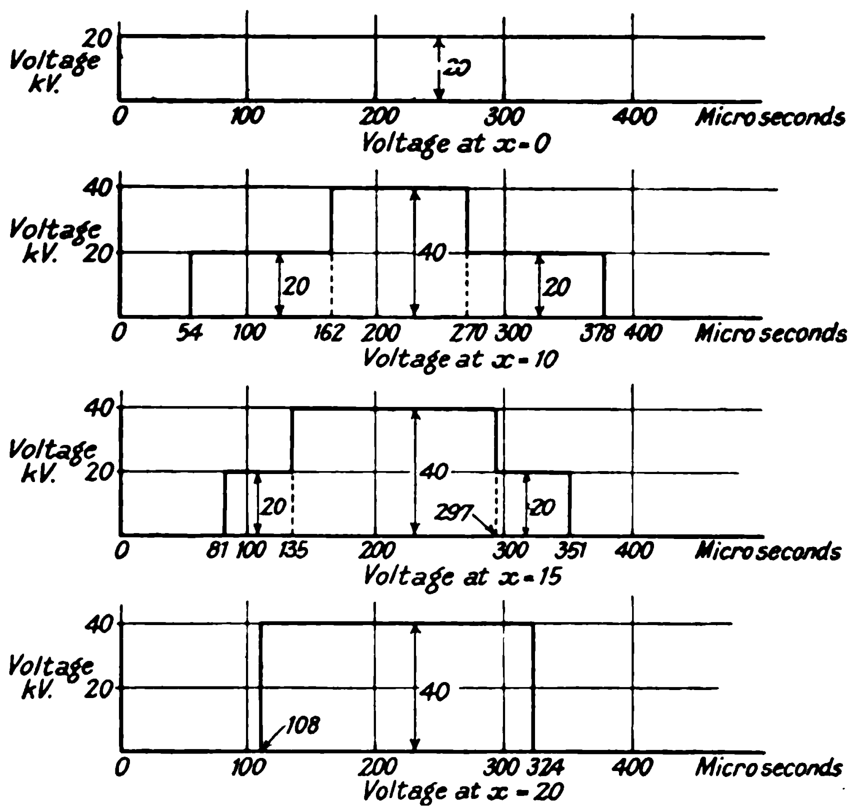


FIG. 67.—Solution to Question 5. Voltage along open-circuited line.

3. A 10,000 volt generator is suddenly switched on to the overhead line (see note above). Calculate the values of the current and voltage associated with the switch-in wave.

Answers : Voltage = 10,000 ; current = 14·7 amperes.

4. A 10,000-volt generator is suddenly switched on to the underground cable (see note above). Calculate the values of the current and voltage associated with the switch-in wave.

Answers : Voltage = 10,000 ; current = 135 amperes.

5. A voltage of 20,000 is suddenly connected to a 20-mile length of overhead line (see note above). The far end of the line is open. Plot curves showing voltage and current at various times after switching-in for the following values of x where x is the distance from generator.

(a) x = 0 miles ; (b) x = 10 miles ; (c) x = 15 miles.
(d) x = 20 miles.

(Solution is given in Figs. 67 and 68.)

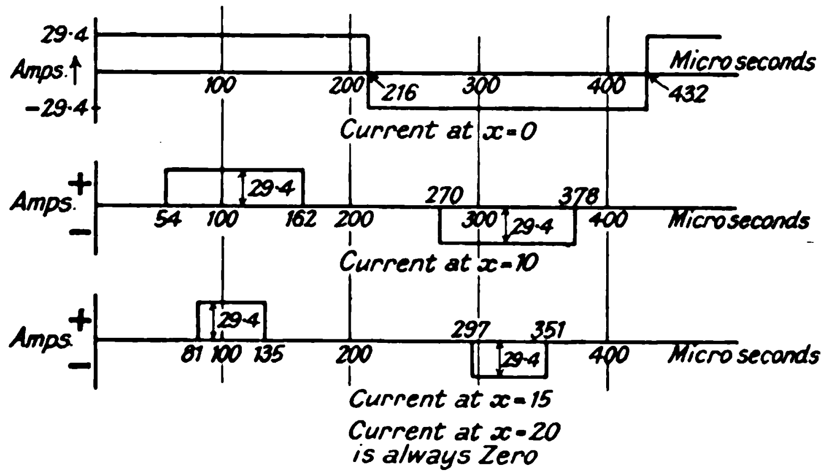


FIG. 68.—Solution to Question 5. Current along open-circuited line.

6. Repeat question 5 above, but assume now that the line is short-circuited at the far end.

(Solution is given in Figs. 69 and 70.)

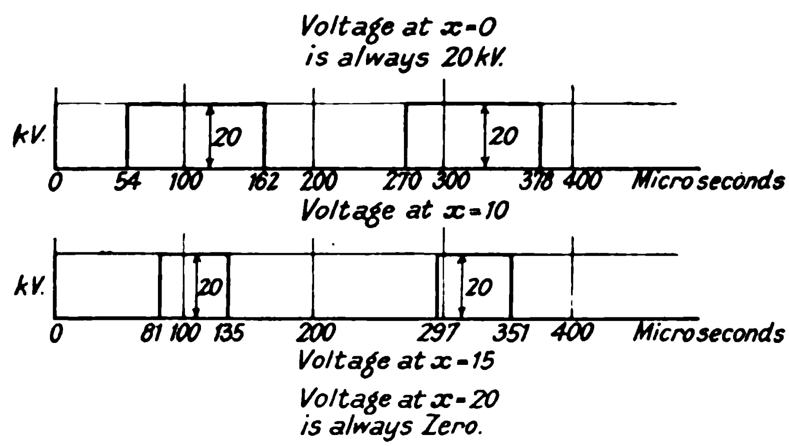


FIG. 69.—Solution to Question 6. Voltage along short-circuited line.

7. In question 5 above, assume that a non-inductive resistance of 100 ohms is connected across the far end of the line. Calculate (a) the magnitude of reflected waves of voltage and current;

(*b*) the resultant voltage at the terminals of the resistance immediately after the arrival of the switch-in wave ; (*c*) the current in the resistance.

Answers (*a*) Reflected wave of voltage = − 14,880 volts.
„ „ current = + 21·8 amperes.
(*b*) Resultant voltage = 5120.
(*c*) Current in resistance = 51·2 amperes.

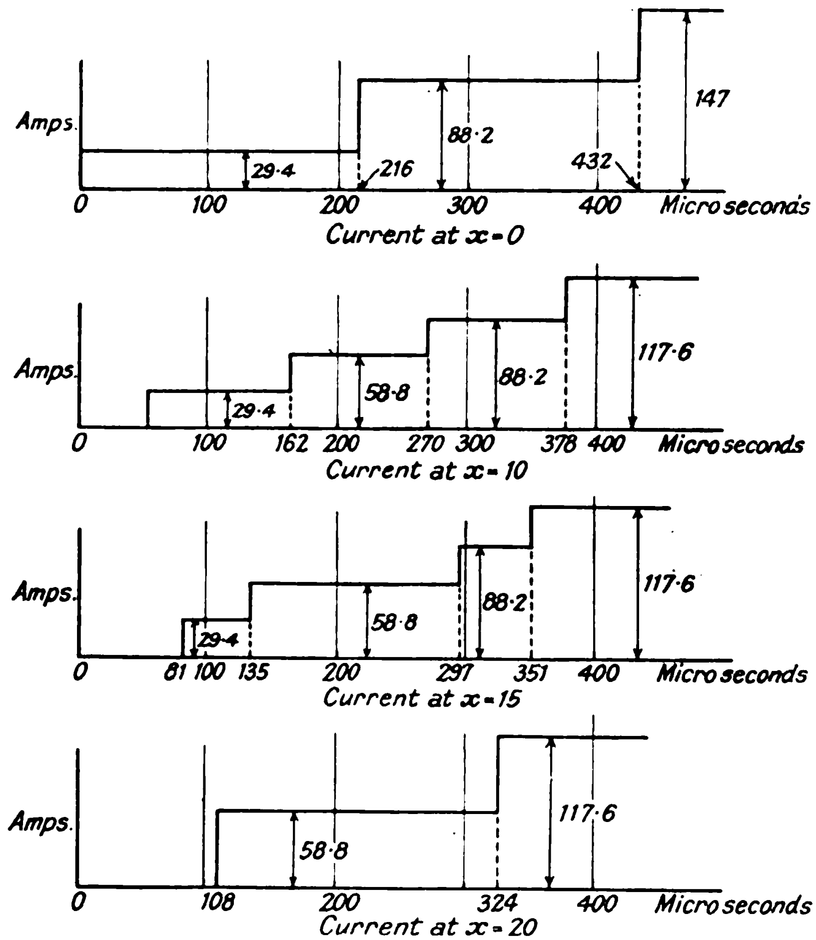


FIG. 70.—Solution to Question 6. Current along short-circuited line.

8. A 10-mile length of overhead line and a 5-mile length of cable (see note above) are connected in series. A travelling voltage wave of constant magnitude = 20,000 volts is in the overhead line moving towards the junction. When front of wave strikes junction, calculate :—

(*a*) The voltage of wave transmitted into cable.
(*b*) The voltage of the wave reflected back into overhead line.
(*c*) The voltage at the junction.

Answers : (*a*) 3930 volts ; (*b*) − 16,070 volts ; (*c*) 3930 volts.

9. A 10-mile length of overhead line and a 5-mile length of cable (see note above) are connected in series. A travelling voltage wave of constant magnitude 20,000 volts is in the cable moving towards the junction. When front of wave strikes junction, calculate :—

(*a*) The voltage of wave transmitted into overhead line.
(*b*) The voltage of wave reflected back into cable.
(*c*) The voltage at the junction.

Answers : (*a*) 36,000 volts ; (*b*) 16,000 volts ; (*c*) 36,000 volts.

10. A $\frac{1}{4}$-mile length of cable is inserted between two very long lengths of overhead line (see note above). A travelling wave whose voltage distribution is uniform and equal to 10,000 volts is travelling in one length of overhead line and is approaching the cable. Calculate :—

(*a*) The voltage of the first and second waves transmitted through cable into the second length of overhead line.
(*b*) The voltage in the second length of overhead line at a point 1 mile from its junction with the cable at times $t = 10$ and 20 micro-seconds after first transmitted wave enters second length of line.

Answers (*a*) 3550 volts, 2280 volts.
(*b*) 3550 volts, 5830 volts.

11. A 10-mile length of overhead line and a 5-mile length of cable (see note above) are connected in series. A voltage of 20,000 is impressed on the end of the overhead line, the end of the cable being open.

Calculate the voltage and current 80 micro-seconds after switching at the following places :—

(*a*) In the overhead line 6 miles from the generator.
(*b*) In the underground cable 12 miles from the generator.

Answers : (*a*) 3930 volts ; 5·73 amps. ; (*b*) zero.

12. If in the previous question the voltage is applied to the cable and the end of the overhead line is open, calculate the voltage and current 170 micro-seconds after switching-in, at the following places :—

(*a*) In the cable $\frac{1}{2}$ mile from the generator.
(*b*) In the overhead line 10 miles from the generator.

Answers : (*a*) 20,000 volts ; 270 amps. ; (*b*) 72,000 volts ; 0.

13 Obtain the law for the behaviour of a voltage surge with vertical wave-front which, after travelling in a transmission line of inductance L and capacity C per unit length, reaches a fork where the line splits into two sections having line constants L_1C_1, and

L_2C_2 respectively. Neglect resistance and attenuation and obtain the distribution of voltage and current immediately after the wave-front has reached the fork.

An overhead transmission line has a surge impedance of 700 ohms, and a voltage wave of 10,000 volts travelling along it. The wave is to be assumed of infinite length and the wave-front is vertical. At a certain point the overhead line terminates and the circuit is continued by two cables in parallel. The surge impedance of one cable is 100 ohms, and that of the other is 200 ohms. Calculate the voltage and current in the overhead line and in the two cables, immediately after the travelling wave has reached the fork.

[London University, 1927.]

Answer: Voltage transmitted into each fork

$$E' = 2E\frac{\sqrt{\frac{C}{L}}}{\sqrt{\frac{C}{L}} + \sqrt{\frac{C_1}{L_1}} + \sqrt{\frac{C_2}{L_2}}}$$

where E = incident voltage,

E′ = 1740 volts.

Reflected wave into overhead line = − 8260 volts.

Current in L_1C_1 = 17·4 amps. Current in L_2C_2 = 8·7 amps.

Original current in LC = 14·2 amps.
Reflected current in LC = 11·8 amps.
} Total current in LC = 26 amps.

CHAPTER VIII.

TYPICAL EXAMPLES OF TRAVELLING WAVES IN TRANSMISSION LINES.

THE travelling waves discussed in the previous chapter were all produced by switching a transmission line into service. Such waves may, however, be produced in a great variety of ways, and in this chapter we will study a few typical examples of travelling waves which are produced by other means than switching-in.

The three following causes of the formation of travelling waves in transmission lines will be considered :—

(*a*) Switching out an inductive load.
(*b*) Earthing the line.
(*c*) Atmospheric influences.

Method of Determining Characteristics of Waves Set Up by any Change in Circuit Conditions.—The distribution of voltage and current along a transmission line must at all times and at all places conform to the following essential requirements :—

(1) The fundamental differential equations for a transmission line must be satisfied.
(2) The initial conditions as to voltage and current must be satisfied.
(3) The boundary conditions must be satisfied.

The differential equations referred to are given in their general form as equations (124) and (125) (p. 182). Since it is assumed that $R = G = 0$, they reduce to equations (126) and (127).

A pure travelling wave, or a series of pure travelling waves, satisfies these differential equations. If this were not so, the solutions of Chapter VII. would not be valid. It can, moreover, be proved mathematically that a pure wave *does* satisfy the differential equations, and more will be said concerning this in Chapter IX.

It does not necessarily follow that the solution giving the voltage and current distributions along a line under any given set of conditions is always a pure wave. Unless requirements (2) and (3) are satisfied it will certainly *not* be the solution. As far as the differential equations are concerned, however, it is important to realise that *one or more pure waves may be injected into a line, adding their voltage and current distributions to those already existing, and the differential equations will still be satisfied.* This follows because the pure wave is itself a solution of the differential equations.

As far as the present study is concerned, time begins with the alteration of circuit conditions. Thus at zero time an inductive load is switched out of circuit, or an earth appears on the line. The initial conditions are those existing at zero time. In the first of the above cases the initial condition is that the current is zero at the switch ; in the second case, the voltage is zero at the earthed point. The previously existing voltage and current distributions no longer satisfy the initial conditions. Suppose now that a pure wave is injected into the line at the point where circuit conditions are altered, i.e. at the switch or at the earthed point. Suppose, also, that this wave is so selected that it makes the initial conditions correct. For instance, in the first of the above cases of switching out an inductive load, inject into the line at the switch a pure wave having a current distribution $-$ I, where I is the original current in the line. The current at the switch will now become zero by the addition of the two current distributions $[I + (-I) = 0]$. Thus the initial conditions will be satisfied, and the differential equations will also be satisfied. Remains only the boundary conditions. The original distributions, whatever they were, must have satisfied the boundary conditions. The injected travelling wave will also satisfy the boundary conditions by obeying the laws as to reflection, etc., which were deduced in Chapter VII.

In the problems we are here considering it will be proved that the initial conditions will be satisfied by the injection of a single pure travelling wave. The voltage and current distribution associated with it will vary with the individual problems, but in all cases it must satisfy one of the two essential requirements for pure waves, viz.,

$$I = \frac{E}{w}, \text{ or } I = -\frac{E}{w}, \text{ where } w = \text{wave resistance.}$$

The injected wave will add its voltage and current distributions to those already present. As it travels along the line it will obey the laws obtained in Chapter VII. for its reflection or transmission. In general, by reflection at the ends of the line, or at points of discontinuity, additional pure waves will be created, and at any time the total voltage and current distributions will be the sum of those due to—

(1) The original conditions.
(2) The first injected pure wave.
(3) A number of additional pure waves caused by reflection.

For all the pure waves $I = \pm \frac{E}{w}$.

Switching Out an Inductive Load.

In Fig. 71 a generator G is shown supplying a current I at voltage E through a switch S and a transmission line to an inductive load M.

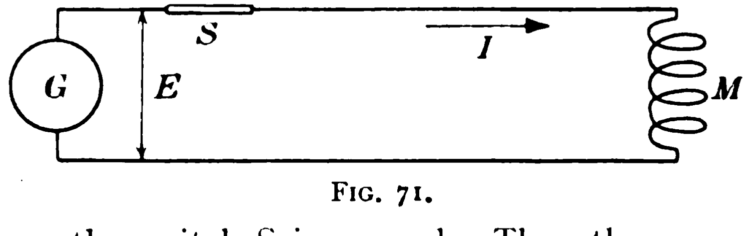


Fig. 71.

Suppose the switch S is opened. Then the current at the open end of the line must become zero The disappearance of the current and its electromagnetic energy gives rise to a wave of voltage of magnitude e where $e = I\sqrt{\frac{L}{C}}$,

L being the inductance of the line per unit length,
C being the capacitance of the line per unit length.

The disappearance of the current I starts at the switch and travels back to the load, and it may be considered as due to a pure travelling wave having current $-$ I associated with voltage e. This wave is called into existence immediately the

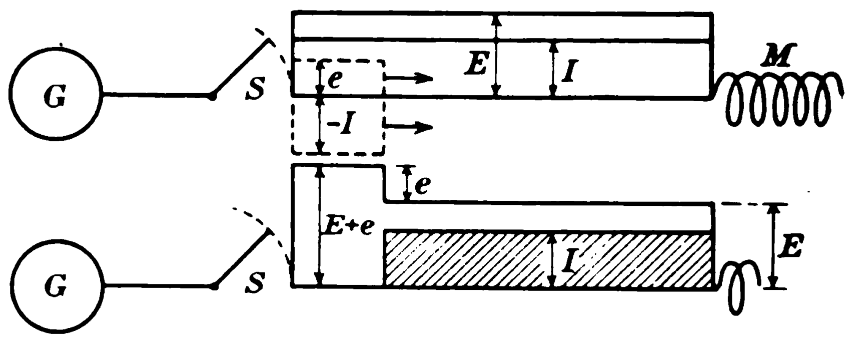


FIG. 72.

switch is opened; it is the injected pure wave which makes initial conditions correct.

In Fig. 72 the wave is drawn in the position it will occupy a little while after the switch is opened, its voltage and current being superimposed on those previously existing. The resulting voltage and current distribution along the line is also shown. When the wave reaches the load the current will be zero all along the line, and the voltage will be $E + e$. The phenomena now resulting depend upon the fact that the current in an inductance cannot change instantaneously.

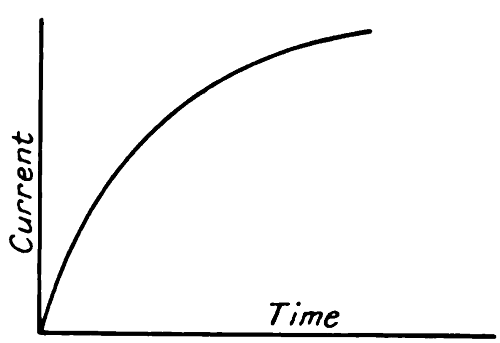


FIG. 73.

If a voltage is applied to an inductance the current in it grows according to a well-known exponential law. The shape of the current time curve is illustrated in Fig. 73. At the instant the incident wave impinges on the inductance the current is zero all along the line, but it is still I in the inductance, and this current cannot change instantaneously. Hence, since there can be no accumulation of current, there must be a wave having a current — I amperes reflected into the line from the load. Associated with this reflected wave there is a voltage distribution whose magnitude at the first instant of reflection may be found as follows. Since no additional current enters the inductance at the first instant after the arrival of the incident wave, it behaves as if w_B were infinite in the formula for the reflected voltage given on page 146, viz.,

$$\text{Reflected voltage} = Z\frac{w_B - w_A}{w_A + w_B}.$$

When $w_B = \infty$, reflected voltage $= Z =$ incident voltage. So that the reflected wave from the load starts off with a current

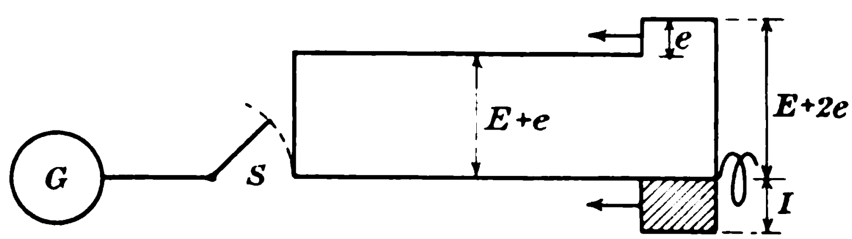


FIG. 74.

— I and voltage e, the total voltage at the load terminals being E + $2e$.

This is shown in Fig. 74.

After a very small interval of time the additional voltage which is now present at the load terminals will cause an addi-

FIG. 75.

tional current i to flow in the inductance and this current will keep on increasing up to a certain maximum (see Fig. 75).

As shown in Fig. 73, this means that the reflected current flowing from the load to the point of disconnection will steadily increase with time. Also, since current is now being transmitted into the inductance it means that in formula 115 w_B must now have some value less than ∞, and the voltage of the reflected wave will be less than e. The state of affairs will be as shown in Fig. 76, which is drawn for an instant of time a little later than Fig. 75. When the reflected wave arrives at the open end it will be reflected, the voltage at the end of the line becoming $E + 3e$. This reflected wave travels back to

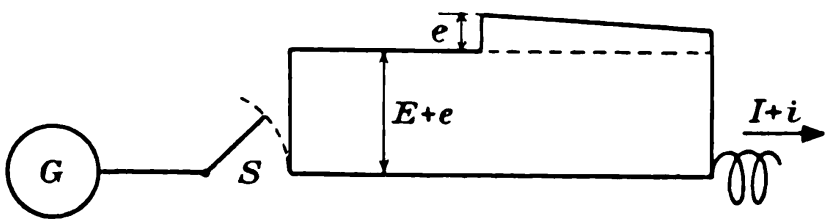


FIG. 76.

the load and the process is repeated. There is thus an incremental rise of voltage on the line.

The magnitude of the final voltage is determined from the consideration that the total electromagnetic energy which was associated with the original current must be converted into electrostatic energy. The amount of the former is $\frac{1}{2}I^2[Ll + L_0]$ where L is the inductance of the line per unit length and L_0 that of the load. The original electrostatic energy was $\frac{1}{2}CE^2l$ where C is the capacity of the line per unit length and E the original pressure. If E_f is the voltage finally attained, then

$$I^2\left[\frac{Ll + L_0}{2}\right] + E^2 \cdot \frac{Cl}{2} = E_f^2 \cdot \frac{Cl}{2},$$

the small capacity of the load being neglected.

If X is the *additional* voltage due to the disappearance of electromagnetic energy, then

$$I^2\left[\frac{Ll + L_0}{2}\right] = X^2 \cdot \frac{Cl}{2}$$

$$X = I\sqrt{\frac{Ll + L_0}{Cl}} \quad . \quad . \quad (118)$$

Since Ll and Cl are dependent upon the length of the line, the value of X, and therefore of the final pressure E_f, will be dependent upon the length of the line.

It is clear that the resulting voltage depends upon the original current. Thus inductance loads should, whenever possible, not be switched out when carrying current, as travelling waves are formed, and the voltage associated with these may be destructive. In any case, the current should be reduced to the smallest possible value before the switch is opened.

Earthing of a Line.—Travelling waves are produced when a line is earthed, and the effect will be considered by examining the special case of a line at a voltage E open at both ends. Such a case may occur when the switch connecting the line to the generator is opened. Let this line be earthed at one end, the pressure becoming zero at the earthed point.

The vanishing of the pressure signifies a liberation of electrostatic energy. This changes immediately into electromagnetic energy and provides the current i flowing to earth. The magnitude of this current is obtained from the equation of energy

$$\tfrac{1}{2}Li^2 = \tfrac{1}{2}CE^2$$

$$i = E\sqrt{\frac{C}{L}}. \qquad . \qquad . \qquad . \qquad . \qquad (119)$$

The phenomena is very similar to that associated with the switching of a voltage E on to an open-circuited line as dis-

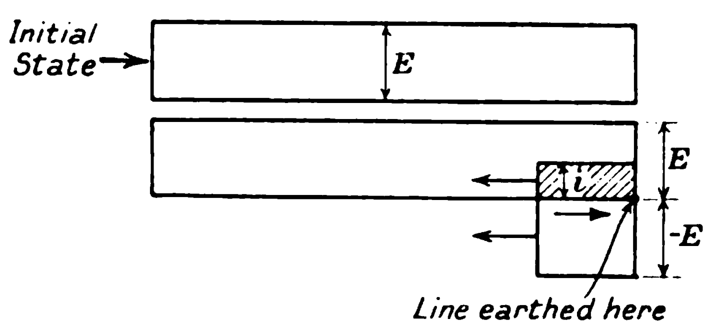


FIG. 77.

cussed on page 128. The reduction of the voltage to zero may be considered as due to the effect of a reducing wave having a voltage $-$ E injected into the line at the earthed point (Fig. 77).

The direction of current is to the right in figure, which is a positive current according to the convention adopted on page 141 (Fig. 78).

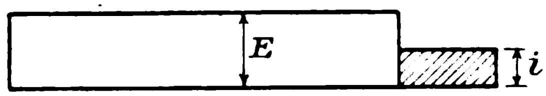


FIG. 78.

The reducing wave is reflected at the open end, where its pressure is doubled and the current drops again to zero (Fig. 79).

The current flow in the conductor is now from right to left, i.e. it is negative. The passage of the reflected wave (Fig. 79),

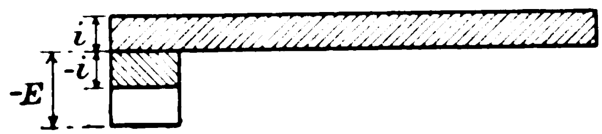


FIG. 79.

beginning from the open end, charges the system up to — E volts. When the reflected wave arrives at the earth point the whole system is at the pressure — E and the current is zero. At this point the first half period finishes (Fig. 80).

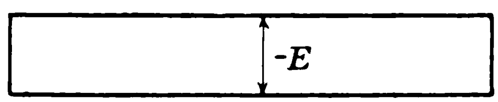


FIG. 80.

There is now again a liberation of electrostatic energy which changes to electromagnetic and produces a current — *i*, i.e. flowing from right to left in the conductor (Fig. 81). The

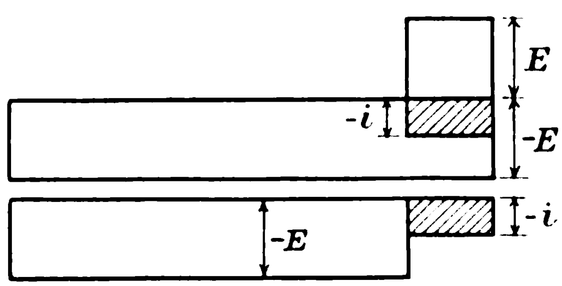


FIG. 81.

changing of the pressure from — E to zero may be considered the effect of a positive charging wave which starts at the earthed point. This wave doubles its pressure by reflection at the open end of the line, and the current disappears (Fig. 82).

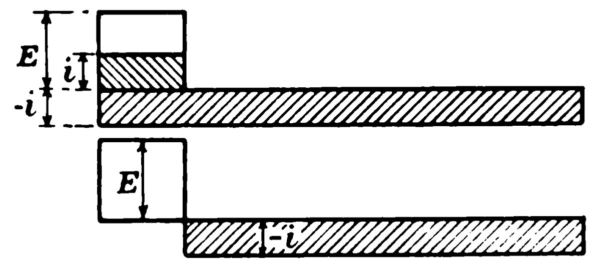


FIG. 82.

When the wave reaches the earthed point the system is again at pressure + E and the current is zero (Fig. 83). The initial

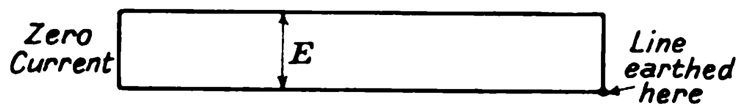


FIG. 83.

condition is now re-established, and the process can repeat itself.

The pressure on the line, therefore, makes the jumps—

E to o
o to − E
− E to o
o to E.

The current in the line is zero before earthing. At the instant of earthing a current i flows to earth. This current to earth remains constant for a time t and then becomes zero again

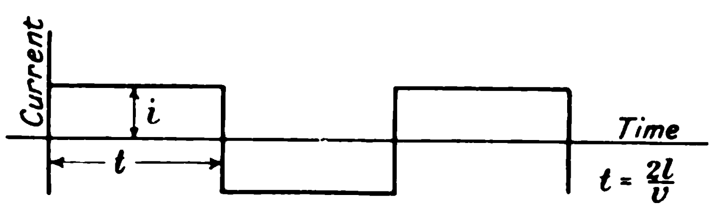


FIG. 84.

momentarily. The current now flows from earth into the conductor. It remains constant for a time and then becomes zero momentarily. This cycle repeats itself indefinitely (Fig. 84).

THE EFFECTS OF ATMOSPHERIC DISCHARGES ON OVERHEAD TRANSMISSION LINES.

In connection with transmission by overhead lines the effects associated with atmospheric electrical discharges have to be carefully studied. If a line receives a direct stroke of lightning serious damage will be caused. Fortunately this rarely occurs. Electrical disturbances in the atmosphere are, however, a source of other line phenomena which, unless guarded against, may give trouble. The disturbances give rise to two classes of phenomena, one due to the effects of indirect charges in causing the voltage of the line to increase, and the other due to effects associated with the flash-over of insulators should the induced voltage be sufficient to cause this. This latter phenomenon gives rise to " Arcing Grounds," and is a very fruitful source of trouble in overhead lines.

RISE OF VOLTAGE DUE TO ATMOSPHERIC DISTURBANCES.

Thunderstorms are caused by electrostatic charges in clouds. Owing to their electrical charges the clouds have a definite potential, so a thundercloud is a " charged body." An overhead transmission line is a conductor insulated from earth. The insulation is not perfect, the insulators allowing a small leakage current to flow to earth. In Fig. 85 a thundercloud (" charged body ") is represented as approaching the insulated conductor (overhead line). Assume that the thundercloud is positively charged. As it approaches the overhead line it will by the laws of electrostatics induce a negative charge in the parts of the conductor adjacent to the cloud and repel a positive charge to the part of the conductor remote from the cloud. This state of affairs is indicated in Fig. 85. The positive and

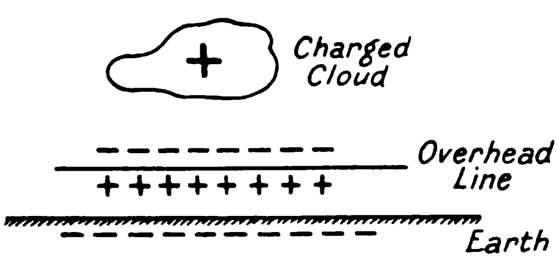


FIG. 85.—Charged cloud approaching perfectly insulated overhead line.

negative charges have this important difference. The negative charge is " bound " by the positive charge on the cloud. The positive charge on the conductor is not bound, and if a path is provided, it will escape to earth. The leakage paths over the insulators permit of a slow discharge to earth, and very often special means (such as water jets) are provided for allowing the charge to escape (Fig. 86). The approach of the cloud is so slow that the positive charge is drained off by the leakage to earth, and as the cloud approaches there will be no change in the potential of the line. When the cloud discharges, however (either to earth or to another cloud), the potential of the part

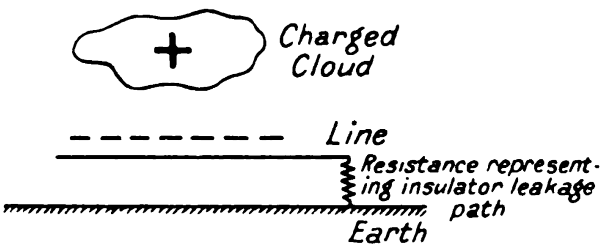


FIG. 86.—Charged cloud approaching overhead line not perfectly insulated.

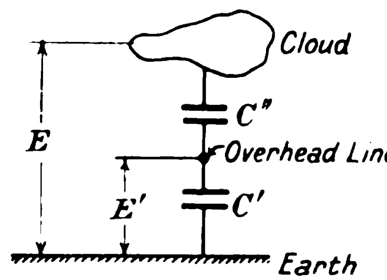


FIG. 87.—Voltage between perfectly insulated line and earth. Voltage induced by thunder-cloud.

of the line under the influence of the cloud *suddenly* rises and important phenomena result.

If the overhead line is completely insulated with no path to earth, we may represent it diagrammatically as in Fig. 87.

E = difference of potential between cloud and earth.
E_1 = difference of potential between line and earth.
C' = capacity of transmission line to earth.
C'' = capacity of cloud to transmission line.

By the laws for condensers in series

$$E_1 = E\frac{C''}{C' + C''} \quad . \quad . \quad . \quad (120)$$

As the cloud approaches the line, C'' increases, C' remains constant, hence E_1 increases. If then the line were completely insulated, with the approach of the thundercloud there would be a slow rise in the voltage of the line. As we have seen, however, in an actual line with leaky insulators the repelled positive charge escapes to earth through whatever paths are available for it, and no increase of line voltage due to the thundercloud results.

The bound charge is confined to a short length and is not evenly distributed over this length. The intensity of the charge is greatest near the centre of the cloud, and diminishes towards the edges of the cloud.

If the cloud discharges, e.g. by a stroke of lightning to earth or between clouds, the bound charge is suddenly liberated and the line assumes a certain potential. The leaky insulators immediately begin to draw this charge off, but this takes time.

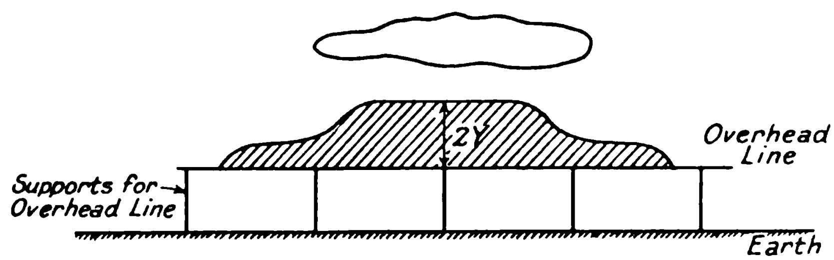


Fig. 88.—Voltage of overhead line raised by sudden discharge of thundercloud.

Assuming that the charge of the same polarity as the cloud has completely escaped to earth, the electrical state of the conductor immediately following the discharge of the cloud is as shown in Fig. 88, where the induced bound charge is assumed to have a new voltage 2Y.

The phenomena which follow are very similar to that associated with the switching on of a transmission line, discussed on page 128.

The bound atmospheric charge has, on release, two paths available for its transmission. The paths are identical and therefore both will be utilised equally. Consequently, the charge spreads in both directions, two pure travelling waves being set up, one to the right and the other to the left, as shown in Fig. 89. As the bound charge does not have a steep front, neither will these travelling waves have steep fronts. This is fortunate, as the stresses induced by the travelling waves in transformers and other machines will be reduced.

To deal with the effects of the travelling waves in the manner developed in the preceding pages we will assume, however, that the charge liberated on the conductor has rectangular form. The charge cannot remain at the place where it was liberated, but will be propagated along the line with the velocity

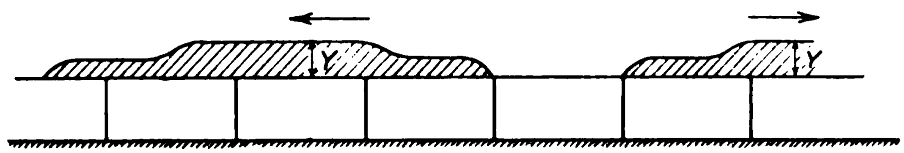


FIG. 89.—Travelling waves set up by discharge of thundercloud.

of light. Since there are two identical paths offered, the charge will divide equally between the two paths. If we assume that the original charge at a pressure 2Y is made up of two equal charges each at a pressure Y, then these two charges will move one to the right and the other to the left.

If C is the capacity per unit length of the line and X the length of the original bound charge, then the energy stored in the original bound charge is all electrostatic, and is given by

$$A = \frac{4CXY^2}{2} = 2CXY^2.$$

The two travelling partial waves possess both electrostatic and electromagnetic energy. The electrostatic energy of each wave is

$$A'_e = \frac{CXY^2}{2}.$$

If L is the inductance of the line per unit length, and i the current in each wave, then the electromagnetic energy of each is

$$A'_m = \frac{LXi^2}{2}.$$

The total energy of the travelling waves is $2[A'_e + A'_m]$. This total energy must equal A.

$$\therefore A = 2[A'_e + A'_m]$$
$$2CXY^2 = CXY^2 + LXi^2.$$

The well-known equation results

$$Y = i\sqrt{\frac{L}{C}}, \text{ or}$$

$$i = Y\sqrt{\frac{C}{L}} \quad . \quad . \quad . \quad . \quad (121)$$

So that when a thundercloud discharges in the neighbourhood of an overhead line, two equal travelling waves moving in opposite directions are set up in the line. Each of these waves has a voltage Y and a current $i = Y\sqrt{\frac{C}{L}}$ associated with it. These waves oscillate backwards and forwards along the line subject to the laws already discussed for travelling waves. They are reflected from ends of the line, and at a certain point in the line the reflected waves meet again and form the original wave again at voltage 2Y. The process then begins again. In practice, as the waves oscillate backwards and forwards along the line they suffer an increasing damping effect, due to the resistance of the line.

It should be carefully noted that 2Y is the voltage of the bound charge, and has no connection whatever with the ordinary voltage of the line.

Arcing Grounds.

In the discussion of the previous section it was assumed that the rise of voltage due to the bound charge was not sufficiently great to cause flash-over of the line insulators. It was pointed out that the voltage rise was quite arbitrary, having no relation to the normal working voltage of the line. Whether any thundercloud will or will not cause the insulators to flash over and give rise to an arcing ground, will depend upon two factors :—

(1) The voltage rise induced by the cloud.

(2) The flash-over voltage of the line insulators.

Regarding (2), the following figures for flash-over are abstracted from the B.E.S.A. specification for overhead line insulators :—

Working Voltage.	Flash-over Voltage.	
	Insulator Dry.	Insulator Wet.
3 K.V.	40 K.V.	20 K.V.
6 ,,	50 ,,	30 ,,
10 ,,	62 ,,	39 ,,
20 ,,	95 ,,	62 ,,
30 ,,	125 ,,	84 ,,
60 ,,	165 ,,	140 ,,
100 ,,	300 ,,	220 ,,

It will be seen that a reasonable factor of safety is allowed between flash-over and working voltage. As, however, the rise of voltage induced by a thundercloud is arbitrary, it will frequently happen that an insulator will flash over during a thunderstorm.

A transmission line normally has a certain potential with respect to earth. If an insulator breaks down, the potential of the line at this point suddenly drops to zero. The resulting phenomena are similar to those discussed on page 169 ("Earthing of the Line"). Since two identical paths are offered, discharge waves will be propagated in both directions along the line. These waves have a steep front and an amplitude equal to the flash-over voltage of the insulators. Thus, following on the insulator flash-over, there will be travelling waves moving in each direction from the earthed point. In each direction the waves may be considered to be divided into two distinct categories, viz. those associated with the movement of the original bound charge and those due to the earthing of the line through the insulator flash-over.

The waves due to the flash-over are particularly dangerous, as the wave-front is steep, while that of the bound charge is sloping.

Steep-fronted Waves Produced by the Screening Effect of Buildings, Transformer Tanks, etc.—Travelling waves having steep-fronted characteristics may be produced without insulator flash-over if the thundercloud is near a building which the line enters. The building screens all the electrical apparatus within it from the influence of the electrostatic field produced by the thundercloud. Thus the bound charge on the line must end abruptly at the walls of the building.

An outdoor transformer tank acts in the same way as the building, and effectually screens the transformer within it.

When the cloud discharges to earth, say, the bound charge is freed and spreads in both directions. The wave travelling in one direction will enter the building. The steepness of its front will make it especially dangerous to the machinery.

Note regarding travelling waves produced when an alternating voltage is applied to end of line.

The theory of travelling waves due to switching has been developed in Chapter VII. on the assumption that the applied voltage is constant. If an alternating voltage is applied to the end of the line, then the theory obtained will still hold as the alternating voltage is equivalent to a succession of constant voltages each lasting for a very short time. Moreover the velocity of propagation of the waves is so rapid compared with the slow cyclical change in the applied voltage, and in most actual lines the damping effect of resistance and leakance will be so great, that during the time the transient phenomena persist the applied voltage is practically constant.

A more accurate discussion of the transient phenomena associated with the switching of an alternating voltage on to a line will be found in Chapter IX.

CHAPTER IX.

INTRODUCTION TO MATHEMATICAL THEORY OF TRAVELLING WAVES.

General Form of Differential Equations for Voltage and Current along Transmission Line.—In the general theory of transmission given in Chapter II. it was assumed, when studying the alternating current case (p. 24), that

$$i = \mathcal{I} \sin wt \text{ and}$$
$$v = \mathcal{V} \sin (wt + \phi).$$

Based on these assumptions, the differential equations (36) and (37) were obtained, and their solution gave expressions for the voltage and current along the line (equations 41 and 42).

A little consideration will show that in their most general form the differential equations for a transmission line are

$$-\frac{dV}{dx} = IR + L\frac{dI}{dt} \quad . \quad . \quad . \quad (122)$$

$$-\frac{dI}{dx} = GV + C\frac{dV}{dt} \quad . \quad . \quad . \quad (123)$$

Equations (36) and (37) are particular cases of equations (122) and (123), and hold only when current and voltage may be assumed to vary sinusoidally with time. It will be proved in this chapter that the assumption is correct only after some time has elapsed, since circuit conditions were last altered, i.e. when the line is in a " steady state " electrically. Before this state is reached the assumption is incorrect and various transient voltages and currents occur. Equations (41) and (42) take no account of these, since they hold only for the steady state.

Similarly, for the direct current case studied in pages 19 to 23, the assumption has been made that at any point along the line, current and voltage are independent of time, i.e.

$$\frac{dI}{dt} = 0, \text{ and}$$

$$\frac{dV}{dt} = 0.$$

Equations (122) and (123) now reduce to

$$-\frac{dV}{dx} = IR$$

$$-\frac{dI}{dx} = GV$$

which are equations (28) and (29) (p. 19). The assumption that $\frac{dI}{dt}$ and $\frac{dV}{dt}$ are zero will be proved to be correct only when the line is in a steady state. As in the alternating current case, we will see that before the steady state is reached various transient phenomena occur.

The general equations for the distribution of voltage and current along a transmission line are equations (122) and (123). A general solution of these equations, satisfying the boundary conditions for each case, would give the values of voltage and current along the line at *all instants*, and must therefore include the transient conditions. In the treatment followed in the preceding pages the transient conditions have been dealt with in Chapter VII, and the steady conditions in Chapter V, and the relation between the two is by no means obvious. In the present chapter the theory which connects the solutions for the transient and steady conditions will be given.

The mathematics involved in a complete general solution of the differential equations (122) and (123) is difficult and beyond the scope of this work, and for such solution reference must be made to more advanced works.* It is in order to avoid the mathematical difficulties involved in a complete solution that

* "The Propagation of Electric Currents in Telephone and Telegraph Conductors," J. A. Fleming; also, "Electromagnetic Theory," Oliver Heaviside: Benn Bros.

the methods of Chapters V. and VII. have been followed. All the solutions given there are really particular solutions of the differential equations, the solutions of Chapter V. holding when all the transient terms have disappeared and the solutions of Chapter VII. holding only for an exceedingly small space of time after circuit conditions have been altered. In actual fact, the solutions for transient and steady conditions merge into one another and are contained in a general solution of the differential equations. Immediately after switching a voltage on to a line the phenomena described under transient conditions occur, while some time later a steady state is reached and the solutions of Chapter V. are correct; but between the transient state and the steady state neither method gives correct solutions.

In practice this is of no great importance. The steady conditions are those with which we are usually concerned, and the methods of Chapter V. apply for these. The transient conditions are chiefly of importance because of their effect in causing the building up of voltage which may result in insulation breakdown. Now it can be shown that the voltage rises actually occurring will (owing to leakance and the damping effect of the resistance of the line) always be less than those calculated by the methods of Chapter VII. So if transients are thus calculated and provision made for dealing with them, the transients which arise in practice should not give any trouble. Hence the methods and solutions of Chapters V. and VII. should meet most practical cases, and they have the advantage of greatly simplifying the mathematical treatment.

A study of the complete mathematical treatment of the problem of travelling waves is helpful in several important respects. It serves to show how the transient and steady conditions merge into one another. Further, it will be proved that the solutions for the transients as obtained from the differential equations agree with those given in Chapter VII. and obtained by a quite different method. Also, the mathematical treatment will be helpful by indicating the effect on the transients of the leakance and resistance of the line, which have been neglected in Chapter VII.

Notation.—In Part I. the following symbols were used :—

V = voltage at any point.
e = base of Napierian logarithms.
I = current at any point.

In this chapter the notation will be altered, it being considered advisable to do this to make it agree with that usually employed by writers when dealing with the mathematical theory of travelling waves. In what follows the notation will be

e = voltage at any point.
ϵ = base of Napierian logarithms.
i = current at any point.

General Solution of Fundamental Differential Equations for Transmission Line :—

The fundamental equations are

$$-\frac{de}{dx} = i\mathrm{R} + \mathrm{L}\frac{di}{dt} \quad . \quad . \quad . \quad (124)$$

$$-\frac{di}{dx} = \mathrm{G}e + \mathrm{C}\frac{de}{dt} \quad . \quad . \quad . \quad (125)$$

The general solution of these simultaneous differential equations for any set of boundary conditions (open-circuited line, short-circuited line, etc.) is difficult, and beyond the scope of this work, but the following special cases are susceptible of simple treatment.

Solution when Resistance and Leakance are neglected.—The assumption of no resistance and no leakance has been made in the theory already given for transients in Chapter VII., and the mathematical solution should agree with the results obtained therein.

The differential equations for current and voltage are now

$$\frac{di}{dx} = -\mathrm{C}\frac{de}{dt} \quad . \quad . \quad . \quad (126)$$

$$\frac{de}{dx} = -\mathrm{L}\frac{di}{dt} \quad . \quad . \quad . \quad (127)$$

For the complete mathematical treatment leading to a solution of these equations, reference should be made to the works previously mentioned (p. 180). The solutions only will be given here without proof.

The complete solution of the equations gives e as

$$e = F(x - vt) + G(x + vt) \text{ where } v = \sqrt{\frac{1}{CL}} \quad . \quad (128)$$

The solution $e = F(x - vt)$ represents a voltage distribution given at time $t = 0$ by $e = F(x)$, and which moves in the direction of *increasing* values of x at velocity v without change of magnitude or shape.

Similarly, the solution $e = G(x + vt)$ represents a voltage distribution given at time $t = 0$ by $e = G(x)$, and which moves in the direction of *decreasing* values of x at velocity v without change of magnitude or shape.

The values of the functions $F(x)$ and $G(x)$ must be determined by reference to the conditions of the problem (line open-circuited or short-circuited, etc.).

As it is of importance to understand this part of the work clearly we will consider one or two examples. Suppose, for example, the terminal conditions are such that at time $t = 0$, $F(x) = x$. The solution $e = F(x - vt)$ is now $e = x - vt$ at any time t, and can be determined by plotting thus—

vt.	Equation of volts along line.
0	$e = x$.
1	$e = x - 1$.
2	$e = x - 2$.
3	$e = x - 3$.

The distribution of voltage along the line at various instants is shown in Fig. 90.

An inspection of Fig. 90 makes it clear that the voltage distribution is moving to the right, without change, at a velocity v.

As a further example, suppose that the terminal conditions necessitate a solution where $F(x) = \sin x$ when $t = 0$. We then have

vt.	Equation of volts along line.
0	$e = \sin x$.
1	$e = \sin (x - 1)$.
2	$e = \sin (x - 2)$.
3	$e = \sin (x - 3)$.

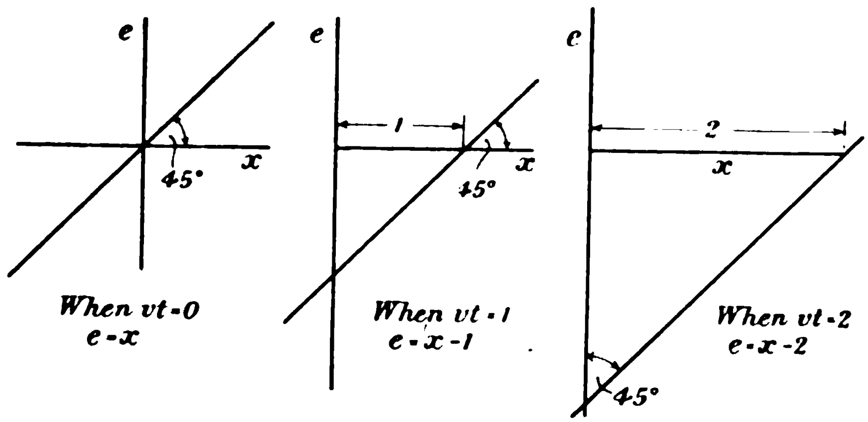


FIG. 90.—Equation $e = x - vt$ plotted for various values of vt.

These equations are plotted in Fig. 91, and an inspection shows that, as before, the voltage distribution is moving to the right, without change, at velocity v.

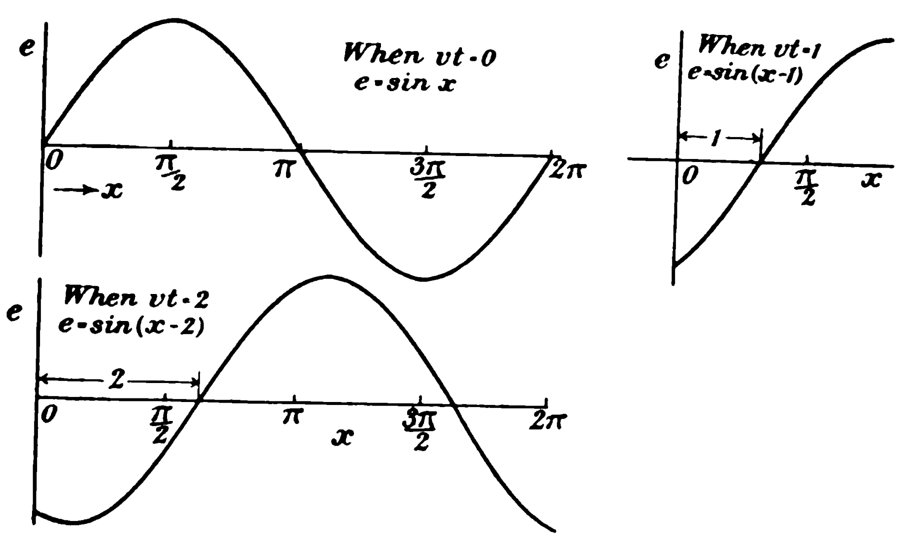


FIG. 91.—Equation $e = \sin (x - vt)$ plotted for various values of vt.

Similarly, $G(x + vt)$ can be shown to be a voltage distribution which moves to the left with velocity v.

It follows that where the voltage distribution at any time t is given by $e = F(x - vt) + G(x + vt)$, this distribution can be determined thus: Obtain the values of $F(x)$ and $G(x)$ which satisfy the boundary conditions. Then allow these voltage distributions to move, the one to the left and the other to the right, with velocity v. Their addition will give the resultant distribution at any instant.

Before the values of $F(x)$ and $G(x)$ can be found, it is necessary to fix the boundary conditions, and they can then be mathematically determined. In the following discussion the values of $F(x)$ and $G(x)$ will be given, but without proof, for each set of boundary conditions. It should be understood that the theory which follows is based upon the acceptance without proof of two things, viz. :—

(1) The expression for e as $e = F(x - vt) + G(x + vt)$.
(2) The values of $F(x)$ and $G(x)$ for each particular set of boundary conditions.

It should be noted that x in the differential equations can have any value between plus and minus infinity. The actual physical length of the line includes only those values of x which lie between 0 and l, but values of $F(x)$ and $G(x)$ exist outside these limits.

Distant-end of Line Open-circuited.—Suppose a voltage E is applied to a line of length l. The boundary conditions are

$e = E$	when	$x = 0$	for all values of	t,
$e = 0$	,,	$t = 0$	,, ,,	x
$i = 0$	,,	$x = l$	,, ,,	t

We know the solution is of the form

$$e = F(x - vt) + G(x + vt),$$

and it can be proved that the boundary conditions necessitate that at time $t = 0$ the values of $F(x)$ and $G(x)$ are as given in the table :—

VALUES OF F(x) AND G(x). $t = 0$.

Range of x.	F(x).	G(x).
↑	↑	
$-4l$ to $-2l$	0	—
$-2l$ „ 0	E	—
0 „ l	0	0
l „ $2l$	—	0
$2l$ „ $4l$	—	E
$4l$ „ $6l$	—	0
↓		↓

These values for F(x) and G(x) are more complex than those taken in the typical examples given on page 183 [F(x) = x and F(x) = sin x], but the results there obtained still hold.

Thus the two infinite series of rectangles represented by the above values for F(x) and G(x) move, one to the right and the other to the left, with velocity v. At time $t = 0$ they will be as represented in Fig. 92. At various other times they will be as represented in Fig. 93. It is seen that these two infinite series of rectangles, the one series moving to the right with velocity v, and the other to the left at the same speed, produce all the effects in the length 0 to l (the actual physical line) which have previously been described in Chapter VII., pages 132 to 136. As before, it is seen that the voltage wave upon striking the open end is reflected without change of sign, the voltage at the open end being thereby doubled.

Current Waves.—So far we have dealt only with the voltage along the line. We will now consider the current.

The solution for e is

$$e = \mathrm{F}(x - vt) + \mathrm{G}(x + vt).$$

The general differential equations [R = G = 0] are

$$\frac{di}{dx} = -\mathrm{C}\frac{de}{dt},$$

$$\frac{de}{dx} = -\mathrm{L}\frac{di}{dt}.$$

Now, if $(x - vt) = u$ say,

then $$\frac{d}{dx}\mathrm{F}(u) = \frac{d}{du}\mathrm{F}(u) \times \frac{du}{dx} = \mathrm{F}'(u) \cdot \frac{du}{dx}$$

where $\mathrm{F}'(u) = \dfrac{d}{du}\mathrm{F}(u)$,

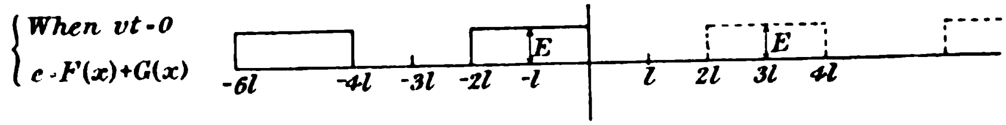


FIG. 92.—Values of F(x) and G(x) for open-circuited line.

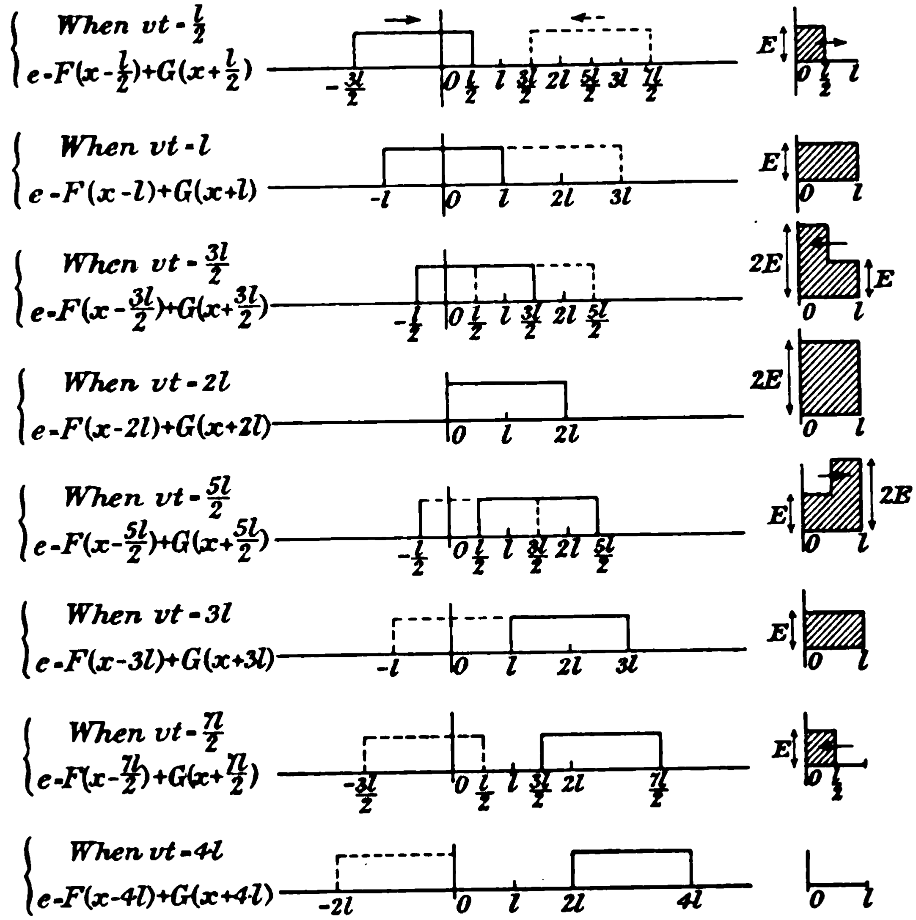


FIG. 93.—Voltage distribution along open-circuited line $e = \mathrm{F}(x - vt) + \mathrm{G}(x + vt)$. $\mathrm{F}(x - vt)$ and $\mathrm{G}(x + vt)$ are shown for various values of vt. Voltage distributions from $x = 0$ to $x = l$ are on right of figure.

i.e. $$\frac{d}{dx}\mathrm{F}(x - vt) = \mathrm{F}'(x - vt) \times 1 = \mathrm{F}'(x - vt).$$

Also, $$\frac{d}{dt}\mathrm{F}(u) = \frac{d}{du}\mathrm{F}(u) \times \frac{du}{dt} = \mathrm{F}'(u)\,.\,\frac{du}{dt},$$

i.e. $$\frac{d}{dt}\mathrm{F}(x - vt) = \mathrm{F}'(x - vt) \times -v = -v\mathrm{F}'(x - vt).$$

Now, $$\frac{di}{dx} = -\mathrm{C}\frac{de}{dt} \text{ and } e = \mathrm{F}(x - vt) + \mathrm{G}(x + vt).$$

$$\therefore \frac{di}{dx} = -\mathrm{C}\left[\frac{d}{dt}\mathrm{F}(x - vt) + \frac{d}{dt}\mathrm{G}(x + vt)\right]$$

$$= \mathrm{C}v[\mathrm{F}'(x - vt) - \mathrm{G}'(x + vt)] \quad . \quad (129)$$

Further, $\frac{de}{dx} = -\mathrm{L}\frac{di}{dt}.$

Since $e = \mathrm{F}(x - vt) + \mathrm{G}(x + vt)$

$$\frac{de}{dx} = \mathrm{F}'(x - vt) + \mathrm{G}'(x + vt).$$

$$\therefore \frac{di}{dt} = -\frac{1}{\mathrm{L}}[\mathrm{F}'(x - vt) + \mathrm{G}'(x + vt)] \quad . \quad (130)$$

Also, $\frac{di}{dx} = \mathrm{C}v[\mathrm{F}'(x - vt) - \mathrm{G}'(x + vt)] \quad . \quad (129)$

Integrating equation (129),

$$i = \mathrm{C}v[\mathrm{F}(x - vt) - \mathrm{G}(x + vt)] \quad . \quad . \quad (131)$$

This also has to satisfy equation (130).

Differentiating (131) with respect to t we have

$$\frac{di}{dt} = \mathrm{C}v[-v\mathrm{F}'(x - vt) - v\mathrm{G}'(x + vt)].$$

$$= -\mathrm{C}v^2[\mathrm{F}'(x - vt) + \mathrm{G}'(x + vt)].$$

This satisfies equation (130) when

$$\mathrm{C}v^2 = \frac{1}{\mathrm{L}},$$

i.e. $v = \sqrt{\frac{1}{\mathrm{CL}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$

$$\therefore i = \sqrt{\frac{\mathrm{C}}{\mathrm{L}}}[\mathrm{F}(x - vt) - \mathrm{G}(x + vt)] \quad . \quad . \quad (133)$$

Comparing this with $e = \mathrm{F}(x - vt) + \mathrm{G}(x + vt)$, we see that the current associated with the wave of voltage $\mathrm{F}(x - vt)$ is in phase with it for all values of t, and its value will be the corresponding value of the voltage divided by the surge impedance $\sqrt{\frac{\mathrm{L}}{\mathrm{C}}}$. The current associated with the wave of voltage $\mathrm{G}(x + vt)$ is obtained by dividing the voltage by $\sqrt{\frac{\mathrm{L}}{\mathrm{C}}}$ and then reversing it.

It should be noted that this theorem connecting voltage and current is independent of the boundary conditions.

In Fig. 94 the current waves are given for the open-circuited line.

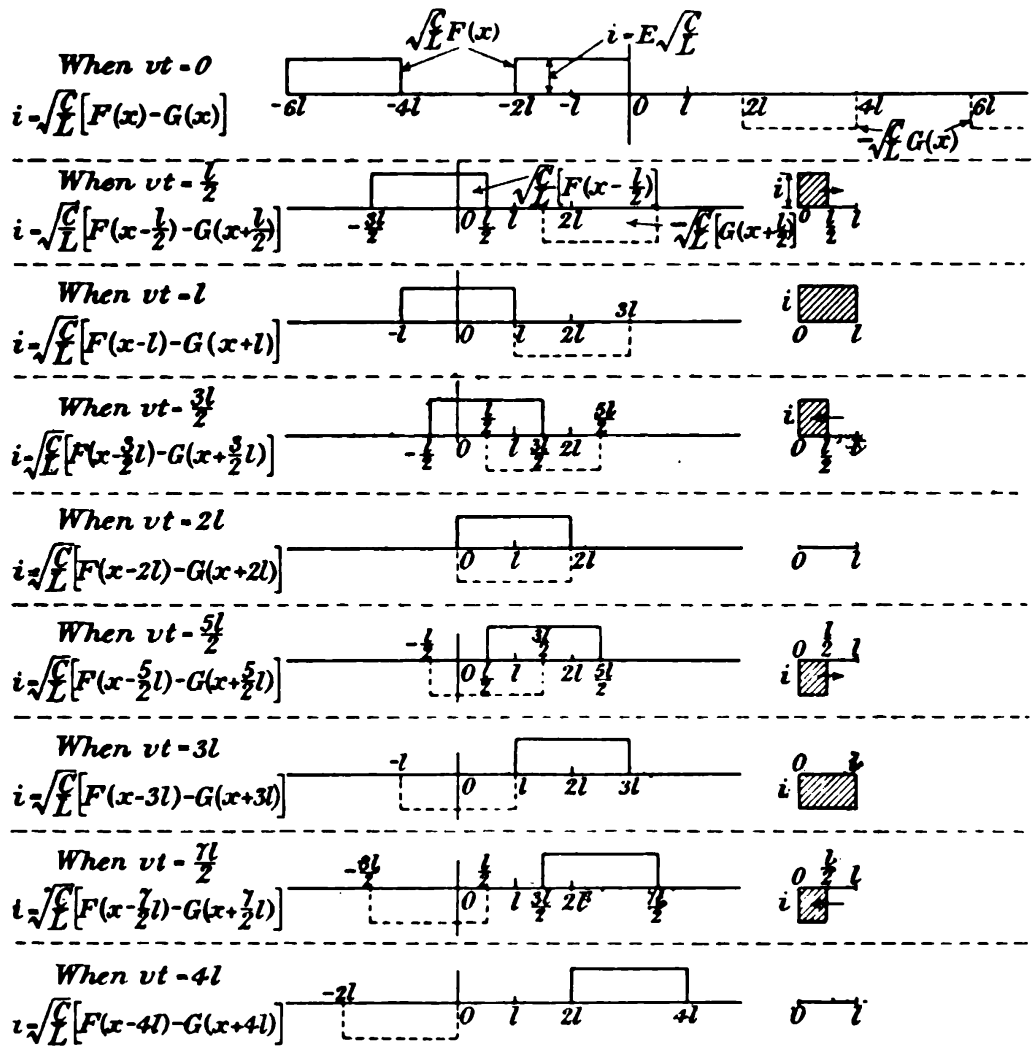


FIG. 94.—Current distribution along open-circuited line

$$i = \sqrt{\frac{C}{L}}\Big[F(x - vt) - G(x + vt)\Big]. \quad \sqrt{\frac{C}{L}}F(x - vt) \text{ and } \sqrt{\frac{C}{L}}G(x + vt)$$

are shown for various values of vt. Current distributions from $x = 0$ to $x = l$ are on right of figure.

Far-end of Line Short-circuited.—Here the boundary conditions are

$e = E$ when $x = 0$,
$e = 0$,, $t = 0$ for all values of x between 0 and l.
$e = 0$,, $x = l$ for all values of t.

The solutions for $F(x)$ and $G(x)$ which satisfy these boundary conditions are found to be as given in the following table :—

AT TIME $t = 0 \begin{cases} F(x - vt) = F(x). \\ G(x + vt) = G(x). \end{cases}$

Range of x.	F(x).	G(x).
↑	↑	
$-6l$ to $-4l$	3E	—
$-4l$,, $-2l$	2E	—
$-2l$,, 0	E	—
0 ,, l	0	0
l ,, $2l$	—	0
$2l$,, $4l$	—	$-$ E
$4l$,, $6l$	—	$-$ 2E
↓		↓

As before, the solution is represented by two infinite series of flat-topped figures, and they are shown in Fig. 95. The distribution of voltage at any other instant than $t = 0$ can be obtained in the usual way, by allowing the figure representing F(x) to move to the right and that representing G(x) to move to the left, and then adding the two together. The actual physical length of the line being l, we are concerned with obtaining results only for values of x between 0 and l. In Fig. 95 the distribution of voltage from 0 to l at various instants is given. As in the case of the line open-circuited at the end, the results are found to agree with those obtained in Chapter VII. (p. 136).

The current can be obtained from the differential equations as on page 188. It is there proved that, independent of boundary conditions, the current associated with any voltage e is $i = \pm \frac{e}{\sqrt{\frac{L}{C}}}$, which is the formula on page 129.

The positive sign is taken for the F distribution and the negative sign for the G distribution (see Fig. 96).

Solution when Resistance and Leakance are Included and when $\frac{R}{L} = \frac{G}{C}$.—The general solution including resistance and leakance is dismissed on page 182 as being too difficult for inclusion in this book. When $\frac{R}{L} = \frac{G}{C}$, however, the mathematical work is greatly simplified, and some space will now be devoted to considering the solution when the above relation holds between the primary constants of the transmission line.

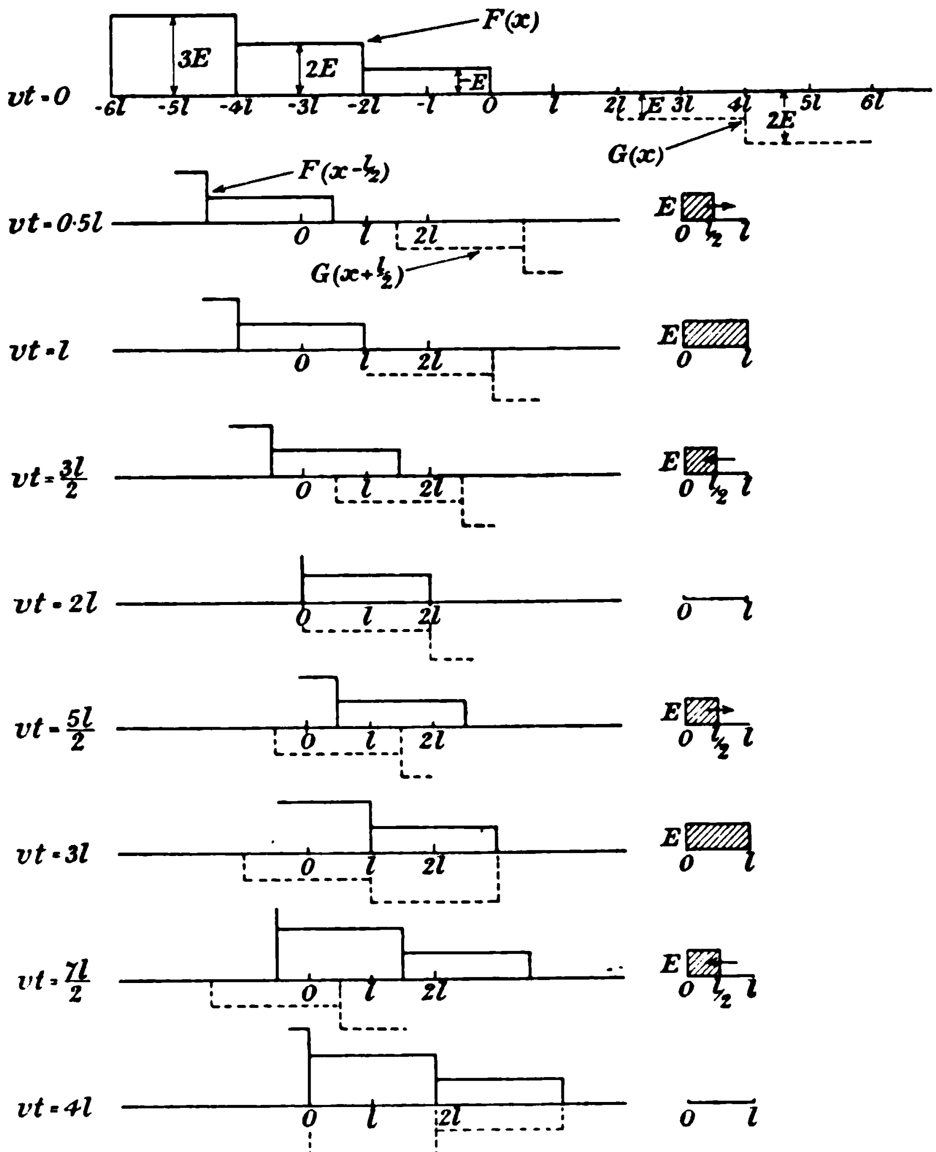


FIG. 95.—Voltage distribution along short-circuited line $e = F(x - vt) + G(x + vt)$. $F(x - vt)$ and $G(x + vt)$ are shown for various values of vt. Voltage distributions from $x = 0$ to $x = l$ are on right of figure.

Since the amount of leakage indicated by this relation is much greater than would be present on a power transmission line, the results obtained have no direct practical application to such lines. They are, however, interesting and instructive as showing qualitatively the effect of resistance and leakage in

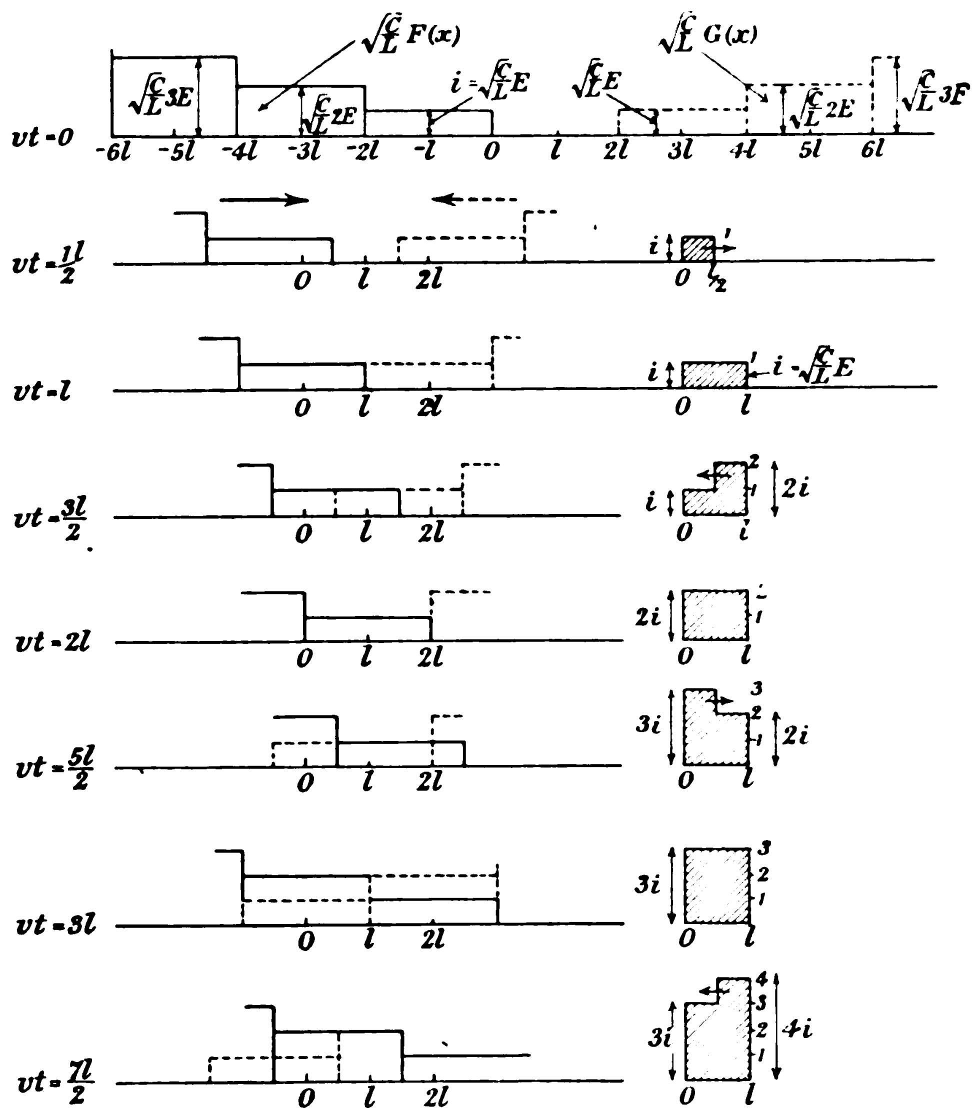


FIG. 96.—Current distribution along short-circuited line

$$i = \sqrt{\frac{C}{L}}\left[F(x - vt) - G(x + vt)\right]. \quad \sqrt{\frac{C}{L}}F(x - vt) \text{ and } \sqrt{\frac{C}{L}}G(x + vt)$$

are shown for various values of vt. Current distributions from $x = 0$ to $x = l$ are on right of figure.

modifying the results obtained when these quantities are neglected.

The solution of the differential equations

$$\left.\begin{aligned} -\frac{de}{dx} &= i\mathrm{R} + \mathrm{L}\frac{di}{dt} \\ -\frac{di}{dx} &= \mathrm{G}e + \mathrm{C}\frac{de}{dt} \end{aligned}\right\} \quad \text{when } \frac{\mathrm{R}}{\mathrm{L}} = \frac{\mathrm{G}}{\mathrm{C}}$$

is
$$e = \epsilon^{-\frac{\mathrm{R}}{\mathrm{L}}t}[\mathrm{F}(x - vt) + \mathrm{G}(x + vt)] \quad . \quad . \quad (134)$$

for the voltage, and a similar expression for the current. The solution for a line open-circuited at its far end is subject to the terminal conditions—

$e = \mathrm{E}$ at $x = 0$,
$i = 0$,, $x = l$,
$e = 0$ when $t = 0$ for all values of x between 0 and l,
$i = 0$,, $t = 0$,, ,, ,, ,,

These boundary conditions result in the following three equations :—

$$\mathrm{F}(x) = \mathrm{G}(x) = 0 \text{ for values of } x \text{ between 0 and } l \quad (134a)$$
$$\mathrm{F}(x) = -\mathrm{G}(-x) + \mathrm{E}\epsilon^{-ax} \quad . \quad . \quad . \quad . \quad (134b)$$
$$\mathrm{F}(l - x) = \mathrm{G}(l + x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (134c)$$

where $a = \sqrt{\mathrm{RG}}$ = attenuation constant β for distortionless lines (see page 39).

In order that these equations may be satisfied, the values of F(x) and G(x) must be as given in the following table :—

VALUES OF F(x) AND G(x).

NOTE.—When $t = 0$, $\epsilon^{-\frac{\mathrm{R}}{\mathrm{L}}t}\mathrm{F}(x - vt) = \mathrm{F}(x)$ and $\epsilon^{-\frac{\mathrm{R}}{\mathrm{L}}t}\mathrm{G}(x + vt) = \mathrm{G}(x)$.

Range of x.	F(x).	G(x).
↑		
$-4l$ to $-2l$	$\mathrm{E}[\epsilon^{-ax} - \epsilon^{-a(x+2l)}]$	
$-2l$,, 0	$\mathrm{E}\epsilon^{-ax}$	
0 ,, l	0	0
l ,, $2l$		0
$2l$,, $4l$		$\mathrm{E}\epsilon^{a(x-2l)}$
$4l$,, $6l$		$\mathrm{E}[\epsilon^{a(x-2l)} - \epsilon^{a(x-4l)}]$
↓		

The way in which the values of F(x) and G(x) are obtained is explained on page 205.

13

Unlike the previous examples, the distribution of voltage along the line at any time cannot now be found by allowing the distributions represented by F(x) and G(x) to move with velocity v and without change of shape. The distributions move (one to the right and the other to the left) with velocity v, but, at the same time, are attenuated by the factor $\epsilon^{-\frac{R}{L}t}$.

The equation representing the distribution at any time t can under these circumstances be most conveniently found by direct substitution in equation (134), thus :—

To determine the voltage distribution at time $t = \frac{l}{v}$ which is the time taken for the function F(x) to move to the right by a distance l, and the function G(x) to move to the left by an equal distance.

$$e = [F(x - vt) + G(x + vt)]\epsilon^{-\frac{R}{L}t}.$$

When $\quad vt = l \quad F(x - vt) = F(x - l).$

$$\therefore e = [F(x - l) + G(x + l)]\,\epsilon^{-\frac{R}{L}\cdot\frac{l}{v}}.$$

Since $\quad a = \frac{R}{Lv} = \sqrt{RG},$

$$e = [F(x - l) + G(x + l)]\epsilon^{-al} \quad . \quad (135)$$

To get the values of the functions between the limits $x = 0$ and $x = l$, the procedure is thus :—

For values of x between 0 and l the limits for $x - l$ lie between $- l$ and 0.

At time $t = 0 \quad$ F(x) is $E\epsilon^{-ax}$ between—l and 0.

,, ,, $t = \frac{l}{v} \quad$ F($x - l$) is $E\epsilon^{-a(x-l)}$ between 0 and l.

Also ,, ,, $t = \frac{l}{v} \quad$ G($x + l$) is 0 between 0 and l.

$$\therefore e = \epsilon^{-al}\,[E\epsilon^{-a(x-l)}] = E\epsilon^{-ax} \quad . \quad . \quad (136)$$

The same method gives the voltage distribution at any other time, and the results obtained are :—

EQUATION OF VOLTS ALONG LINE AT VARIOUS INSTANTS.

t.	From $x = 0$ to $x = \frac{1}{2}l$.	From $x = \frac{1}{2}l$ to $x = l$.
0	0	0
$0{\cdot}5\frac{l}{v}$	$E\epsilon^{-ax}$	0
$1{\cdot}0\frac{l}{v}$	$E\epsilon^{-ax}$	$E\epsilon^{-ax}$
$1{\cdot}5\frac{l}{v}$	$E\epsilon^{-ax}$	$E[\epsilon^{-ax} + \epsilon^{-a(2l-x)}]$
$2{\cdot}0\frac{l}{v}$	$E[\epsilon^{-ax} + \epsilon^{-a(2l-x)}]$	As from 0 to $\frac{1}{2}l$
$2{\cdot}5\frac{l}{v}$	$E[\epsilon^{-ax} + \epsilon^{-a(2l-x)} - \epsilon^{-a(2l+x)}]$	$E[\epsilon^{-ax} + \epsilon^{-a(2l-x)}]$
$3{\cdot}0\frac{l}{v}$	$E[\epsilon^{-ax} + \epsilon^{-a(2l-x)} - \epsilon^{-a(2l+x)}]$	As from 0 to $\frac{1}{2}l$
↓	↓	↓

An examination of this table shows how the mathematics fits in with the theory of successive reflections which occur each time the wave-front strikes either the open end of the line or the generator end. From time $t = 0$ to $t = \frac{l}{v}$ only one wave is present, viz. $E\epsilon^{-ax}$. From time $t = \frac{l}{v}$ to time $t = \frac{2l}{v}$ there are present the original wave $E\epsilon^{-ax}$ and a single reflected wave $E\epsilon^{-a(2l-x)}$. From time $t = \frac{2l}{v}$ to time $t = \frac{3l}{v}$ there are present the original wave $E\epsilon^{-ax}$, the original reflected wave $E\epsilon^{-a(2l-x)}$, and a second negative reflected wave $E\epsilon^{-a(2l+x)}$. This process continues indefinitely.

The first reflected wave starts off with a value $\epsilon^{-a(2l-l)} = \epsilon^{-al}$ equal in value to the incident wave when it strikes the end of the line. This we would expect from previous work. At a time later by $0{\cdot}5\frac{l}{v}$ it will have moved halfway back to the generator, and the original wave-front ϵ^{-al} will be attenuated by $\epsilon^{-0\cdot5al}$, and its value will now be $\epsilon^{-1\cdot5al}$. After a further interval of time $0{\cdot}5\frac{l}{v}$ (making $\frac{2l}{v}$ from beginning) it will be attenuated to $\epsilon^{-1\cdot5al} \times \epsilon^{-0\cdot5al} = \epsilon^{-2al}$, and the voltage distribution over the whole line will be given by equation $E[\epsilon^{-ax} + \epsilon^{-a(2l-x)}]$. After arrival at the generator there must be an equal but opposite reflected wave into the line to keep the voltage at the generator equal to E. Hence initial value of this reflected wave will be ϵ^{-2al}. At time $t = 3\cdot0\frac{l}{v}$ the second reflected wave will have got to the open end of the line, will be attenuated by ϵ^{-al}, and its wave-front will be ϵ^{-3al}. These successive reflections are illustrated in Fig. 97.

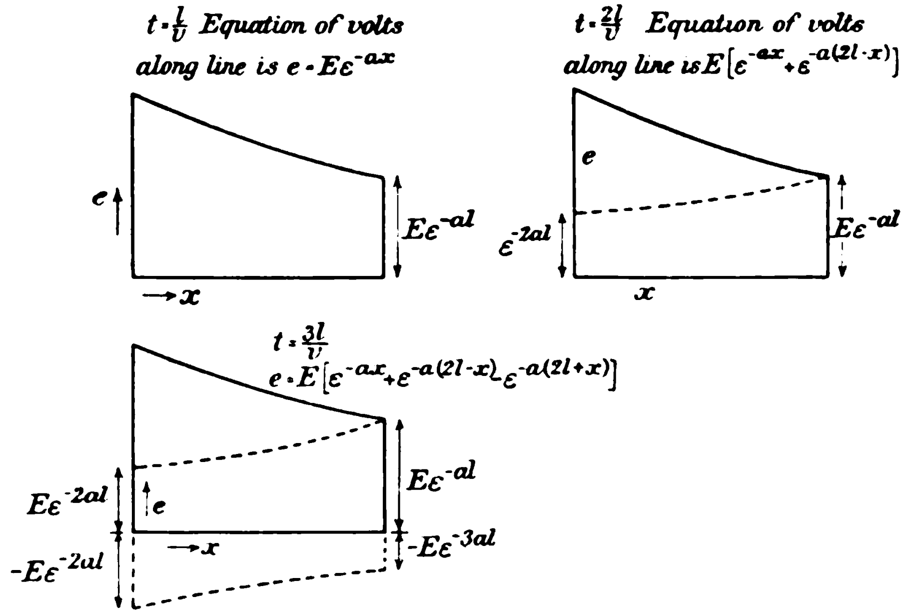


FIG. 97.—Open-circuited line for which $\frac{R}{L} = \frac{G}{C}$. Distribution of voltage along line at various instants.

The only moving thing is the wave-front either incident or reflected. As it moves it leaves behind it a voltage distribution which remains constant independent of time, but to this constant distribution is being continually added other distributions associated with further reflections. Thus, after the first passage of the original wave-front from 0 to l, the voltage distribution represented by $E\epsilon^{-ax}$ persists for all time as a component of any future voltage distribution along the line. After the passage of the first reflected wave $E\epsilon^{-a(2l-x)}$ remains as a further component of all future voltage distributions, and so on.

Passing from Transient to Steady State.—The voltage at any point x changes with each passage of a wave-front, the amount of the change getting less with each successive passage. The voltage will eventually settle down to a steady amount given by

$$\mathrm{E}\{\epsilon^{-ax} + \epsilon^{-a(2l-x)} - \epsilon^{-a(2l+x)} - \epsilon^{-a(4l-x)} + \epsilon^{-a(4l+x)} + \epsilon^{-a(6l-x)} + \ldots\}$$

$$= \mathrm{E}\epsilon^{-ax}\{1 - \epsilon^{-2al} + \epsilon^{-4al} \ldots\} + \mathrm{E}\{\epsilon^{-2al+ax} - \epsilon^{-4al+ax} + \epsilon^{-6al+ax} \ldots\}$$

$$= \mathrm{E}\left\{\frac{\epsilon^{-ax}}{1+\epsilon^{-2al}}\right\} + \mathrm{E}\left\{\frac{\epsilon^{ax}\epsilon^{-2al}}{1+\epsilon^{-2al}}\right\}$$

$$= \mathrm{E}\left\{\frac{\epsilon^{-ax}\epsilon^{al}}{\epsilon^{al}+\epsilon^{-al}} + \frac{\epsilon^{ax}\epsilon^{-al}}{\epsilon^{al}+\epsilon^{-al}}\right\}$$

$$= \mathrm{E}\,\frac{\cosh a(l-x)}{\cosh al} \ldots$$

$$= \mathrm{E}\left\{\frac{\cosh \gamma l \cosh \gamma x - \sinh \gamma l \sinh \gamma x}{\cosh \gamma l}\right\} \text{ since } a = \sqrt{\mathrm{RG}} = \gamma$$

$$= \mathrm{E}\{\cosh \gamma x - \tanh \gamma l \sinh \gamma x\} \,. \qquad (137)$$

This agrees with the equation for the " steady state " voltage given on page 22, viz.,

$$\mathrm{V} = \mathrm{V}_o(\cosh \gamma x - \tanh \gamma l \sinh \gamma x).$$

Example.—A transmission line has a resistance per mile of loop of 0·5 ohm, and an inductance per mile of loop of 0·004 henry. The leakance is $8{\cdot}75 \times 10^{-7}$ mho per mile of loop, and the capacitance is 7×10^{-9} farads per mile of loop. The line is 500 miles long, and a continuous voltage of 100,000 is maintained at one end, the other end being open. Calculate the distribution of voltage and current along the line at various times after applying the voltage.

Since $\frac{\mathrm{R}}{\mathrm{L}} = \frac{\mathrm{G}}{\mathrm{C}}\left[\frac{0{\cdot}5}{0{\cdot}004} = \frac{8{\cdot}75 \times 10^{-7}}{7 \times 10^{-9}}\right]$, the solution given by equation (134) holds.

$$\sqrt{\frac{L}{C}} = \sqrt{\frac{0{\cdot}004}{7 \times 10^{-9}}} = 748 \text{ ohms.}$$

$$v = \frac{1}{\sqrt{LC}} = 186000 \text{ miles per second.}$$

$$a = \sqrt{RG} = \sqrt{0{\cdot}5 \times 8{\cdot}75 \times 10^{-7}}$$
$$= 0{\cdot}000662.$$

$$l = 500 \text{ miles.}$$

Time for wave to travel a distance equal to the length of the line $= \dfrac{500}{186000} = 0{\cdot}00269$ seconds $= \dfrac{l}{v}$.

At time $t = \dfrac{l}{v}$ equation of volts along line $= E\epsilon^{-ax}$.

$x =$	0	100	200	300	400	500	Miles.
ax .	0	0·0662	0·1324	0·1986	0·2648	0·331	
ϵ^{-ax} .	1	0·937	0·878	0·818	0·767	0·719	
$E\epsilon^{-ax}$.	100,000	93,700	87,800	81,800	76,700	71,900	Volts.

At time $t = \dfrac{2l}{v}$ equation of volts along line $= E\epsilon^{-ax} + E\epsilon^{-a(2l-x)}$.

x.	0	100	200	300	400	500
$(2l - x)$.	1,000	900	800	700	600	500
$a(2l - x)$.	0·662	0·5958	0·5296	0·4634	0·3972	0·331
$\epsilon^{-a(2l-x)}$.	0·517	0·554	0·588	0·631	0·67	0·719
$E\epsilon^{-a(2l-x)}$.	51,700	55,400	58,800	63,100	67,000	71,900
$E\epsilon^{-ax} + E\epsilon^{-a(2l-x)}$	151,700	149,100	146,600	144,900	143,700	143,800

In the same way, the voltage at any other instant can be found. The voltage distributions at the two instants for which the tables are calculated are given in Fig. 98.

If the resistance and leakance had been neglected the voltage at time $t = \dfrac{l}{v}$ would have been 100,000 all along the line, and at time $t = \dfrac{2l}{v}$ it would have been 200,000 all along the line.

Current along Line when $\frac{R}{L} = \frac{G}{C}$ (Open-circuited Line). —The general solution for the voltage along the line has been considered. Regarding the current it can be shown that the current along the line is everywhere in phase with the voltage and is in magnitude equal to the voltage divided by $w = \sqrt{\frac{L}{C}}$, the surge impedance. At the open end of the line the current is reflected *with* change of sign in order that the current at the end of the line may always be zero. Corresponding tables may be constructed for the current thus :—

At time $t = \frac{l}{v}$

equation of volts along line is $e = E\epsilon^{-ax}$

equation of current along line is $i = E\epsilon^{-ax}/w$,

where w surge impedance $= 748$ ohms.

Therefore, equation of current along line is $i = \frac{E}{748}\epsilon^{-ax}$

$= 133{\cdot}7\ \epsilon^{-ax}$.

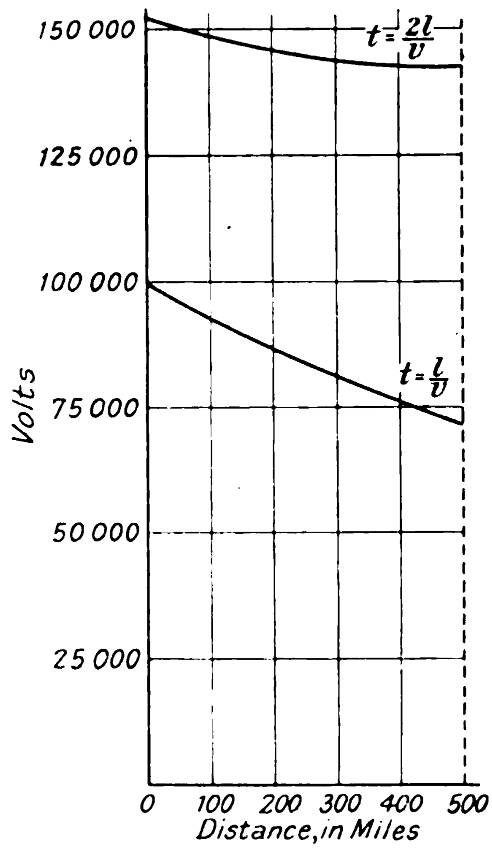


FIG. 98.—Voltage distribution along open-circuited 500-mile line for which $\frac{R}{L} = \frac{G}{C}$, when 100,000 volts are continuously applied.

x	0	100	200	300	400	500
ax	0	0·0662	0·1324	0·1986	0·2648	0·331
ϵ^{-ax}	1	0·937	0·878	0·818	0·767	0·719
$133{\cdot}7\ \epsilon^{-ax}$	133·7	125	117·2	109·1	102·5	96

At time $t = \frac{2l}{v}$ equation of volts along line is

$$e = \mathrm{E}\epsilon^{-ax} + \mathrm{E}\epsilon^{-a(2l-x)},$$

equation of current along line is

$$i = \frac{\mathrm{E}}{w}\epsilon^{-ax} - \frac{\mathrm{E}}{w}\epsilon^{-a(2l-x)}.$$

x	0	100	200	300	400	500
$(2l - x)$ . . .	1000	900	800	700	600	500
$a(2l - x)$ . .	0·662	0·5958	0·5296	0·4634	0·3972	0·331
$\epsilon^{-a(2l-x)}$ . .	0·517	0·554	0·588	0·631	0·67	0·719
$\frac{\mathrm{E}}{w}\left\{\epsilon^{-a(2l-x)}\right\}$.	69·1	74	78·7	84·5	89·5	96
$\frac{\mathrm{E}}{w}\left\{\epsilon^{-ax} - \epsilon^{-a(2l-x)}\right\}$	64·6	51	38·5	24·6	13	0

In the same way the current distribution along the line can be found for any other time.

t.	Equation of Current along Line.	
	From 0 to $\frac{1}{2}l$.	From $\frac{1}{2}l$ to l.
0	0	0
$0·5\frac{l}{v}$	$\frac{\mathrm{E}}{w}\epsilon^{-ax}$	0
$1·0\frac{l}{v}$	$\frac{\mathrm{E}}{w}\epsilon^{-ax}$	$\frac{\mathrm{E}}{w}\epsilon^{-ax}$
$1·5\frac{l}{v}$	$\frac{\mathrm{E}}{w}\epsilon^{-ax}$	$\frac{\mathrm{E}}{w}\left[\epsilon^{-ax} - \epsilon^{-a(2l-x)}\right]$
$2·0\frac{l}{v}$	$\frac{\mathrm{E}}{w}\left[\epsilon^{-ax} - \epsilon^{-a(2l-x)}\right]$	As from 0 to $\frac{1}{2}l$
$2·5\frac{l}{v}$	$\frac{\mathrm{E}}{w}\left[\epsilon^{-ax} - \epsilon^{-a(2l-x)} - \epsilon^{-a(2l+x)}\right]$	$\frac{\mathrm{E}}{w}\left[\epsilon^{-ax} - \epsilon^{-a(2l-x)}\right]$
$3·0\frac{l}{v}$	$\frac{\mathrm{E}}{w}\left[\epsilon^{-ax} - \epsilon^{-a(2l-x)} - \epsilon^{-a(2l+x)}\right]$	As from 0 to $\frac{1}{2}l$.

The current distributions at the two instants for which the tables are calculated are given in Fig. 99.

The current waves corresponding to the voltage curves of Fig. 97 are—

At time $t = 0$ the current is zero all along the line (Fig. 100).

At time $t = 0{\cdot}5\frac{l}{v}$ the wave of current accompanying the wave of voltage has reached a point halfway along the line (Fig. 101).

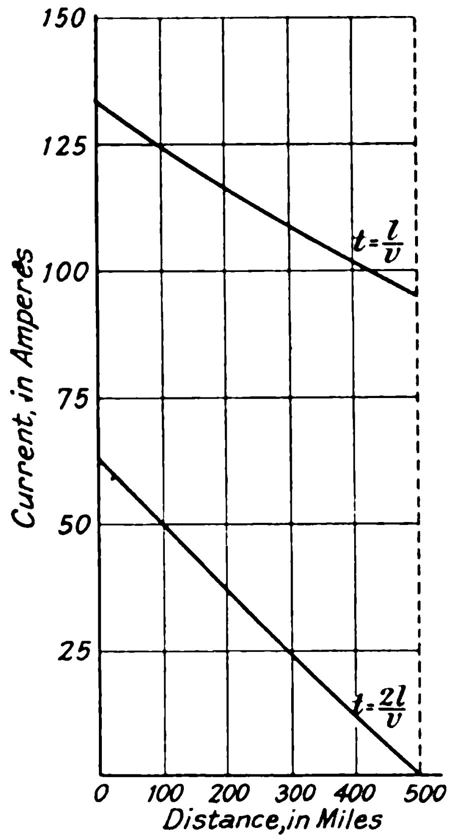


FIG. 99.—Current distribution along open-circuited 500-mile line for which $\frac{R}{L} = \frac{G}{C}$, when 100,000 volts are continuously applied.

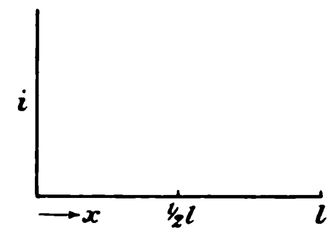


FIG. 100.

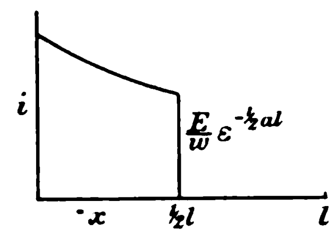


FIG. 101.

Equation to current along line is

$$i = \frac{E}{w}\epsilon^{-ax} \text{ from } 0 \text{ to } \tfrac{1}{2}l,$$

$$i = 0 \text{ from } \tfrac{1}{2}l \text{ to } l.$$

At time $t = \frac{l}{v}$ the wave of current has reached the open end of the line, where the current must be zero; i when $x = l$ is $i = \frac{E}{w}\epsilon^{-al}$. There must now be a reflected wave of current of initial amplitude $\frac{E}{w}\epsilon^{-al}$. Equation of current along line is $i = \frac{E}{w}\epsilon^{-ax}$ (Fig. 102).

At time $t = 1 \cdot 5 \frac{l}{v}$ the reflected wave of current has got half-way back to the generator again. Equation of current along line is

$$x = 0 \text{ to } x = \tfrac{1}{2}l \qquad i = \frac{E}{w}\epsilon^{-ax}$$

$$x = \tfrac{1}{2}l \text{ to } x = l \qquad i = \frac{E}{w}[\epsilon^{-ax} - \epsilon^{-a(2l-x)}] \text{ (Fig. 103).}$$

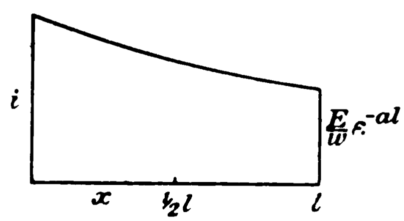


FIG. 102.

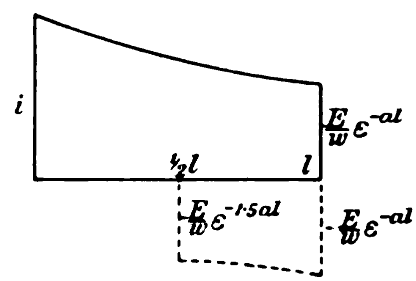


FIG. 103.

At time $t = \frac{2l}{v}$ the reflected wave of current has reached the generator.

Equation of current along line is

$$i = \frac{E}{w}[\epsilon^{-ax} - \epsilon^{-a(2l-x)}].$$

Now equation of voltage along line is

$$e = E[\epsilon^{-ax} + \epsilon^{-a(2l-x)}]$$

and this is made equal to E at the generator by a wave of volts $e = -E\epsilon^{-a(2l+x)}$ (see p. 195), and accompanying this is a wave of current

$$i = -\frac{E}{w}\epsilon^{-a(2l+x)} \text{ (Fig. 104).}$$

At time $t = 2{\cdot}5\frac{l}{v}$ the wave of current as given by equation $i = -\frac{E}{w}\epsilon^{-a(2l+x)}$ has got halfway to open end of line, and equation to current is

from 0 to $\frac{1}{2}l$ $\quad i = \frac{E}{w}\epsilon^{-ax} - \frac{E}{w}\epsilon^{-a(2l-x)} - \frac{E}{w}\epsilon^{-a(2l+x)}.$

From $\frac{1}{2}l$ to l it is

$$i = \frac{E}{w}\epsilon^{-ax} - \frac{E}{w}\epsilon^{-a(2l-x)} \text{ (Fig. 105).}$$

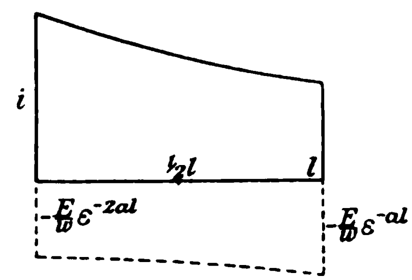


FIG. 104.

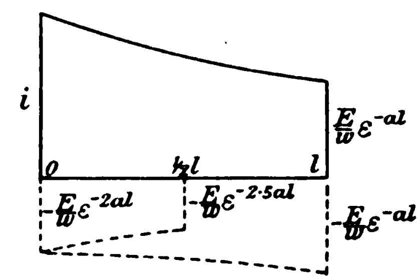


FIG. 105.

At time $t = \frac{3l}{v}$ the wave of current $i = -\frac{E}{w}\epsilon^{-a(2l+x)}$ has got back to open end of line and equation to current along line is

$$i = \frac{E}{w}\epsilon^{-ax} - \frac{E}{w}\epsilon^{-a(2l-x)} - \frac{E}{w}\epsilon^{-a(2l+x)} \text{ (Fig. 106).}$$

We have seen, then, that when the current wave reaches the open end of the line, the reflected wave of current is of opposite sign to the reflected wave of voltage $E\epsilon^{-a(2l-x)}$ for voltage and $-\frac{E}{w}\epsilon^{-a(2l-x)}$ for current. When, however, a current wave reaches the generator end of the line the reflected wave has the same sign as the reflected wave of voltage ($-E\epsilon^{-a(2l+x)}$ for voltage and $-\frac{E}{w}\epsilon^{-a(2l+x)}$ for current).

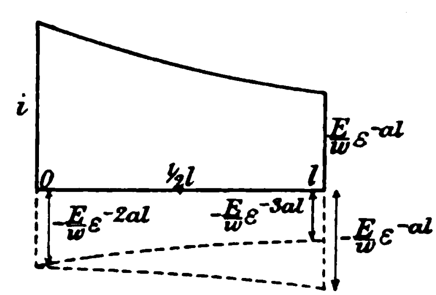


FIG. 106.

These results are embodied in the mathematical solution for the current.

Alternating Current Transients.—The only transients which have been considered in this chapter are those associated with the switching-on of a continuous voltage E. It remains to consider the switching-on of an alternating voltage.

Only lines where $\frac{R}{L} = \frac{G}{C}$ will be considered. Suppose an alternating voltage $e = E \cos wt$ is switched on to an open line of length l; the problem is, to determine the voltage distribution along the line at any desired instant after switching on.

As on page 193, the voltage distribution at any instant t is expressed by

$$e = \epsilon^{-avt}[F(x - vt) + G(x + vt)] \quad . \quad . \quad (138)$$

The boundary conditions are the same as those holding for the switching on of a continuous voltage (p. 185), excepting that when $x = 0$, $e = E \cos wt = \epsilon^{-avt}[F(- vt) + G(vt)]$. It can be proved that these boundary conditions result in the three following equations :—

$$F(x) = G(x) = 0 \text{ for } x \text{ between } 0 \text{ and } l \quad . \quad (138A)$$

$$F(x) = - G(- x) + E\epsilon^{-ax} \cos \frac{w}{v} x \quad . \quad . \quad (138B)$$

$$F(l - x) = G(l + x) \quad . \quad . \quad . \quad . \quad . \quad (138C)$$

When $t = 0$, the expression $e^{-avt}[F(x - vt) + G(x + vt)]$ becomes $[F(x) + G(x)]$. It will be proved that in order to satisfy the boundary conditions, the values of $F(x)$ and $G(x)$ must be as given in the table (p. 205).

The values for $F(x)$ and $G(x)$ are built up from the equations 138A, B, and C in a manner which will now be explained. It should be noted that, excepting for the cosine term in equation 138B, the equations 138A, B, and C are identical with those given on page 193 for the case of a continuous applied voltage E. The method now to be described for the building-up of the values of $F(x)$ and $G(x)$, as given in the table on page 205, can be applied to the conditions on page 193. The results are given in the table on page 193.

VALUES OF F(x) AND G(x).

NOTE.—F(x) = F($x - vt$) at time t = 0 and G(x) = G($x + vt$) at t = 0.

Range of x.	F(x).	G(x).
↑		
$-4l$ to $-2l$	$E\epsilon^{-ax}\cos\frac{w}{v}x - E\epsilon^{-a(x+2l)}\cos\frac{w}{v}(x+2l)$	—
$-2l$ to 0	$E\epsilon^{-ax}\cos\frac{w}{v}x$	—
0 to $2l$	0	0
$2l$ to $4l$	—	$E\epsilon^{a(x-2l)}\cos\frac{w}{v}(x-2l)$
↓		

Values for F(x) and G(x) to Suit Boundary Conditions as Expressed in Equations 138A, B, and C.—Since F(x) = G(x) = 0 between $x = 0$ and $x = l$, this enables a start to be made as shown in Fig. 107a. Applying equation 138B, it follows that from $x = 0$ to $x = -l$

$$F(x) = 0 + E\epsilon^{-ax}\cos\frac{w}{v}.x$$

$$= E\epsilon^{-ax}\cos\frac{w}{v}.x.$$

The building-up can now be carried to the stage indicated in Fig. 107b.

Also, from equation 138C, G(x) must be 0 for x between l and $2l$, since F(x) is 0 for x between 0 and l (Fig. 107c). From this it follows that $F(x) = E\epsilon^{-ax}\cos\frac{w}{v}x$ between l and $2l$. This is indicated in Fig. 107d.

Further progress can be made by using equation 138B thus: For any value of x (say $x = z$), $F(z) = E\epsilon^{-az}\cos\frac{w}{v}z$ for z between 0 and $-2l$.

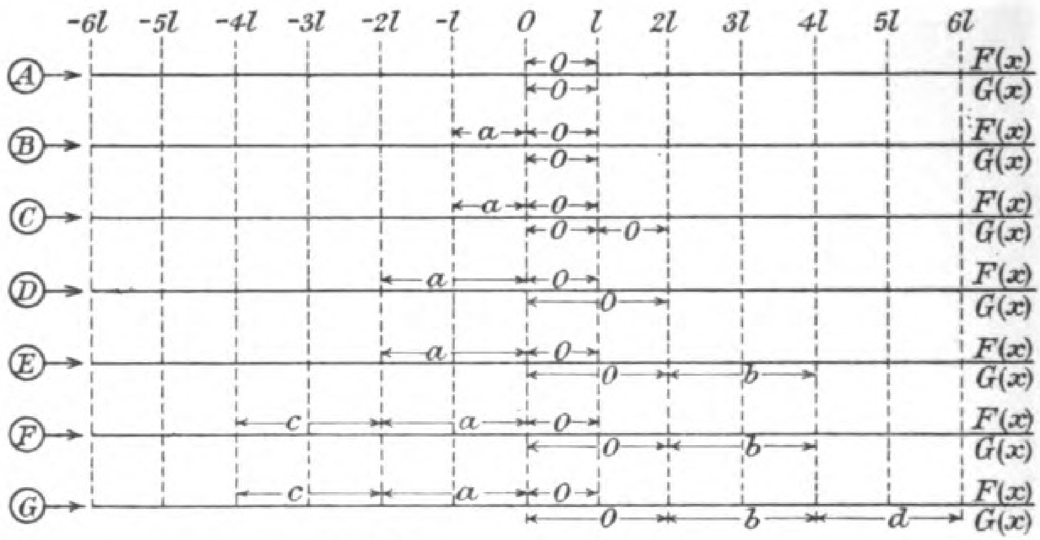
Hence, if $x = l - z$, then $F(l - z) = E\epsilon^{-a(l-z)}\cos\frac{w}{v}(l-z)$ for $(l - z)$ between 0 and $-2l$.

Since by equation (138*c*) $F(l - z) = G(l + z)$

$$G(l + z) = E\epsilon^{-a(l-z)} \cos \frac{w}{v}(l - z) \text{ for } (l - z) \text{ between 0 and } - 2l,$$

i.e. for z between l and $3l$.

$$\therefore G(l + z) = E\epsilon^{-a\{-(l+z)+2l\}} \cos \frac{w}{v}\{-(l + z) + 2l\} \text{ for } (l + z) \text{ between } 2l \text{ and } 4l.$$



$$a = E\epsilon^{-ax} \cos \frac{w}{v}x.$$

$$b = E\epsilon^{a(x-2l)} \cos \frac{w}{v}(x - 2l).$$

$$c = E\epsilon^{-ax} \cos \frac{w}{v}x - E\epsilon^{-a(x+2l)} \cos \frac{w}{v}(x + 2l).$$

$$d = E\epsilon^{a(x-2l)} \cos \frac{w}{v}(x - 2l) - E\epsilon^{a(x-4l)} \cos \frac{w}{v}(x - 4l).$$

FIG. 107.—Illustrating process of building up values of F(x) and G(x).

So that in general, writing $(l + z) = x$,

$$G(x) = E\epsilon^{-a(-x+2l)} \cos \frac{w}{v}(- x + 2l) \text{ for } x \text{ between } 2l \text{ and } 4l.$$

This relation enables the building-up to be carried to the stage indicated in Fig. 107*e*.

Using equation 138c again, it follows that since

$$G(x) = E\epsilon^{a(x-2l)} \cos \frac{w}{v}(x - 2l) \text{ between } 2l \text{ and } 4l$$

$$F(x) = -E\epsilon^{a(-x-2l)} \cos \frac{w}{v}(- x - 2l) + E\epsilon^{-ax} \cos \frac{w}{v}x$$

from $- 2l$ to $- 4l$,

i.e. $F(x) = E\epsilon^{-ax}\cos\frac{w}{v}x - E\epsilon^{-a(x+2l)}\cos\frac{w}{v}(x+2l)$

from $-2l$ to $-4l$.

This is shown in Fig. 107*f*.

The next step is to use equation 138B again thus :—

For z between $-2l$ and $-4l$,

$$F(z) = E\epsilon^{-az}\cos\frac{w}{v}z - E\epsilon^{-a(z+2l)}\cos\frac{w}{v}(z+2l).$$

$$\therefore\ F(l-z) = E\epsilon^{-a(l-z)}\cos\frac{w}{v}(l-z) - E\epsilon^{-a(3l-z)}\cos\frac{w}{v}(3l-z)$$

for $(l-z)$ between $-2l$ and $-4l$, i.e. for z between $3l$ and $5l$.

$$\therefore\ G(l+z) = E\epsilon^{-a(l-z)}\cos\frac{w}{v}(l-z) - E\epsilon^{-a(3l-z)}\cos\frac{w}{v}(3l-z)$$

for $(l+z)$ between $4l$ and $6l$

$$= E\epsilon^{-a\{-(l+z)+2l\}}\cos\frac{w}{v}\{-(l+z)+2l\}$$

$$- E\epsilon^{-a\{-(l+z)+4l\}}\cos\frac{w}{v}\{-(l+z)+4l\}.$$

Or in general

$$G(x) = E\epsilon^{-a(-x+2l)}\cos\frac{w}{v}(-x+2l) - E\epsilon^{-a(-x+4l)}\cos\frac{w}{v}(-x+4l)$$

for x between $4l$ and $6l$.

This is indicated in Fig. 107*g*.

In this way, by using equations 138B and 138C alternately the process of building-up may be indefinitely continued.

The Values of $F(x-vt)$ and $G(x+vt)$.—$F(x)$ and $G(x)$ are the values of $F(x-vt)$ and $G(x+vt)$ at time $t = 0$. They are as given in the table on page 205, and are shown in Fig. 108*a*. In Figs. 108*b*, *c*, and *d* the expressions for the functions $F(x-vt)$ and $G(x+vt)$ are given for other values of t, and the value of $F(x-vt) + G(x+vt)$ can be obtained at any time t by adding corresponding expressions as obtained from this figure. Multiplying by ϵ^{-avt} will then give the expression for the voltage distribution along the line. It is simpler, however, to proceed thus :—

Distribution of Voltage along Line when $t = 0{\cdot}5\frac{l}{v}$.

$$e = \epsilon^{-avt}\left[F(x - vt) + G(x + vt)\right]$$

when $\quad t = 0{\cdot}5\frac{l}{v} \quad e = \epsilon^{-\frac{1}{2}al}\left[F(x - \tfrac{1}{2}l) + G(x + \tfrac{1}{2}l)\right].$

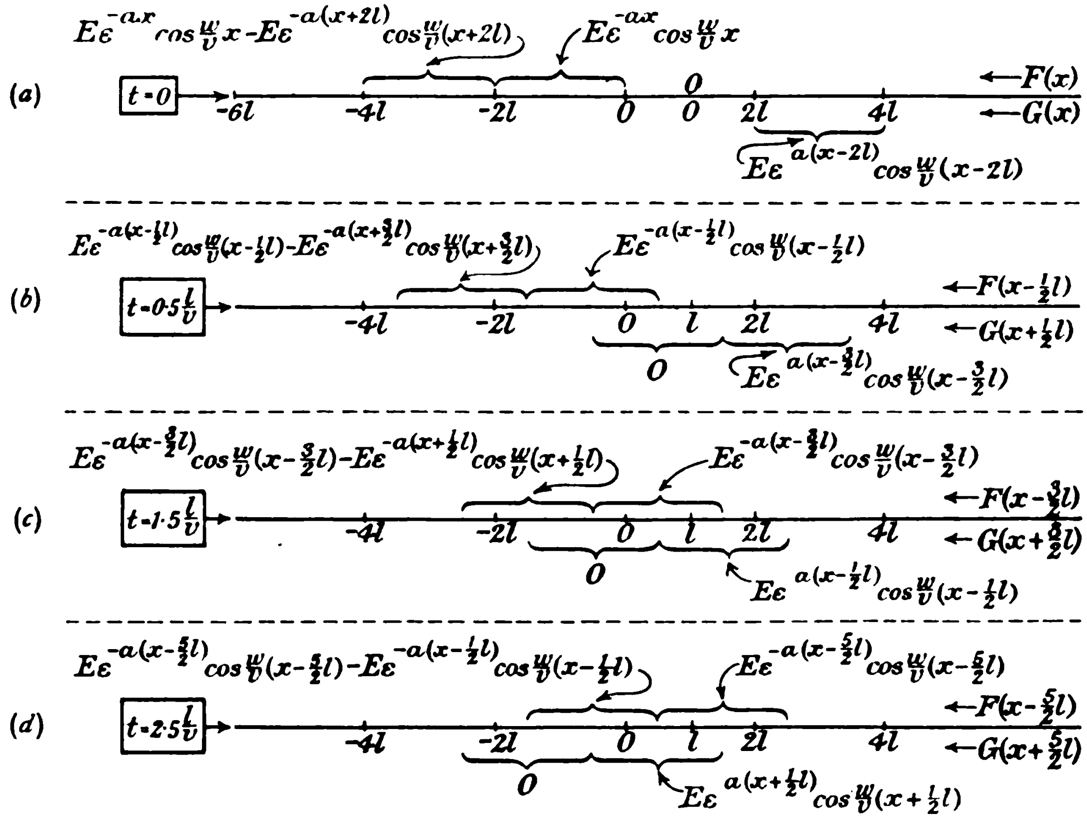


FIG. 108.—Values of $F(x - vt)$ and $G(x + vt)$ for various values of vt.

From Fig. 108 $F(x - \frac{1}{2}l) = E\epsilon^{-a(x-\frac{1}{2}l)}\cos\frac{w}{v}(x - \frac{1}{2}l)$ for x between 0 and $\frac{1}{2}l$.

$F(x - \frac{1}{2}l) = 0$ for x between $\frac{1}{2}l$ and l.

$G(x + \frac{1}{2}l) = 0$ for x between 0 and l.

$\therefore\ e = \epsilon^{-\frac{1}{2}al}\left[E\epsilon^{-a(x-\frac{1}{2}l)}\cos\frac{w}{v}(x - \frac{1}{2}l)\right]$ for x between 0 and $\frac{1}{2}l$.

$e = 0$ for x between $\frac{1}{2}l$ and l.

Now $\epsilon^{-\frac{1}{2}al}\left[E\epsilon^{-a(x-\frac{1}{2}l)}\cos\frac{w}{v}(x - \frac{1}{2}l)\right]$

$= E\epsilon^{-ax}\cos\frac{w}{v}(x - \frac{1}{2}l)$ = voltage distribution from $x=0$ to $x=\frac{1}{2}l$. (139)

Distribution of Voltage along Line when $t = \frac{l}{v}$.

$$e = \epsilon^{-avt}[F(x - l) + G(x + l)].$$

From Fig. 108 $[F(x - l) + G(x + l)]$

$$= E\epsilon^{-a(x-l)}\cos\frac{w}{v}(x - l)] \text{ for } x \text{ between 0 and } l.$$

$$\therefore e = \epsilon^{-al}[E\epsilon^{-a(x-l)}\cos\frac{w}{v}(x - l)]$$

$$= \underline{E\epsilon^{-ax}\cos\frac{w}{v}(x - l)} \quad . \quad . \quad . \quad . \quad (140)$$

Distribution of Voltage along Line when $t = 1{\cdot}5\frac{l}{v}$.

$$e = \epsilon^{-\frac{3}{2}al}[F(x - 1{\cdot}5l) + G(x + 1{\cdot}5l)].$$

From Fig. 108 $[F(x - 1{\cdot}5l) + G(x + 1{\cdot}5l)]$

$$= E\epsilon^{-a(x-\frac{3}{2}l)}\cos\frac{w}{v}(x - \tfrac{3}{2}l) \text{ for } x \text{ between 0 and } \tfrac{1}{2}l.$$

$$\therefore e = \epsilon^{-\frac{3}{2}al}[E\epsilon^{-a(x-\frac{3}{2}l)}\cos\frac{w}{v}(x - \tfrac{3}{2}l)]$$

$$= \underline{E\epsilon^{-ax}\cos\frac{w}{v}(x - \tfrac{3}{2}l)} \text{ for } x \text{ between 0 and } \tfrac{1}{2}l. \quad (141)$$

Also from Fig. 108, $F(x - 1{\cdot}5l) + G(x + 1{\cdot}5l)$

$$= E\epsilon^{-a(x-\frac{3}{2}l)}\cos\frac{w}{v}(x - \tfrac{3}{2}l) + E\epsilon^{a(x-\frac{1}{2}l)}\cos\frac{w}{v}(x - \tfrac{1}{2}l) \text{ for } x$$

between $\frac{1}{2}l$ and l.

$$\therefore \underline{e = E\epsilon^{-ax}\cos\frac{w}{v}(x - \tfrac{3}{2}l) + E\epsilon^{-a(2l-x)}\cos\frac{w}{v}(x - \tfrac{1}{2}l)}$$

for x between $\frac{1}{2}l$ and l. (142)

The distribution of voltage along the line at other instants may be found in a similar way.

Example.—Taking line particulars as given in the example on page 197, calculate the voltage distribution along the line at various instants after the switching on of an alternating E.M.F. expressed as 100,000 cos $314t$. (This is, of course, an E.M.F. of 100,000 volts maximum value, and frequency 50 cycles per second.) The line is, as before, open-circuited.

I. *At time* $t = \frac{1}{2}\frac{l}{v}$ (i.e. 0·00134 secs. after switching on),

$$w = 314. \qquad v = 186{,}000.$$

$$\therefore \frac{w}{v} = \frac{314}{186000} = 0{\cdot}001689.$$

x.	0	100	200	250	Miles.
ax	0	0·0662	0·1324	0·1655	
ϵ^{-ax}	1	0·937	0·878	0·847	
$E\epsilon^{-ax}$	100,000	93,700	87,800	84,700	
$x - \frac{1}{2}l$	− 250	− 150	− 50	0	
$\frac{w}{v}(x - \frac{1}{2}l)$ . . .	− 0·422	− 0·253	− 0·084	0	
$\text{Cos}\,\frac{w}{v}(x - \frac{1}{2}l)$ . .	0·913	0·968	0·996	1	
$E\epsilon^{-ax}\cos\frac{w}{v}(x - \frac{1}{2}l)$.	91,300	92,500	87,400	84,700	

This voltage distribution is plotted on Fig. 109. It should be noted that at $x = 0$, the voltage must be that given by

$$\begin{aligned} e &= 100{,}000 \cos 314 \times \frac{500}{2 \times 186000} \\ &= 100{,}000 \cos 0{\cdot}422 \\ &= 100{,}000 \cos 24{\cdot}3^\circ \\ &= \underline{91{,}300 \text{ volts.}} \end{aligned}$$

II. *At time* $t = \frac{l}{v}$ (i.e. 0·00269 secs. after switching on).

x.	0	100	200	250	300	400	500
ax . . .	0	0·0662	0·1324	0·1655	0·1986	0·2648	0·331
ϵ^{-ax} . . .	1	0·937	0·878	0·847	0·818	0·767	0·718
$E\epsilon^{-ax}$ . .	100,000	93,700	87,800	84,700	81,800	76,700	71,800
$(x - l)$ . .	− 500	− 400	− 300	− 250	− 200	− 100	0
$\frac{w}{v}(x - l)$ . .	− 0·845	−0·6756	− 0·5067	− 0·4225	− 0·3378	− 0·1689	0
$\cos\frac{w}{v}(x - l)$.	0·663	0·786	0·873	0·913	0·943	0·986	1
$E\epsilon^{-ax}\cos\frac{w}{v}(x-l)$	66,300	73,600	76,700	77,300	77,200	75,600	71,800

This voltage distribution is shown in Fig. 110. At $x = 0$, $e = 100{,}000 \cos 0{\cdot}845 = 100{,}000 \cos 48{\cdot}6^\circ = 66{,}300$.

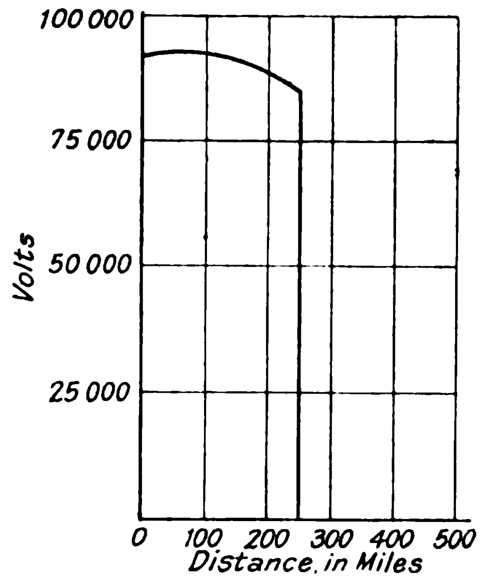


FIG. 109.—Voltage distribution along open-circuited 500-mile line for which $\frac{R}{L} = \frac{G}{C}$, when an alternating E.M.F. of 100,000 volts maximum value is applied at peak of wave. Figure gives distribution $\frac{1}{2}\frac{l}{v}$ seconds after switching on.

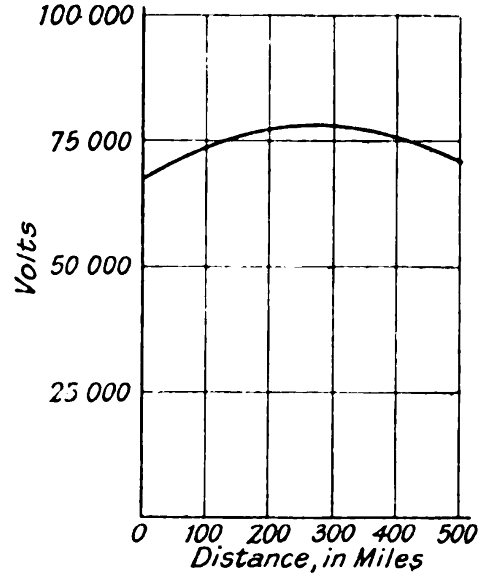


FIG. 110.—As in Fig. 109, but figure gives voltage distribution $\frac{l}{v}$ seconds after switching on.

III. *At time* $t = 1{\cdot}5\frac{l}{v}$ (i.e. 0·00403 secs. after switching on).

x.	0	100	200	250	300	400	500
ax	0	0·0662	0·1324	0·1655	0·1986	0·2648	0·331
ϵ^{-ax}	1	0·937	0·878	0·847	0·819	0·767	0·719
$E\epsilon^{-ax}$	100,000	93,700	87,800	84,700	81,900	76,700	71,900
$x - \frac{3}{2}l$	− 750	− 650	− 550	− 500	− 450	− 350	− 250
$\frac{w}{v}(x - \frac{3}{2}l)$	1·266	1·098	0·929	0·844	0·76	0·591	0·422
$\cos\frac{w}{v}(x - \frac{3}{2}l)$	0·298	0·454	0·598	0·662	0·725	0·83	0·91
$A = E\epsilon^{-ax}\cos\frac{w}{v}(x - \frac{3}{2}l)$	29,800	42,600	52,500	56,000	59,300	63,600	65,500
$2l - x$	—	—	—	750	700	600	500
$a(2l - x)$	—	—	—	0·4965	0·4634	0·397	0·331
$\epsilon^{-a(2l-x)}$	—	—	—	0·609	0·630	0·672	0·718
$E\epsilon^{-a(2l-x)}$	—	—	—	60,900	63,000	67,200	71,800
$x - \frac{1}{2}l$	—	—	—	0	50	150	250
$\frac{w}{v}(x - \frac{1}{2}l)$	—	—	—	0	0·0845	0·253	0·422
$\cos\frac{w}{v}(x - \frac{1}{2}l)$	—	—	—	1	0·996	0·968	0·913
$B = E\epsilon^{-a(2l-x)}\cos\frac{w}{v}(x - \frac{1}{2}l)$	—	—	—	60,900	62,750	65,050	65,500
$A + B$	—	—	—	116,900	122,050	128,650	131,000

This voltage distribution is shown in Fig. 111. At $x = 0$, $e = 100{,}000 \cos 1{\cdot}266 = 100{,}000 \cos 72{\cdot}9^\circ = 29{,}800$.

The voltage distribution at other instants may be obtained in a similar manner.

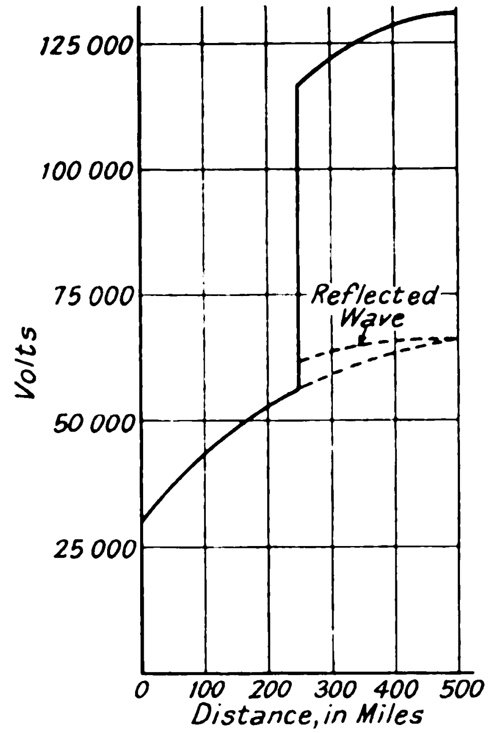


FIG. 111.—As in Figs. 109 and 110, but figure gives voltage distribution $1{\cdot}5\frac{l}{v}$ seconds after switching on.

ALTERNATING CURRENT TRANSIENTS: REFLECTION OF WAVES.

The mathematical formulæ given in the preceding section enable the voltage at any point along the line to be determined at certain instants. If an attempt is made to include other instants such as $t = 2\frac{l}{v}, 3\frac{l}{v}, 4\frac{l}{v}$, and so on, the formulæ become increasingly complicated. A study of equations (139) to (142) will show, however, that the labour associated with the use of such complicated formulæ can be avoided by means of a theory of "reflected waves." This theory will give results identical with the mathematical theory, but in a much more convenient form.

At a certain instant $t = 0$ it was assumed that an alternating voltage of magnitude E was switched on to the line. At this instant a wave having a voltage E entered the line and, in the distortionless case now being considered, this wave travels along the line with velocity v where v is the velocity of light. At time $t = \frac{1}{2}\frac{l}{v}$ the voltage distribution is given by equation (139) as:—

$$e = \mathrm{E}\epsilon^{-ax}\cos\frac{w}{v}(\tfrac{1}{2}l - x) \text{ since } \cos\frac{w}{v}(x - \tfrac{1}{2}l) = \cos\frac{w}{v}(\tfrac{1}{2}l - x).^*$$

The front of the wave is located at the point $x = \frac{1}{2}l$, and the voltage there is $e = \mathrm{E}\epsilon^{-a\frac{l}{2}}$. The voltage at the generator is now no longer E, but $\mathrm{E}\cos\frac{w}{v}\frac{l}{2}$. It follows, then, that the voltage at the wave-front has been attenuated from E to $\mathrm{E}\epsilon^{-\frac{al}{2}}$, i.e. by the factor $\epsilon^{-\frac{1}{2}al}$ in travelling the distance $\frac{1}{2}l$. Moreover, it started in time phase with the generator voltage, but it is now $\frac{w}{v}\frac{1}{2}l$ behind it.

At time $t = \frac{l}{v}$ the wave-front will have reached the end of the line, and the voltage there (from equation (140)) is $e = \mathrm{E}\epsilon^{-al}$, the voltage of the generator being $\mathrm{E}\cos\frac{w}{v}l$. Hence, in traversing the additional distance $\frac{1}{2}l$, the voltage at the wave-front has been attenuated by an additional $\epsilon^{-\frac{1}{2}al}$, and lags by an additional $\frac{w}{v}\frac{1}{2}l$, making $\frac{w}{v}l$ total lag. It would appear, then, that in travelling any distance the voltage at the wave-front is attenuated and altered in phase with respect to the generator voltage. The amount of attenuation for unit distance is ϵ^{-a}, and the amount of phase displacement for unit distance is $-\frac{w}{v}$.

The equation which gives the voltage distribution for the next period is equation (142). The first term is of the same form as equations (139) and (140). It will now be proved that the second term may be considered as representing the voltage distribution associated with a wave reflected from the open end of the line, this reflected wave having the following properties :—

* In this section it is convenient to change the form of equations (139) to (142) by using the law that $\cos\phi = \cos(-\phi)$ where ϕ is any angle. Thus

$$\cos\frac{w}{v}(x - \tfrac{1}{2}l) = \cos\frac{w}{v}(\tfrac{1}{2}l - x), \text{ also } \cos\frac{w}{v}(x - l) = \cos\frac{w}{v}(l - x),$$

and so on.

(*a*) It is created when the front of the original wave reaches the end of the line, i.e. when $t = \frac{l}{v}$.

(*b*) The voltage at the front of the reflected wave has, at the instant of reflection $\left(t = \frac{l}{v}\right)$, a value equal to that at the front of the original wave, i.e. $E\epsilon^{-al}$.

(*c*) The reflected wave shares with the original wave the properties that its front is attenuated by ϵ^{-a} and retarded in phase by $\frac{w}{v}$ per unit distance travelled.

We will first study the voltage at the front of the reflected wave as it moves along the line, and show that the voltage there agrees with that obtained from equation (142). In the next section the general case of the voltage at any point along the line will be discussed in terms of the reflected wave theory.

In time $t = \frac{1}{2}\frac{l}{v}$, after leaving the open end, the front of the reflected wave will have travelled a distance $\frac{1}{2}l$. It will be attenuated by $\epsilon^{-\frac{1}{2}al}$ and retarded in phase by $\frac{w}{v}\frac{1}{2}l$. So that its voltage is $\epsilon^{-al}\epsilon^{-\frac{1}{2}al}\ E = E\epsilon^{-\frac{3}{2}al}$, lagging behind the generator voltage by $\frac{w}{v}l + \frac{w}{v}\frac{1}{2}l = \frac{w}{v}\frac{3l}{2}$. The generator voltage at this instant $\left(t = 1{\cdot}5\frac{l}{v}\right)$ is $E \cos\frac{w}{v}\frac{3l}{2}$, so that if the above procedure has been correct, the voltage at the reflected wave-front $(x = \frac{1}{2}l)$ is $E\epsilon^{-\frac{3}{2}al}\cos\frac{w}{v}\left(\frac{3l}{2} - \frac{3l}{2}\right) = E\epsilon^{-\frac{3}{2}al}$. From equation (142) we see that the voltage is

$$e = E\epsilon^{-ax}\cos\frac{w}{v}\left(\frac{3}{2}l - x\right) + E\epsilon^{-a(2l-x)}\cos\frac{w}{v}(x - \tfrac{1}{2}l).$$

The second term corresponds to the reflected wave, and substituting $x = \frac{1}{2}l$ we obtain the voltage at the wave-front as $E\epsilon^{-\frac{3}{2}al}$, agreeing with the reflected wave theory. At time

$t = \frac{2l}{v}$ the reflected wave arrives at the generator, and agreement with the mathematical analysis will be obtained if we say that the wave is again reflected, but this time with change of sign. This reflected wave starts off with the characteristics of a wave which has already travelled a distance $2l$. When it reaches the open end another reflected wave will be set up, and so on indefinitely.

As the wave-fronts move along the line they leave voltage distributions behind them, and the voltage at any point x is the sum of the voltages due to each distribution. The way in which this voltage can be calculated will now be explained.

Voltage at any Point along the Line.—Equation (139) for the voltage at any point along the line at the instant $t = \frac{1}{2}\frac{l}{v}$ is

$$e = \mathrm{E}\epsilon^{-ax} \cos \frac{w}{v}(\tfrac{1}{2}l - x),$$

x being the distance from the generator.

At this instant the generator voltage is $e_g = \mathrm{E} \cos \frac{w}{v}\frac{1}{2}l$. Comparing the equations for e and e_g, we see that in magnitude $e = e_g\epsilon^{-ax}$, while it lags behind e_g in phase by $\frac{w}{v} . x$. Equation (140) gives the voltage along the line at $t = \frac{l}{v}$ as

$$e = \mathrm{E}\epsilon^{-ax} \cos \frac{w}{v}(l - x).$$

The generator voltage e_g is now $\mathrm{E} \cos \frac{w}{v}l$. As before, we find that $e = e_g\epsilon^{-ax}$ in magnitude and lags by $\frac{w}{v} . x$. Equation (141) gives the voltage along the line at $t = 1{\cdot}5\frac{l}{v}$ as

$$e = \mathrm{E}\epsilon^{-ax} \cos \frac{w}{v}\left(\frac{3}{2}l - x\right)$$

for x between 0 and $\frac{1}{2}l$. At this instant $e_g = \mathrm{E} \cos \frac{w}{v}\frac{3}{2}l$. As in the two previous cases, $e = e_0\epsilon^{-ax}$ in magnitude and lags by $\frac{w}{v}x$. The equation for the voltage between $\frac{1}{2}l$ and l is more complicated. It is

$$e = \text{E}\epsilon^{-ax}\cos\frac{w}{v}\left(\frac{3}{2}l - x\right) + \text{E}\epsilon^{-a(2l-x)}\cos\frac{w}{v}(x - \tfrac{1}{2}l).$$

The first part is identical with the distribution from 0 to $\frac{1}{2}l$, and will give the same results. It will now be proved that the second part also obeys the same laws when, as discussed in the previous section, we consider it as representing the voltage distribution due to a wave reflected from the open end at the instant $t = \frac{l}{v}$. Thus the wave-front of the distribution

$$e = \text{E}\epsilon^{-ax}\cos\frac{w}{v}(l - x)$$

reaches the open end of the line when $t = \frac{l}{v}$, the voltage at the end being $\text{E}\epsilon^{-al}$. If the reflected wave is set up of equal value, then the voltage at the end of the line becomes $2\text{E}\epsilon^{-al}$. The reflected wave travels towards the generator, and assuming that it obeys the same laws as have been deduced for the other voltage distributions, at time $t = 1{\cdot}5\frac{l}{v}$, its front will have travelled a distance $\frac{1}{2}l$, and the voltage at any point x due to the reflected wave will have a magnitude

$$\text{E}\epsilon^{-al}\epsilon^{-a(l-x)} = \text{E}\epsilon^{-a(2l-x)}.$$

In phase it will have fallen $\frac{w}{v}(l - x)$ behind its starting phase, i.e. $\frac{w}{v}(l - x)$ behind the voltage at the end of the line. The phase of this voltage has been proved to be $\frac{w}{v}l$ behind that of the generator. So that at the point x the voltage due to the reflected wave is $\frac{w}{v}(l - x) + \frac{w}{v}l = \frac{w}{v}(2l - x)$ behind the generator. At the instant $t = 1{\cdot}5\frac{l}{v}$ the phase of the generator voltage is $\frac{w}{v}\frac{3l}{2}$. Hence, if the reflected wave obeys the laws assumed, then the voltage at x will be $E\epsilon^{-a(2l-x)}$ in magnitude, and

$$\frac{w}{v}\frac{3l}{2} - \frac{w}{v}(2l - x) = \frac{w}{v}\left(\frac{3l}{2} - 2l + x\right) = \frac{w}{v}(x - \tfrac{1}{2}l)$$

in phase. It will accordingly be written

$$E\epsilon^{-a(2l-x)} \cos\frac{w}{v}(x - \tfrac{1}{2}l),$$

and since this agrees with equation (142), the assumptions made as to the behaviour and nature of the reflected wave are correct. If other instants are considered, it will be found that formulæ such as those given on page 209 can be translated into terms of reflected waves, thus :—

From time $t = 0$ to $t = \frac{l}{v}$ only the voltage distribution due to one wave exists.

From time $t = \frac{l}{v}$ to $t = \frac{2l}{v}$ the voltage distributions due to two waves exist, the second being associated with a wave reflected from the open end.

From time $t = \frac{2l}{v}$ to $t = \frac{3l}{v}$ the voltage distributions due to three waves exist, the third being associated with a wave reflected from the generator end, and so on.

All waves obey the law that the voltage is attenuated per unit of length by ϵ^{-a} and retarded in phase by $\frac{w}{v}$.

The length to be taken in calculating the total attenuation and phase displacement of the voltage at any point distant x from the generator will vary with the different waves. That this must be so is due to the fact that the first reflected wave from open end starts with all the characteristics of a wave which has already travelled a distance l (see Fig. 112). In the same way the first reflection from the generator starts as a wave which has travelled a distance $2l$, and so on. The voltages at a point distant x from the generator for the various waves may be expressed in tabular form, thus :—

Voltage at distance x from generator due to various reflected waves.
Voltage at generator = E.

Wave.	Voltage.
First from open end . .	$E\epsilon^{-a(2l-x)}\Big/ -\frac{w}{v}(2l-x)$
,, ,, generator . .	$E\epsilon^{-a(2l+x)}\Big/ -\frac{w}{v}(2l+x)$
Second from open end . .	$E\epsilon^{-a(4l-x)}\Big/ -\frac{w}{v}(4l-x)$
,, ,, generator . .	$E\epsilon^{-a(4l+x)}\Big/ -\frac{w}{v}(4l+x)$
Third from open end . .	$E\epsilon^{-a(6l-x)}\Big/ -\frac{w}{v}(6l-x)$
,, ,, generator . . and so on.	$E\epsilon^{-a(6l+x)}\Big/ -\frac{w}{v}(6l+x)$

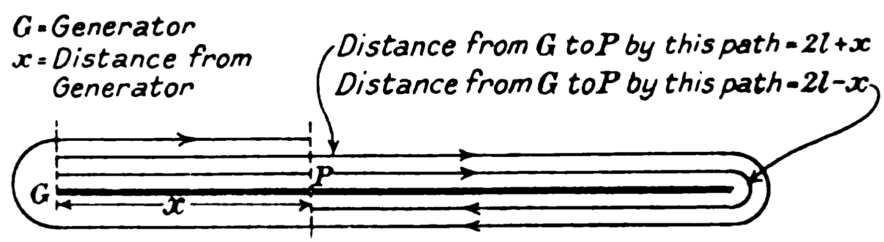


FIG. 112.—Showing effective lengths for attenuation, etc., in first reflected wave from open end and first reflected wave from generator.

It must be remembered that at the open end reflection occurs without change of sign, while at the generator end the sign is changed.

The voltage at x due to the original wave is $E\epsilon^{-ax}\Big/ -\frac{w}{v}x$.

Now an operator $\epsilon^{j\theta}$ turns the vector on which it operates through the angle θ. So that if we operate on the vector E with $\epsilon^{-j\frac{w}{v}x}$ we will turn it through the angle $-\frac{w}{v}x$. Hence $E\epsilon^{-ax}\underline{/-\frac{w}{v}x}$ may be written $E\epsilon^{-ax}\epsilon^{-j\frac{w}{v}x} = E\epsilon^{-\left(a+j\frac{w}{v}\right)x}$. In this theory

$a = \sqrt{RG}$ = attenuation constant for a distortionless line
= β on page 39.
$\frac{w}{v}$ = wave-length constant for a distortionless line
= α on page 39,

so that

$$E\epsilon^{-ax}\underline{/-\frac{w}{v}x} = E\epsilon^{-(\beta+j\alpha)x} = E\epsilon^{-\gamma x} \quad . \quad . \quad (143)$$

$$E\epsilon^{-a(2l-x)}\underline{/-\frac{w}{v}(2l-x)} = E\epsilon^{-(\beta+j\alpha)(2l-x)} = E\epsilon^{-\gamma(2l-x)} \quad (144)$$

and so on, where γ is the propagation constant for a distortionless line.

So that we can repeat the statement on page 26, viz. "the voltage at any point along the line is $V_0 \times \epsilon^{-\beta x}$, while it differs in phase from the voltage V_0 at the sending end by the angle $-\alpha x$."

x must be given its correct value for each wave from the table on page 218, and α and β must have the values which hold for a distortionless line.

Passing from Transient to Steady State with Alternating Voltage applied to the Line.—When only one wave is present, voltage V at any point x is

$$V = E\epsilon^{-\gamma x}.$$

When first reflected wave is also present,

$$V = E\epsilon^{-\gamma x} + E\epsilon^{-\gamma(2l-x)}.$$

When second reflected wave (i.e. first from generator) is also present,

$$V = E\epsilon^{-\gamma x} + E\epsilon^{-\gamma(2l-x)} - E\epsilon^{-\gamma(2l+x)}.$$

When third reflected wave (i.e. second from open end) is also present,

$$V = E\epsilon^{-\gamma x} + E\epsilon^{-\gamma(2l-x)} - E\epsilon^{-\gamma(2l+x)} - E\epsilon^{-\gamma(4l-x)}.$$

When fourth reflected wave (i.e. second from generator) is also present,

$$V = E\epsilon^{-\gamma x} + E\epsilon^{-\gamma(2l-x)} - E\epsilon^{-\gamma(2l+x)} - E\epsilon^{-\gamma(4l-x)} + E\epsilon^{-\gamma(4l+x)}.$$

This process repeats itself indefinitely, and the voltage at any point x eventually settles down to

$$\begin{aligned} V &= E\epsilon^{-\gamma x} + E\epsilon^{-\gamma(2l-x)} - E\epsilon^{-\gamma(2l+x)} - E\epsilon^{-\gamma(4l-x)} \\ &\qquad + E\epsilon^{-\gamma(4l+x)} + \ldots \text{ to infinity} \\ &= E\epsilon^{-\gamma x}[1 - \epsilon^{-2\gamma l} + \epsilon^{-4\gamma l} \ldots] \\ &\qquad + E[\epsilon^{-2\gamma l+\gamma x} - \epsilon^{-4\gamma l+\gamma x} + \epsilon^{-6\gamma l+\gamma x} \ldots]. \end{aligned}$$

Now $[1 - \epsilon^{-2\gamma l} + \epsilon^{-4\gamma l} \ldots]$ is a geometrical series of ratio $\epsilon^{-2\gamma l}$ and its sum to infinity is $\dfrac{1}{1 + \epsilon^{-2\gamma l}}$.

Also $[\epsilon^{-2\gamma l+\gamma x} - \epsilon^{-4\gamma l+\gamma x} + \epsilon^{-6\gamma l+\gamma x} \ldots]$ is a geometrical series of ratio $-\epsilon^{-2\gamma l}$, and its sum to infinity is $\dfrac{\epsilon^{-2\gamma l+\gamma x}}{1 + \epsilon^{-2\gamma l}}$.

Therefore,

$$\begin{aligned} V &= \frac{E\epsilon^{-\gamma x}}{1 + \epsilon^{-2\gamma l}} + \frac{E\epsilon^{-2\gamma l+\gamma x}}{1 + \epsilon^{-2\gamma l}} \\ &= E\left[\frac{\epsilon^{\gamma(l-x)} + \epsilon^{-\gamma(l-x)}}{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}\right] \\ &= \frac{E \cosh \gamma(l-x)}{\cosh \gamma l}. \end{aligned}$$

Now, $\cosh \gamma(l - x) = \cosh \gamma l \cosh \gamma x - \sinh \gamma l \sinh \gamma x.$

$$\therefore V = E\{\cosh \gamma x - \tanh \gamma l \sinh \gamma x\} \quad . \quad (145)$$

which agrees with equation (48), page 29.

Voltage Waves in Transmission Line : General Case.—The theory of the voltage waves produced when an alternating voltage is switched on to a distortionless line open at the distant end has now been developed. In the general case, where R, L, G, and C may have any values, the mathematical work corresponding with that which resulted in equations (139) to (142) on page 209 is much more difficult. The method of treatment by means of the theory of reflected waves which has been developed for the distortionless line is, however, most helpful in dealing with the general case. It is found, moreover, that with but small modifications the results obtained still hold. Thus, instead of the statement on page 217, "All waves obey the law that the voltage is attenuated per unit length by ϵ^{-a} and retarded in phase by $\frac{w}{v}$," must be substituted the statement, "All waves obey the law that the voltage is attenuated per unit length by $\epsilon^{-\beta}$ and retarded in phase by α." This latter statement is made on page 26, and β and α are the attenuation and wave-length constants obtained from formulæ (63) and (64) on page 39. The former statement is merely a special case of the latter, since for distortionless lines $\beta = a$ and $\alpha = \frac{w}{v}$.

The theory developed for distortionless lines can now be applied almost in its entirety to the general case. Thus, if at time $t = 0$, an alternating voltage E is switched on to an open line a wave of voltage E enters the line and travels along it with velocity v as determined from equation (46). This wave has the property that at any point in it distant x from the generator, the voltage is $E\epsilon^{-\gamma x}$. When the wave-front reaches the open end at time $t = \frac{l}{v}$, the wave is reflected and travels back towards the generator. At the generator it is again reflected, and so on. Hence, after the elapse of some time the voltage at any point distant x from the generator is

$$V = E\epsilon^{-\gamma x} + E\epsilon^{-\gamma(2l - x)} - E\epsilon^{-\gamma(2l + x)} \ . \ . \ . \text{ and so on,}$$

as on page 220, but now γ is the propagation constant in its general sense.

The application of these methods to the general case is illustrated by the following example:—

A transmission line is 500 miles long, and its propagation constant γ is $1{\cdot}82 \times 10^{-3} \underline{/76^\circ \, 13'}$. Calculate the various component voltages which go to make up the total voltage at the open end of the line at the instant when the applied alternating voltage has a value of 100,000.

$$\gamma = 1{\cdot}82 \times 10^{-3} \underline{/76^\circ\ 13'}$$
$$= 0{\cdot}435 \times 10^{-3} + j\,1{\cdot}78 \times 10^{-3}.$$
$$\therefore \gamma l = 0{\cdot}2175 + j\,0{\cdot}89.$$
$$\beta l = 0{\cdot}2175.$$
$$\alpha l = 0{\cdot}89 \text{ radians} = 51^\circ.$$

Substituting in equation for V, page 220, the voltage at end of line with voltage E at generator

$$= 2E\epsilon^{-\gamma l} - 2E\epsilon^{-3\gamma l} + 2E\epsilon^{-5\gamma l} \ldots \text{ and so on.}$$

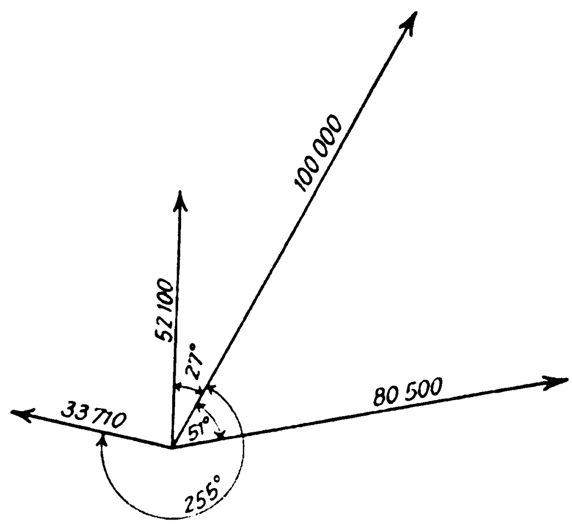


FIG. 113.—Illustrating numerical example. 100,000 volts at generator. Remaining vectors represent component voltages at end of line.

Now,
$$\epsilon^{-\beta l} = \epsilon^{-0{\cdot}2175} = 0{\cdot}805.$$
$$\epsilon^{-3\beta l} = \epsilon^{-0{\cdot}6525} = 0{\cdot}521.$$
$$\epsilon^{-5\beta l} = \epsilon^{-1{\cdot}0875} = 0{\cdot}3371.$$

∴ Voltage at end of line is

$$100{,}000\{0{\cdot}805\underline{/-51^\circ} - 0{\cdot}521\underline{/-153^\circ} + 0{\cdot}3371\underline{/-255^\circ} + \ldots\}$$
$$= 80{,}500\underline{/-51^\circ} + 52{,}100\underline{/27^\circ} + 33{,}710\underline{/-255^\circ} + \ldots$$

These vectors are shown in Fig. 113, and their addition will give the total voltage at the end of the line.

Quarter Wave Resonance.—It is clear from the preceding example that if $\alpha l = \frac{\pi}{2}$ radians, then all the various voltages which go to make up the total potential at the end of the line will be in phase, the arguments of the various vectors being

$$\underline{/-90^\circ} - \underline{/-270^\circ} + \underline{/-450^\circ} + \dots$$
$$= \underline{/-90^\circ} + \underline{/-90^\circ} + \underline{/-90^\circ} + \dots$$

Under these circumstances there will be a steady building up of voltage at the end of the line, and the total voltage reached may be greatly in excess of the voltage impressed by the generator. This phenomenon is called "Quarter Wave Resonance."

The condition of $\alpha l = \frac{\pi}{2}$ is obtained when the length of the line is one-quarter of its wave-length.

From formula (45), page 28,

$$\text{wave-length} = \frac{2\pi}{\alpha}.$$

If $\alpha l = \frac{\pi}{2}$, then wave-length $= \frac{2\pi}{\pi/2l} = 4l$, where l is the length of the line.

Hence $$l = \frac{\text{Wave-length}}{4} \quad . \quad . \quad . \quad (146)$$

It is on this account that the phenomenon is called "Quarter-Wave Resonance."

In the problem just considered one-quarter wave-length will be obtained with a line of length l, where

$$l = \frac{\pi}{2\alpha} = \frac{\pi}{2 \times 1{\cdot}78 \times 10^{-3}} = 840 \text{ miles.}$$

Considering the special case of R = G = O,

$$\alpha = w\sqrt{\text{CL}} \text{ from formula (67), page 35,}$$
$$= 2\pi f\sqrt{\text{CL}}.$$

Now, $$\frac{1}{\sqrt{CL}} = 186{,}000 \text{ for an overhead line.}$$

$$\therefore \alpha = \frac{2\pi f}{186000}.$$

$$\therefore \text{Length for resonance} = \frac{\pi}{2\alpha} = \frac{\pi \times 186000}{4\pi f}$$

$$= \frac{46500}{f} \text{ miles} \quad (147)$$

where f is the frequency in cycles per second.

The following table gives the length of line required for quarter-wave resonance at various commercial frequencies under the assumed conditions of $R = G = O$:—

Frequency.	Length for Resonance.
15 cycles per second.	3100 miles.
25 ,, ,,	1860 ,,
50 ,, ,,	930 ,,
60 ,, ,,	775 ,,

If harmonics are present in the impressed voltage wave, then the lengths of line which give resonance with the harmonics are shorter than those given above. For example, the lengths for 50 cycle fundamental frequency are given in the following table :—

Frequency.	Length for Resonance.
50 cycles per second.	930 miles.
150 ,, ,,	310 ,,
250 ,, ,,	186 ,,
350 ,, ,,	133 ,,

The effect of line resistance and leakance is to reduce the length of line at which resonance occurs. As a rough approximation, it may be taken that if the length of the line is 90 per cent. of the above figures, then there is danger of quarter-wave resonance. The calculation of the exact length for resonance may be made as in the example on page 223.

Travelling Waves in Three-phase System.—Travelling waves in the three parallel conductors of a 3-phase transmission line can be dealt with by elaborating the theory for the single phase case which has been given in the preceding pages. The problem is complicated by the interaction of the currents in the various conductors. The mathematical solution is given in a paper by Dr. S. Bekku, published in the "Proceedings of the Japanese Institution of Electrical Engineers," February, 1923. The solution of a very similar problem, "Electric Oscillations in the Double-circuit Three-phase Transmission Line," is given by T. Satoh in the *Journal of the American Institution of Electrical Engineers*, September, 1927.

APPENDIX.

TABLE I.

Sizes of Hard-drawn Copper Telephone Conductors (at 60° F.). English, American, and Continental.

British Standard Wire Gauge.	Brown and Sharpe Gauge. American.	Diameter.		Weight.		Resistance.	
		Inches.	Millimetres.	Lbs. per mile.	Kilogrammes per kilometre.	Ohms per mile.	Ohms per kilometre.
4·4		0·2237	5·683	800·0	225·5	1·098	0·6825
		0·1968	5·0	620·98	175·0	1·416	0·879
5·9		0·1937	4·921	600·0	169·1	1·465	0·910
		0·1772	4·5	502·93	141·75	1·747	1·085
	6	0·162	4·1147	420·6	118·53	2·089	1·297
8·1		0·1582	4·018	400·0	112·7	2·197	1·365
		0·1575	4·0	397·4	112·0	2·212	1·374
		0·1378	3·5	304·27	85·75	2·886	1·792
9·4		0·1370	3·48	300·0	84·56	2·929	1·820
		0·1181	3·0	223·56	63·0	3·932	2·442
	8	0·1285	3·26	264·6	74·56	3·321	2·062
11·3		0·1119	2·841	200·0	56·37	4·394	2·730
		0·0984	2·5	155·24	43·75	5·663	3·517
	10	0·1019	2·59	166·4	46·91	5·28	3·279
12·6		0·0969	2·461	150·0	42·28	5·858	3·640
	11	0·09079	2·306	131·8	37·15	6·667	4·14
		0·07874	2·0	99·35	28·0	8·847	5·494
14·1		0·0791	2·009	100·0	28·19	8·787	5·460
	12	0·0808	2·05	104·6	29·48	8·4	5·22
	13	0·0720	1·829	83·09	23·42	10·58	6·57
15·7		0·0662	1·681	70·0	19·73	12·55	7·80
	14	0·06408	1·63	65·85	18·58	13·34	8·28
		0·0591	1·5	55·88	15·75	15·72	9·78
	15	0·05706	1·45	52·26	14·73	16·81	10·44
	16	0·05082	1·29	41·36	11·66	21·25	13·19
Between 17-18		0·050	1·27	40·0	11·28	21·96	13·63
	17	0·0453	1·151	32·89	9·27	26·70	16·59
	18	0·0403	1·017	26·03	7·34	33·73	20·97
		0·0394	1·0	24·84	7·00	35·37	21·97
	19	0·0359	0·91	20·30	5·79	43·26	26·57
		0·0354	0·9	20·12	5·67	43·66	27·13
Between 19-20		0·0352	0·893	20·0	5·64	43·97	27·28
		0·0315	0·8	15·90	4·48	55·2	34·33
		0·0276	0·7	12·17	3·43	72·2	44·8
	22	0·0253	0·64	10·259	2·89	85·6	53·2
Between 22-23		0·025	0·635	10·0	2·82	87·8	54·5
		0·0236	0·6	8·942	2·52	98·2	61·0
	23	0·0226	0·57	8·186	2·31	107·3	66·6
	24	0·0201	0·51	6·475	1·82	135·7	84·5
		0·0197	0·5	6·21	1·75	141·5	87·8

This table is also approximately correct for resistance according to the I.E.C. standard for annealed copper conductors at 20° C. The error is less than half of one per cent.

TABLE II.

TELEPHONE TRANSMISSION DATA FOR CABLE AND AERIAL CONDUCTORS USED IN THE UNITED KINGDOM.

Type of Line.	Constants per mile Loop. R Ohms.	L Henrys.	G Micro-mhos.	C Micro-farads.	Propagation Constant.	Attenuation Constant.	Wave-length Constant.	Characteristic Impedance.	No. of miles of Standard Cable equal in attenuation to 1 mile of Line.	No. of miles of Line equal in attenuation to 1 mile of Standard Cable.
Underground A.S.P.C. Cable.										
Standard Cable	88·00	0·001	1	0·054	0·15427/46° 31′	0·10616	0·11193	571·4\43° 16′	1·0000	1·0000
6½-lb. cable .	270·77	0·001	1	0·065	0·29668/45° 26′	0·20819	0·21136	912·9\44° 22′	1·9611	0·5099
10-lb. ,, .	176·00	0·001	1	0·065	0·23921/45° 43′	0·16702	0·17126	736·0\44° 5′	1·5733	0·6356
20-lb. ,, .	88·00	0·001	1	0·065	0·16926/46° 32′	0·11643	0·12284	520·8\43° 17′	1·0965	0·9117
40-lb. ,, .	44·00	0·001	1	0·065	0·11997/48° 9′	0·08004	0·08935	369·1\41° 40′	0·7540	1·3263
50-lb. ,, .	35·20	0·001	1	0·065	0·10749/48° 57′	0·07059	0·08106	330·7\40° 52′	0·6650	1·5038
70-lb. ,, .	25·14	0·001	1	0·065	0·09127/50° 32′	0·05802	0·07046	280·8\39° 17′	0·5465	1·8298
100-lb. ,, .	17·60	0·001	1	0·065	0·07711/52° 50′	0·04659	0·06145	237·3\36° 59′	0·4388	2·2787
150-lb. ,, .	11·73	0·001	1	0·065	0·06437/56° 27′	0·03558	0·05365	198·1\33° 22′	0·3351	2·9840
200-lb. ,, .	8·80	0·001	1	0·065	0·05735/59° 42′	0·02894	0·04952	176·5\30° 6′	0·2726	3·6685
300-lb. ,, .	5·87	0·001	1	0·065	0·05006/65° 7′	0·02106	0·04541	154·0\24° 41′	0·1984	5·0398
Aerial Wire										
40-lb. Bronze .	91·00	0·00419	1	0·0075	0·05928/50° 43′	0·03755	0·04857	1581·0\37° 45′	0·3540	2·825
70-lb. ,, .	52·00	0·00419	1	0·0079	0·04727/55° 15′	0·02694	0·03884	1197·0\33° 18′	0·2539	3·939
100-lb. Copper .	17·60	0·00390	1	0·0081	0·03259/68° 20′	0·01205	0·03029	804·7\20° 15′	0·1136	8·803
150-lb. ,, .	11·73	0·00376	1	0·0084	0·03050/73° 22′	0·00874	0·02922	725·3\15° 16′	0·0824	12·139
200-lb. ,, .	8·80	0·00366	1	0·0086	0·02955/76° 31′	0·00690	0·02878	687·0\12° 8′	0·0650	15·375
300-lb. ,, .	5·87	0·00355	1	0·0089	0·02888/80° 16′	0·00488	0·02846	647·3\8° 26′	0·0460	21·743
400-lb. ,, .	4·40	0·00344	1	0·0092	0·02858/82° 13′	0·00387	0·02834	620·9\6° 32′	0·0365	27·412
600-lb. ,, .	2·93	0·00331	1	0·0096	0·02841/84° 22′	0·00279	0·02827	591·6\4° 25′	0·0263	38·023
800-lb. ,, .	2·20	0·00322	1	0·0099	0·02836/85° 32′	0·00221	0·02827	572·8\3° 19′	0·0208	48·008
400-lb. Iron .	60·00	0·01200	1	0·0100	0·06514/66° 55′	0·02554	0·05993	1303·0\21° 55′	0·2408	4·153

TABLE III.

VALUES OF SINH $(a + jb)$ OR HYPERBOLIC SINES OF VECTORS OF VARIOUS SIZES AND SLOPES GIVEN IN THE FORM $A\underline{/\theta}$.

Ratio b/a 1 to 10. Size $\sqrt{a^2 + b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}$ = 1.		= 2.		= 3.		= 4.		= 10.	
$\sqrt{a^2 + b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0	0·0000	45° 00′	0·0000	63° 26′	0·0000	71° 34′	0·0000	75° 58′	0·0000	84° 17′
0·1	0·1000	45° 05′	0·0999	63° 30′	0·0998	71° 34′	0·0999	76° 01′	0·0998	84° 17′
0·2	0·2000	45° 23′	0·1992	63° 44′	0·1988	71° 48′	0·1991	76° 09′	0·1986	84° 22′
0·3	0·2999	45° 51′	0·2973	64° 08′	0·2965	72° 65′	0·2960	76° 22′	0·2955	84° 28′
0·4	0·4000	46° 31′	0·3937	64° 40′	0·3916	72° 52′	0·3907	76° 41′	0·3897	84° 36′
0·5	0·5003	47° 23′	0·4877	65° 22′	0·4835	73° 01′	0·4818	77° 06′	0·4800	84° 46′
0·6	0·6006	48° 27′	0·5789	66° 13′	0·5718	73° 40′	0·5688	77° 38′	0·5655	84° 59′
0·7	0·7010	49° 40′	0·6668	67° 15′	0·6555	74° 27′	0·6508	78° 14′	0·6454	85° 15′
0·8	0·8016	51° 06′	0·7510	68° 27′	0·7341	75° 22′	0·7272	78° 57′	0·7191	85° 33′
0·9	0·9033	52° 44′	0·8310	69° 50′	0·8070	76° 25′	0·7973	79° 47′	0·7865	85° 55′
1·0	1·0055	54° 32′	0·9066	71° 23′	0·8739	77° 37′	0·8605	80° 45′	0·8448	86° 19′
1·1	1·1089	56° 31′	0·9775	73° 08′	0·9355	79° 03′	0·9219	81° 50′	0·8956	86° 47′
1·2	1·2138	58° 41′	1·0435	74° 46′	0·9877	80° 31′	0·9647	83° 03′	0·9377	87° 19′
1·3	1·3205	61° 02′	1·1047	77° 15′	1·0342	82° 13′	1·0049	84° 25′	0·9706	87° 55′
1·4	1·4297	63° 34′	1·1610	79° 38′	1·0730	84° 07′	1·0565	85° 57′	0·9941	88° 35′
1·5	1·5418	66° 15′	1·2125	82° 14′	1·1047	86° 14′	1·0606	87° 42′	1·0081	89° 20′

Note.—1 = tan 45°, 2 = tan 63° 30′, 3 = tan 71° 35′, 4 = tan 76°, and 10 = tan 84° 20′.
Hence sinh $1{\cdot}5\underline{/76^\circ} = 1{\cdot}0606\underline{/87^\circ\ 42'}$.

TABLE IV.

VALUES OF COSH $(a + jb)$ OR HYPERBOLIC COSINES OF VECTORS OF VARIOUS SIZES AND SLOPES GIVEN IN THE FORM $A/\underline{\theta}$.

Ratio b/a 1 to 10. Size $\sqrt{a^2 + b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}$ = 1.		= 2.		= 3.		= 4.		= 10.	
$\sqrt{a^2 + b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′
0·1	1·0000	0° 17′	0·9970	0° 14′	0·9960	0° 11′	0·9956	0° 08′	0·9951	0° 04′
0·2	1·0001	1° 09′	0·9881	0° 56′	0·9841	0° 41′	0·9824	0° 33′	0·9805	0° 14′
0·3	1·0007	2° 35′	0·9736	2° 06′	0·9645	1° 35′	0·9607	1° 15′	0·9562	0° 31′
0·4	1·0021	4° 35′	0·9538	3° 47′	0·9375	2° 52′	0·9308	2° 16′	0·9227	0° 57′
0·5	1·0051	7° 08′	0·9295	6° 01′	0·9038	4° 36′	0·8930	3° 38′	0·8801	1° 33′
0·6	1·0107	10° 16′	0·9010	8° 52′	0·8637	6° 51′	0·8479	5° 26′	0·8291	2° 19′
0·7	1·0198	13° 53′	0·8706	12° 22′	0·8184	9° 40′	0·7966	7° 44′	0·7702	3° 20′
0·8	1·0336	18° 00′	0·8385	16° 37′	0·7693	13° 13′	0·7399	10° 39′	0·7040	4° 39′
0·9	1·0533	22° 34′	0·8071	21° 41′	0·7177	17° 38′	0·6791	14° 22′	0·6313	6° 22′
1·0	1·0803	27° 29′	0·7782	27° 36′	0·6656	23° 07′	0·6160	19° 09′	0·5533	8° 41′
1·1	1·1157	32° 41′	0·7542	34° 26′	0·6137	30° 02′	0·5531	25° 19′	0·4712	11° 56′
1·2	1·1608	38° 05′	0·7378	42° 06′	0·5716	38° 08′	0·4935	33° 19′	0·3865	16° 44′
1·3	1·2162	43° 35′	0·7316	50° 28′	0·5370	47° 59′	0·4422	43° 39′	0·3026	24° 21′
1·4	1·2832	49° 05′	0·7376	59° 18′	0·5165	59° 15′	0·4052	56° 34′	0·2250	37° 40′
1·5	1·3616	54° 33′	0·7571	68° 18′	0·5138	71° 23′	0·3894	71° 34′	0·1687	62° 16′

TABLE V.

VALUES OF TANH $(a + jb)$ OR HYPERBOLIC TANGENTS OF VECTORS OF VARIOUS SIZES AND SLOPES GIVEN IN THE FORM $A\underline{/\theta}$.

Ratio b/a 1 to 10. Size $\sqrt{a^2 + b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}$ = 1.		= 2.		= 3.		= 4.		= 10.	
$\sqrt{a^2 + b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0	0·0000	45° 00′	0·0000	63° 26′	0·0000	71° 34′	0·0000	75° 58′	0·0000	84° 18′
0·1	0·1000	44° 48′	0·1002	63° 16′	0·1002	71° 23′	0·1004	75° 53′	0·1003	84° 13′
0·2	0·1999	41° 14′	0·2016	62° 48′	0·2020	71° 07′	0·2027	75° 36′	0·2026	84° 08′
0·3	0·2997	43° 16′	0·3054	62° 02′	0·3074	70° 30′	0·3081	75° 07′	0·3091	83° 57′
0·4	0·3991	41° 56′	0·4128	60° 53′	0·4178	70° 00′	0·4197	74° 25′	0·4223	83° 39′
0·5	0·4977	40° 15′	0·5248	59° 21′	0·5350	68° 35′	0·5395	73° 28′	0·5454	83° 13′
0·6	0·5942	38° 11′	0·6425	57° 21′	0·6620	66° 49′	0·6708	72° 12′	0·6820	82° 40′
0·7	0·6874	35° 47′	0·7659	54° 53′	0·8010	64° 47′	0·8170	70° 30′	0·8380	81° 55′
0·8	0·7756	33° 06′	0·8956	51° 50′	0·9542	62° 09′	0·9828	68° 18′	1·0213	80° 54′
0·9	0·8576	30° 10′	1·0295	48° 09′	1·1243	58° 47′	1·1740	65° 25′	1·2460	79° 33′
1·0	0·9306	27° 03′	1·1650	43° 47′	1·3130	54° 30′	1·3967	61° 36′	1·5268	77° 38′
1·1	0·9939	23° 50′	1·2960	38° 42′	1·5242	40° 01′	1·6667	56° 31′	1·9006	74° 51′
1·2	1·0456	20° 36′	1·4141	32° 40′	1·7280	42° 23′	1·9548	49° 44′	2·4258	70° 35′
1·3	1·0857	17° 27′	1·5100	26° 37′	1·9260	34° 14′	2·2722	40° 46′	3·2070	63° 34′
1·4	1·1141	14° 29′	1·5740	20° 20′	2·0777	24° 52′	2·6070	29° 23′	4·4180	50° 55′
1·5	1·1323	11° 42′	1·6015	13° 56′	2·1500	14° 51′	2·7235	16° 08′	5·9750	27° 04′

TABLE VI.

VALUES OF COSECH $(a + jb)$ OR HYPERBOLIC COSECANTS OF VECTORS OF VARIOUS SIZES AND SLOPES GIVEN IN THE FORM $A\underline{/\theta}$.

Ratio b/a 1 to 10. Size $\sqrt{a^2 + b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}$ = 1.		= 2.		= 3.		= 4.		= 10.	
$\sqrt{a^2 + b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0		45° 00′		63° 26′		71° 34′		75° 58′		84° 17′
0·1	10·0025	45° 05′	10·009	63° 30′	10·017	71° 34′	10·010	76° 01′	10·021	84° 17′
0·2	5·0011	45° 23′	6·020	63° 44′	5·0301	71° 48′	5·0228	76° 09′	5·035	84° 22′
0·3	3·3340	45° 51′	3·363	64° 08′	3·3732	72° 05′	3·3780	76° 22′	3·384	84° 28′
0·4	2·5002	46° 31′	2·540	64° 40′	2·5540	72° 52′	2·560	76° 41′	2·566	84° 36′
0·5	1·9989	47° 23′	2·0502	65° 22′	2·0680	73° 01′	2·0754	77° 06′	2·083	84° 46′
0·6	1·6651	48° 27′	1·7273	66° 13′	1·7488	73° 40′	1·7580	77° 38′	1·786	84° 59′
0·7	1·4265	49° 40′	1·4997	67° 15′	1·5255	74° 27′	1·5365	78° 14′	1·549	85° 15′
0·8	1·2475	51° 06′	1·3316	68° 27′	1·3622	75° 22′	1·3752	78° 57′	1·391	85° 33′
0·9	1·1070	52° 44′	1·2034	69° 50′	1·2391	76° 25′	1·2543	79° 47′	1·272	85° 55′
1·0	0·9945	54° 32′	1·1030	71° 23′	1·1442	77° 37′	1·1621	80° 45′	1·184	86° 19′
1·1	0·9018	56° 31′	1·0230	73° 08′	1·0689	79° 03′	1·0847	81° 50′	1·117	86° 47′
1·2	0·8238	58° 41′	0·9583	74° 46′	1·0125	80° 31′	1·0366	83° 03′	1·067	87° 19′
1·3	0·7573	61° 02′	0·9052	77° 15′	0·9670	82° 13′	0·9951	84° 25′	1·030	87° 55′
1·4	0·6995	63° 34′	0·8613	79° 38′	0·9319	84° 07′	0·9465	85° 57′	1·006	88° 35′
1·5	0·6486	66° 15′	0·8247	82° 14′	0·9051	86° 14′	0·9434	87° 42′	0·992	89° 20′

N.B.—All the angles in this table are negative.

TABLE VII.

VALUES OF SECH $(a + jb)$ OR HYPERBOLIC SECANTS OF VECTORS OF VARIOUS SIZES AND SLOPES GIVEN IN THE FORM $A\underline{/\theta}$.

Ratio b/a 1 to 10. Size $\sqrt{a^2 + b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}$ = 1.		= 2.		= 3.		= 4.		= 10.	
$\sqrt{a^2 + b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′
0·1	1·0000	0° 17′	1·0030	0° 14′	1·0040	0° 11′	1·0044	0° 08′	1·0049	0° 04′
0·2	0·9999	1° 09′	1·0120	0° 56′	1·0162	0° 41′	1·0179	0° 33′	1·0199	0° 14′
0·3	0·9993	2° 35′	1·0272	2° 06′	1·0369	1° 35′	1·0409	1° 15′	1·0458	0° 31′
0·4	0·9979	4° 35′	1·0485	3° 47′	1·0667	2° 52′	1·0744	2° 16′	1·0838	0° 57′
0·5	0·9949	7° 08′	1·0759	6° 01′	1·1065	4° 36′	1·1199	3° 38′	1·1362	1° 33′
0·6	0·9894	10° 16′	1·1099	8° 52′	1·1579	6° 51′	1·1793	5° 26′	1·2061	2° 19′
0·7	0·9806	13° 53′	1·1487	12° 22′	1·2219	9° 40′	1·2553	7° 44′	1·2984	3° 20′
0·8	0·9675	18° 00′	1·1926	16° 37′	1·2998	13° 13′	1·3516	10° 39′	1·4205	4° 39′
0·9	0·9494	22° 34′	1·2390	21° 41′	1·3930	17° 38′	1·4726	14° 22′	1·5840	6° 22′
1·0	0·9256	27° 29′	1·2850	27° 36′	1·5025	23° 07′	1·6234	19° 09′	1·8073	8° 41′
1·1	0·8963	32° 41′	1·3259	34° 26′	1·6296	30° 02′	1·8081	25° 19′	2·1222	11° 56′
1·2	0·8614	38° 05′	1·3553	42° 06′	1·7496	38° 08′	2·0262	33° 19′	2·5873	16° 44′
1·3	0·8222	43° 35′	1·3669	50° 28′	1·8623	47° 59′	2·2613	43° 39′	3·3047	24° 21′
1·4	0·7793	49° 05′	1·3557	59° 18′	1·9360	59° 15′	2·4677	56° 34′	4·4703	37° 40′
1·5	0·7344	54° 13′	1·3208	68° 18′	1·9463	71° 23′	2·5682	71° 34′	5·9277	62° 16′

N.B.—All the angles in this table are negative.

TABLE VIII.

VALUES OF COTH $(a + jb)$ OR HYPERBOLIC COTANGENTS OF VECTORS OF VARIOUS SIZES AND SLOPES GIVEN IN THE FORM $A\underline{/\theta}$.

Ratio b/a 1 to 10. Size $\sqrt{a^2 + b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}$ = 1.		= 2.		= 3.		= 4.		= 10.	
$\sqrt{a^2 + b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0		45° 00′		63° 26′		71° 34′		75° 58′		84° 18′
0·1	10·0025	44° 48′	9·980	63° 16′	9·980	71° 23′	9·960	75° 53′	9·970	84° 13′
0·2	5·0018	44° 14′	4·960	62° 48′	4·950	71° 07′	4·933	75° 36′	4·936	84° 08′
0·3	3·3365	43° 16′	3·274	62° 02′	3·253	70° 30′	3·246	75° 07′	3·235	83° 57′
0·4	2·5050	41° 56′	2·422	60° 53′	2·393	70° 00′	2·383	74° 25′	2·370	83° 39′
0·5	2·0092	40° 15′	1·905	59° 21′	1·869	68° 35′	1·854	73° 28′	1·834	83° 13′
0·6	1·6830	38° 11′	1·556	57° 21′	1·511	66° 49′	1·491	72° 12′	1·466	82° 40′
0·7	1·4547	35° 47′	1·306	54° 53′	1·2484	64° 47′	1·224	70° 30′	1·193	81° 55′
0·8	1·2894	33° 06′	1·117	51° 50′	1·0480	62° 09′	1·018	68° 18′	0·9791	80° 54′
0·9	1·1660	30° 10′	0·9714	48° 09′	0·8894	58° 47′	0·8518	65° 25′	0·8026	79° 33′
1·0	1·0746	27° 03′	0·8584	43° 47′	0·7616	54° 30′	0·7160	61° 36′	0·6550	77° 38′
1·1	1·0061	23° 50′	0·7716	38° 42′	0·6560	49° 01′	0·6000	56° 31′	0·5261	74° 51′
1·2	0·9564	20° 36′	0·7071	32° 40′	0·5787	42° 23′	0·5116	49° 44′	0·4122	70° 35′
1·3	0·9211	17° 27′	0·6623	26° 37′	0·5192	34° 14′	0·4401	40° 46′	0·3118	63° 34′
1·4	0·8996	14° 29′	0·6353	20° 20′	0·4815	24° 52′	0·3836	29° 23′	0·2263	50° 55′
1·5	0·8831	11° 42′	0·6244	13° 56′	0·4651	14° 51′	0·3672	16° 08′	0·1674	27° 04′

N.B.—All the angles in this table are negative.

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