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ABSTRACT

An analytical expression is obtained for the aerodynamic torque, which a space vehicle, configured as a finite cylinder and two flat solar panels, encounters while passing through rarefied air at a high altitude. The torque is evaluated separately on each component unit, taking into consideration the front, back, and end surfaces and the general direction of the air velocity with respect to the vehicle. The overall torque is given as the sum of the torques on the component units.

The analytical expression of the torque may be used to investigate the equilibrium and stability of the attitude motion of the vehicle. Also, for a given attitude orientation of the vehicle, the torque may be integrated around an orbit to give the magnitude of angular momentum that an active attitude control system will have to counteract.

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FROM: E. Y. Yu

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TECHNICAL MEMORANDUM

I. INTRODUCTION

At orbital altitudes higher than 100 n. miles, the air density is at most in the order of 10^{-13} gm/cm^3 and the molecular mean-free-path is at least in the order of 2500 ft. If the maximum linear dimension of the vehicle is 125 ft, the Knudsen number (= mean-free-path/body linear dimension) is at least 20. With such a large Knudsen number, the flow of rarefied air over the vehicle is classified as free-molecular flow, in which no collisions occur between molecules impinging to and re-emitted from the vehicle. The flow effects such as drag, torque, and heat transfer, can be computed by assuming that the impinging molecules have a Maxwellian velocity distribution and are partially or perfectly accommodated to the vehicle surface before reemission. The Maxwellian distribution function is an equilibrium solution of the Boltzmann equation (which governs the rarefied gas flow). Therefore, the computation of free-molecular flow can completely bypass the difficult task of solving the Boltzmann equation.

Since free-molecular flow theory is relatively simple, many results on the drag, lift, and heat transfer of bodies of various shapes have appeared in journals, reports, and monographs (see, e.g., References [1], [2]). But very few general results on aerodynamic torque are reported in the literature. Those appearing in articles on dynamics analysis (see, e.g., Reference [3]) are not wholly analytical in that they are obtained by fitting an analytical expression to numerical data. It is therefore the purpose of this report to record a general way of getting analytical expressions of aerodynamic torque on several body shapes.

The essential effects of the aerodynamic torque on the motion of the vehicle are two-fold. First, the torque affects the attitude equilibrium and stability of the rigid body motion. Second, if the vehicle is maintained inertially oriented by means of control moment gyros, as in the case of the AAP (Apollo Applications Program) cluster vehicle, it is of interest to know the nonzero time integral of the aerodynamic torque per orbit for the design of the active control system. To investigate these two problems, it is useful to obtain an analytical expression for the aerodynamic torque.

II. FREE-MOLECULAR TORQUE ON A FINITE CYLINDER

A. Notations and Coordinate Systems

Let us define $0 - X_oY_oZ_o$ as the coordinate system on a finite cylinder of length L and radius R_o, with the origin O situated at the center of geometry. Define O-XYZ as the coordinates with $OZ||OZ_o$ and OX,OY making an angle θ with OX_o,OY_o (see Figure 1). Denote the position vectors of the center of mass M in the $0 - X_oY_oZ_o$ and $0 - XYZ$ coordinate systems by $\vec{R}_{om}(X_o,Y_o,Z_o)$ and $\vec{R}_m(X_m,Y_m,Z_m)$, respectively, so that

$$\begin{aligned} X_m &= X_o \cos\theta + Y_o \sin\theta \\ Y_m &= -X_o \sin\theta + Y_o \cos\theta \\ Z_m &= Z_o \end{aligned} \tag{1}$$

The uniform air velocity is denoted by $\vec{U}$, which lies in the XZ-plane and makes an angle ϕ with OZ. On a front cylindrical surface element we define a local coordinate system xyz, with the x-axis pointing normally into the surface and the z-axis being parallel to OZ. The angle θ that the component of $\vec{U}$ in the XY-plane makes with the x-axis is negative when the surface element is in the first quadrant of the XY-plane and is positive when the surface element is in the fourth quadrant. We define a back surface element as the one symmetric to the front element with respect to the center of symmetry O when projected on the XY-plane. The coordinates fixed to the back surface element are obtained by a translation of those of the front surface element.

The Maxwellian distribution function of the molecules from infinity impinging on a surface element is

$$F_m = \frac{m\, n_\infty}{(2\pi R\, T_\infty)^{3/2}} \exp\left\{-\frac{1}{2R\,T_\infty}\left[(\xi_x-U_x)^2 + (\xi_y-U_y)^2 + (\xi_z-U_z)^2\right]\right\} \tag{2}$$

where n_∞ and T_∞ are the number density and temperature of the air at infinity, R is the gas constant and m is the molecular mass.* Both the molecular velocity, $\vec{\xi}$, and the air velocity, $\vec{U}$, are written in their components in the local coordinates, namely,

$$U_x = U \sin\phi\cos\theta \quad , \quad U_y = U \sin\phi\sin\theta \quad , \quad U_z = -U \cos\phi \qquad (3)$$

Here, θ ranges between 0 and $-\frac{\pi}{2}$ when the local coordinate system is situated in the first quadrant of the XY-plane, and between 0 and π/2 when the local coordinate system is in the fourth quadrant. The unit vectors of the local coordinate system are given in the 0 - XYZ system as

$$\begin{aligned} \hat{x} &= -\cos\theta\, \hat{X} + \sin\theta\, \hat{Y} \\ \hat{y} &= -\sin\theta\, \hat{X} - \cos\theta\, \hat{Y} \\ \hat{z} &= \hat{Z} \end{aligned} \qquad (4)$$

The position vector of the center of a surface element from 0 is denoted by $\vec{R}$ and from M by $\vec{r}$, with subscript f or b to indicate the front or the back side. These position vectors are related to each other by (see Figure 2)

$$\vec{r}_f = \vec{R}_f - \vec{R}_m \quad , \quad \vec{r}_b = \vec{R}_b - \vec{R}_m \qquad (5)$$

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*F_m is actually the product of the gas density ($m\, n_\infty$) and the joint probability density function of molecular velocity. F_m has the significance that $(1/m)F_m dxdydz\, d\xi_x d\xi_y d\xi_z$ is the number of molecules within the volume element dxdydz having component velocities in the range ξ_x and $\xi_x + d\xi_x$, etc. (see [4].

where

$$\vec{R}_F = R_0 \cos\theta \hat{X} - R_0 \sin\theta \hat{Y} + Z\hat{Z} \tag{5a}$$

$$\vec{R}_B = -R_0 \cos\theta \hat{X} + R_0 \sin\theta \hat{Y} + Z\hat{Z} \tag{5b}$$

b. Forces on a Cylindrical Surface Due to Impinging Molecules

The force tensor produced by impinging molecules to a front cylindrical surface element $dA(=R_0 d\theta dZ)$ (written in the local coordinate system) is

$$\vec{P}_i^{(f)} = p_i^{(f)}\hat{x} + \tau_{y,i}^{(f)}\hat{y} + \tau_{z,i}^{(f)}\hat{z} = \iint_{-\infty}^{\infty}\int_0^{\infty} \xi_x(\xi_x\hat{x}+\xi_y\hat{y}+\xi_z\hat{z})F_m d\xi_x d\xi_y d\xi_z$$

$$= [\rho_\infty U^2/2\pi^{1/2}S^2]\Big\{[S_x\hat{x}+S_y\hat{y}+S_z\hat{z}][\pi^{1/2}S_x(1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$

$$+ \exp(-S_x^2)] + (\pi^{1/2}/2)[1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))\hat{x}\Big\} \tag{6a}$$

where $\hat{x}$ is a unit vector normal to dA. The force tensor on a corresponding back surface element dA is

$$\vec{P}_i^{(b)} = p_i^{(b)}\hat{x} + \tau_{y,i}^{(b)}\hat{y} + \tau_{z,i}^{(b)}\hat{z} = -\iint_{-\infty}^{\infty}\int_{-\infty}^{0} \xi_x(\xi_x\hat{x}+\xi_y\hat{y}+\xi_z\hat{z})F_m d\xi_x d\xi_y d\xi_z$$

$$= -(\rho_\infty U^2/2\pi^{1/2}S^2)\Big\{[S_x\hat{x}+S_y\hat{y}+S_z\hat{z}][\pi^{1/2}S_x(1-\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$

$$- \exp(-S_x^2)] + (\pi^{1/2}/2)[1-\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)]\hat{x}\Big\} \tag{6b}$$

where $S = U/(2R\ T_\infty)^{1/2}$ is the speed ratio, $S_{x,y,z} = U_{x,y,z}/(2R\ T_\infty)^{1/2}$, and $\rho_\infty = m\ n_\infty$ is the air density at infinity.

C. Tangential and Normal Forces Due to Reflected Molecules

Tangential and normal components of forces per unit area, τ_r and p_r, produced by molecules reflected from the wall, part diffusely and part specularly, may be computed by assuming a certain value of the accommodation coefficients[4] defined as

$$\sigma = \frac{\tau_i - \tau_r}{\tau_i - \tau_w} \quad , \quad \text{or} \quad \tau_r = (1-\sigma)\tau_i \tag{7a}$$

$$\sigma' = \frac{p_i - p_r}{p_i - p_w} \quad , \quad \text{or} \quad p_r = (1-\sigma')p_i + \sigma' p_w \tag{7b}$$

Here, p_w and $\tau_w(=0)$ are the normal and tangential components of the force resulted from the diffuse reflection of the molecules which assume a Maxwellian distribution function corresponding to a wall temperature T_w and a number density n_w:

$$F_w = \frac{m\, n_w}{(2\pi R\, T_w)^{3/2}} \exp\left[-\frac{1}{2R\, T_w}(\xi_x^2 + \xi_y^2 + \xi_z^2)\right] \tag{8}$$

One can easily compute p_w at the front and back surfaces as

$$p_w^{(f)} = \iint_{-\infty}^{\infty} \int_0^{\infty} \xi_x^2\, F_w\, d\xi_x d\xi_y d\xi_z = \frac{m}{2}(2\pi R\, T_w)^{1/2}\, N_w^{(f)} \tag{9}$$

where

$$N_w^{(f)} = \iint_{-\infty}^{\infty} \int_0^{\infty} \frac{1}{m} F_w \xi_x\, d\xi_x d\xi_y d\xi_z = n_w^{(f)}\, (R\, T_w/2\pi)^{1/2} \tag{9a}$$

is the number flux of the reflected molecules. This should be equal to the number flux $n_i^{(f)}$ of the impinging molecules if no ablation occurs at the surface, where

$$n_i^{(f)} = \iint_{-\infty}^{\infty} \int_0^{\infty} \frac{1}{m} f_m \xi_x \, d\xi_x d\xi_y d\xi_z$$

$$= n_\infty (k T_\infty / 2\pi)^{1/2} \left\{ \pi^{1/2} S_x [1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)] + \exp(-S_x^2) \right\} \tag{10}$$

Hence,

$$\vec{p}_w^{(f)} = (\rho_\infty U^2/4S^2)(T_w/T_\infty)^{1/2} \left\{ \pi^{1/2} S_x [1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)] + \exp(-S_x^2) \right\} \hat{x} \tag{11}$$

For the back surface, it is calculated in a similar way that

$$\vec{p}_w^{(b)} = (\rho_\infty U^2/4S^2)(T_w/T_\infty)^{1/2} \left\{ -\pi^{1/2} S_x [1-\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)] + \exp(-S_x^2) \right\} (-\hat{x}) \tag{12}$$

Therefore, τ_r and p_r for the front surface are

$$\begin{aligned} \tau_{y,r}^{(f)} &= (1-\sigma)\tau_{y,i}^{(f)} \\ \tau_{z,r}^{(f)} &= (1-\sigma)\tau_{z,i}^{(f)} \\ p_r^{(f)} &= (1-\sigma')p_i^{(f)} + \sigma' p_w^{(f)} \end{aligned} \tag{13}$$

Similar expressions are obtained for the back surface.

Here we have assumed the same accommodation coefficient for the y- and z-components of the tangential force. The definition of the accommodation coefficients given in (7a,b) signifies that for a perfect accommodation (i.e., completely diffuse reflection), $\sigma=1$ and $\sigma'=1$, and for zero accommodation (i.e., completely specular reflection), $\sigma=0$ and $\sigma'=0$. For a clean surface, the molecules are partially accommodated, and values of σ and σ' usually range between 0.8 and 0.95.

The foregoing procedure of computing the forces due to molecules reflected from the wall is standard as one may see, e.g., from Reference (5), where the analysis is made only on a front surface element. It should be pointed out that in Reference (5) the local coordinate system is taken in such a way that the incident air velocity $\vec{U}$ has a zero component in one coordinate direction. Consequently, the expressions given there for the normal and tangential components of the force on a surface element cannot be used with an arbitrary direction of $\vec{U}$ for an integration over an arbitrary concave surface to yield analytical results for the force or the torque.

D. The Resultant Force Tensor

The resultant force tensor on a surface element dA due to both the impinging and reflected molecules is, for the front part,

$$\bar{P}^{(f)} = p^{(f)}\hat{x} + \tau_y^{(f)}\hat{y} + \tau_z^{(f)}\hat{z} \tag{14}$$

where

$$p^{(f)} = p_i^{(f)} + p_r^{(f)} = (2-\sigma')p_i^{(f)} + \sigma' p_W^{(f)}$$

$$= \frac{\rho_\infty U^2}{2S^2}\Big\langle(2-\sigma')S_x[S_x(1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$

$$+ \pi^{-1/2}\exp(-S_x^2)] + \tfrac{1}{2}(2-\sigma')[1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)]$$

$$+ \tfrac{1}{2}\,\sigma'(T_W/T_\infty)^{1/2}[\pi^{1/2}S_x(1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)) + \exp(-S_x^2)]\Big\rangle$$

(14a)

$$\tau_y^{(f)} = \tau_{y,i}^{(f)} - \tau_{y,r}^{(f)} = \sigma\tau_{y,i}^{(f)} = \frac{\rho_\infty U^2}{2S^2}\,\sigma S_y[S_x(1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$
$$+ \pi^{-1/2}\exp(-S_x^2)] \quad (14b)$$

$$\tau_z^{(f)} = \tau_{z,i}^{(f)} - \tau_{z,r}^{(f)} = \sigma\tau_{z,i}^{(f)} = \frac{\rho_\infty U^2}{2S^2}\,\sigma S_z[S_x(1+\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$
$$+ \pi^{-1/2}\exp(-S_x^2)] \quad (14c)$$

On the back surface,

$$\vec{p}^{(b)} = p^{(b)}\hat{x} + \tau_y^{(b)}\hat{y} + \tau_z^{(b)}\hat{z} \quad (15)$$

where

$$p^{(b)} = p_i^{(b)} + p_r^{(b)} = (2-\sigma')p_i^{(b)} + \sigma' p_w^{(b)}$$
$$= -\frac{\rho_\infty U^2}{2S^2}\Big\{(2-\sigma')S_x[S_x(1-\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$
$$- \pi^{-1/2}\exp(-S_x^2)] + \tfrac{1}{2}(2-\sigma')[1-\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)]$$
$$+ \tfrac{1}{2}\sigma'(T_w/T_\infty)^{1/2}[-\pi^{1/2}S_x(1-\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$
$$+ \exp(-S_x^2)]\Big\} \quad (15a)$$

$$\tau_y^{(b)} = \sigma\tau_{y,i}^{(b)} = -\frac{\rho_\infty U^2}{2S^2}\,\sigma S_y[S_x(1-\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$
$$- \pi^{-1/2}\exp(-S_x^2)] \quad (15b)$$

$$\tau_z^{(b)} = \sigma\tau_{z,i}^{(b)} = -\frac{\rho_\infty U^2}{2S^2}\,\sigma S_z[S_x(1-\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|))$$

$$-\pi^{-1/2}\exp(-S_x^2)] \qquad (15c)$$

For convenience, let us define the sum and the difference of the forces on the front and back surface elements as

$$p^{(+)} = p^{(f)} + p^{(b)} = \frac{\rho_\infty U^2}{2S^2}\Big\{(2-\sigma')[(1+2S_x^2)\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)$$

$$+2\pi^{-1/2}S_x\exp(-S_x^2)] + \sigma'(\pi T_w/T_\infty)^{1/2}S_x\Big\} \qquad (16a)$$

$$\tau_y^{(+)} = \tau_y^{(f)} + \tau_y^{(b)} = \frac{\rho_\infty U^2}{2S^2}\,\sigma S_y[2S_x\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)$$

$$+2\pi^{-1/2}\exp(-S_x^2)] \qquad (16b)$$

$$\tau_z^{(+)} = \tau_z^{(f)} + \tau_z^{(b)} = \frac{\rho_\infty U^2}{2S^2}\,\sigma S_z[2S_x\mathrm{sgn}(S_x)\mathrm{erf}(|S_x|)$$

$$+2\pi^{-1/2}\exp(-S_x^2)] \qquad (16c)$$

$$\tau_y^{(-)} = \tau_y^{(f)} - \tau_y^{(b)} = \frac{\rho_\infty U^2}{2S^2}\,2\sigma S_x S_y \qquad (17a)$$

$$\tau_z^{(-)} = \tau_z^{(f)} - \tau_z^{(b)} = \frac{\rho_\infty U^2}{2S^2}\,2\sigma S_x S_z \qquad (17b)$$

E. Torque Due to Forces on the Cylindrical Surface

Torque due to the forces on the cylindrical surface is given by the following integration

$$\vec{T}_{C.S.} = R_o \int_{-\pi/2}^{+\pi/2} \int_{-L/2}^{L/2} [\vec{r}_f \times \vec{P}^{(f)} + \vec{r}_b \times \vec{P}^{(f)}] dZ d\theta \tag{18}$$

Integration with respect to Z and substitution of (4), (5), (14) and (15) yield

$$\begin{aligned} T_{C.S.} = R_o L \int_{-\pi/2}^{+\pi/2} \Big\{ & [-R_o \sin\theta \tau_z^{(-)} - Y_m \tau_z^{(+)} + Z_m (\sin\theta p^{(+)} - \cos\theta \tau_y^{(+)})]\hat{X} \\ & + [-R_o \cos\theta \tau_z^{(-)} + X_m \tau_z^{(+)} + Z_m (\cos\theta p^{(+)} + \sin\theta \tau_y^{(+)})]\hat{Y} \\ & + [-R_o \tau_y^{(-)} + (X_m \cos\theta - Y_m \sin\theta) \tau_y^{(+)} \\ & - (X_m \sin\theta + Y_m \cos\theta) p^{(+)}]\hat{Z} \Big\} d\theta \end{aligned} \tag{19}$$

in which the meaning of each term is quite clear. Now, upon substituting (16a,b,c) and (17a,b) into (19) and carrying out the θ-integration with the aid of the formulas tabulated in Appendix A, we finally obtain the expression of the torque due to the force acting on the cylindrical surface,

$$\vec{T}_{C.S.} = T_{SX}\hat{X} + T_{SY}\hat{Y} + T_{SZ}\hat{Z} \tag{20}$$

Here, the components are given as

$$T_{SX} = p_\infty (2R_oL) Y_m H_1(\phi) \tag{20a}$$

$$T_{SY} = p_\infty (2R_oL) [R_o (\pi\sigma/2) S^2 \sin\phi\cos\phi - X_m H_1(\phi) + Z_m H_2(\phi)] \tag{20b}$$

$$T_{SZ} = -p_\infty (2R_oL) Y_m H_2(\phi) \tag{20c}$$

where

$$h_1(\phi) = \pi^{1/2} S\sigma\cos\phi\exp(-S_X^2/2)[(1+S_X^2)I_0(S_X^2/2)+S_X^2 I_1(S_X^2/2)] \tag{20d}$$

$$h_2(\phi) = \frac{1}{4}\pi^{3/2}\sigma'(T_w/T_\infty)^{1/2}S_X + \pi^{1/2}S_X\exp(-S_X^2/2)[(2-\sigma')+\sigma/2]$$

$$[(1+\frac{2}{3}S_X^2)I_0(S_X^2/2) + \frac{1}{3}(1+2S_X^2)I_1(S_X^2/2)] \tag{20e}$$

Here $S_X = S\sin\phi$, $p_\infty = \rho_\infty R T_\infty$, and I_0, I_1 are the zeroth and first order modified Bessel functions of the first kind.

F. Torque Due to Forces on the End Surfaces of the Finite Cylinder

Referring to Figure 2, we denote $0'-x'y'z'$ as the local coordinates on the end surfaces of the finite cylinder. The unit vectors of the local coordinates and those of the $0-XYZ$ coordinate system are related by

$$\hat{x}' = \hat{X} \quad , \quad \hat{y}' = -\hat{Y} \quad , \quad \hat{z}' = -\hat{Z} \tag{21}$$

Similar to the procedure followed in Section B it can be shown that the force (per unit area) produced by impinging molecules on a surface element of the front and back ends is

$$\vec{P}_i^{(f)} = \tau_{x,i}^{(f)}\hat{x}' + p_i^{(f)}\hat{z}' = p_\infty\Big\{(S_{x'}\hat{x}'+S_{z'}\hat{z}')[\pi^{-1/2}\exp(-S_{z'}^2)$$

$$+ S_{z'}(1+\text{sgn}(S_{z'})\text{erf}(|S_{z'}|))] + \frac{1}{2}[1+\text{sgn}(S_{z'})\text{erf}(|S_{z'}|)]\hat{z}'\Big\} \tag{22a}$$

$$\vec{p}_i^{(b)} = \tau_{x,i}^{(b)}\hat{x}' + p_i^{(b)}\hat{z}' = -p_\infty\Big\{(S_{x'}\hat{x}'+S_{z'}\hat{z}')[-\pi^{-1/2}\exp(-S_{z'}^2)$$

$$+ S_{z'}(1-\text{sgn}(S_{z'})\text{erf}(|S_{z'}|))] + \tfrac{1}{2}[1-\text{sgn}(S_{z'})\text{erf}(|S_{z'}|)]\hat{z}'\Big\} \qquad (22b)$$

where $S_{x'} = -S\sin\phi$ and $S_{z'} = S\cos\phi$. The normal pressure produced by the molecules reflected diffusely from the front and back ends are

$$\vec{p}_w^{(f)} = \frac{1}{2}\,p_\infty\sqrt{\frac{T_w}{T_\infty}}\Big\{\sqrt{\pi}\,S_{z'}[1+\text{sgn}(S_{z'})\text{erf}(|S_{z'}|)] + \exp(-S_{z'}^2)\Big\}\hat{z}' \qquad (23a)$$

$$\vec{p}_w^{(b)} = \frac{1}{2}\,p_\infty\sqrt{\frac{T_w}{T_\infty}}\Big\{-\sqrt{\pi}S_{z'}[1-\text{sgn}(S_{z'})\text{erf}(|S_{z'}|)] + \exp(-S_{z'}^2)\Big\}(-\hat{z}') \qquad (23b)$$

Hence, the force tensor on a surface element at the front end produced by both the impinging and reflected molecules is

$$\vec{p}^{(f)} = \tau_x^{(f)}\hat{x}' + p^{(f)}\hat{z}' \qquad (24)$$

where

$$\tau_x^{(f)} = \sigma\tau_{x,i}^{(f)} = \sigma p_\infty S_{x'}[S_{z'}(1+\text{sgn}(S_{z'})\text{erf}(|S_{z'}|))$$

$$+ \pi^{-1/2}\exp(-S_{z'}^2)] \qquad (24a)$$

$$p^{(f)} = (2-\sigma')p_i^{(f)} + \sigma' p_w^{(f)} = p_\infty \Big\{ (2-\sigma')S_{z'}[S_{z'}(1+\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|))$$

$$+ \pi^{-1/2}\exp(-S_{z'}^2)] + \frac{1}{2}(2-\sigma')[1+\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|)]$$

$$+ \frac{\sigma'}{2}\left(\frac{T_w}{T_\infty}\right)^{1/2}[\pi^{1/2}S_{z'}(1+\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|)) + \exp(-S_{z'}^2)]\Big\} \qquad (24b)$$

The force tensor on the back end is

$$\vec{F}^{(b)} = \tau_x^{(b)}\hat{x}' + p^{(b)}\hat{z}' \qquad (25)$$

where

$$\tau_x^{(b)} = -\sigma p_\infty S_{x'}[S_{z'}(1-\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|)) - \pi^{-1/2}\exp(-S_{z'}^2)] \qquad (25a)$$

$$p^{(b)} = -p_\infty \Big\{ (2-\sigma')S_{z'}[S_{z'}(1-\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|))$$

$$- \pi^{-1/2}\exp(-S_{z'}^2)] + \frac{1}{2}(2-\sigma')[1-\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|)$$

$$+ \frac{\sigma'}{2}\left(\frac{T_w}{T_\infty}\right)^{1/2}[-\pi^{1/2}S_{z'}(1-\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|)$$

$$+ \exp(-S_{z'}^2)]\Big\} \qquad (25b)$$

Let us denote the vector from 0 to the center of the front end by $\vec{R}'_f = \frac{1}{2} L\hat{Z}$ and that from 0 to the center of the back end by $\vec{R}'_b = -\frac{1}{2} L\hat{Z}$. Then, the torque due to the forces on both ends is

$$\vec{T}_{END} = \pi R_0^2[(\vec{R}'_f-\vec{R}_m)\times \vec{P}^{(f)} + (\vec{R}'_b-\vec{R}_m)\times \vec{P}^{(b)}]$$

$$= \pi R_0^2\left\{Y_m p^{(+)}\hat{X} + [\tfrac{L}{2}\tau_x^{(-)} - Z_m\tau_x^{(+)} - X_m p^{(+)}]\hat{Y} + Y_m\tau_x^{(+)}\hat{Z}\right\} \qquad (26)$$

where

$$p^{(+)} = p^{(f)} + p^{(b)} = p_\infty\Big\{2(2-\sigma')S_{z'}[S_{z'}\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|) + \pi^{-1/2}\exp(-S_{z'}^2)] + (2-\sigma')\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|) + \sigma'(\pi T_W/T_\infty)^{1/2}S_{z'}\Big\} \qquad (26a)$$

$$\tau_x^{(+)} = \tau_x^{(f)} + \tau_x^{(b)} = 2\sigma p_\infty S_{x'}[S_{z'}\mathrm{sgn}(S_{z'})\mathrm{erf}(|S_{z'}|) + \pi^{-1/2}\exp(-S_{z'}^2)] \qquad (26b)$$

$$\tau_x^{(-)} = \tau_x^{(f)} - \tau_x^{(b)} = 2\sigma p_\infty S_{x'}S_{z'} \qquad (26c)$$

We may also write $\vec{T}_{END}$ as

$$\vec{T}_{END} = T_{EX}\hat{X} + T_{EY}\hat{Y} + T_{EZ}\hat{Z} \qquad (27)$$

Here

$$T_{EX}/\pi R_0^2 p_\infty = Y_m A_2(\phi) \qquad (27a)$$

$$T_{EY}/\pi R_0^2 P_\infty = +2\sigma S_{x'} S_{z'} - X_m A_2(\phi) + Z_m A_1(\phi) \tag{27b}$$

$$T_{EZ}/\pi R_0^2 P_\infty = -Y_m A_1(\phi) \tag{27c}$$

where

$$A_1(\phi) = -2\sigma S_{x'} [S_{z'} \mathrm{sgn}(S_{z'}) \mathrm{erf}(|S_{z'}|) + \pi^{-1/2} \exp(-S_{z'})] \tag{27d}$$

$$A_2(\phi) = (2-\sigma')[(2S_{z'}^2 + 1) \mathrm{sgn}(S_{z'}) \mathrm{erf}(|S_{z'}|)$$
$$+ 2\pi^{-1/2} S_{z'} \exp(-S_{z'}^2)] + \sigma' (\pi T_W/T_\infty)^{1/2} S_{z'} \tag{27e}$$

and $S_{z'} = S\cos\phi$, $S_{x'} = -S\sin\phi$.

G. Free-Molecular Torque on a Finite Cylinder

Free-molecular torque on a finite cylinder is the sum of $\vec{T}_{C.S.}$ in (20) and $\vec{T}_{END}$ in (27), i.e.,

$$\vec{T}_{CYL} = T_{CX}\hat{X} + T_{CY}\hat{Y} + T_{CZ}\hat{Z}$$

where $T_{CX} = T_{SX} + T_{EX}$, $T_{CY} = T_{SY} + T_{EY}$, $T_{CZ} = T_{SZ} + T_{EZ}$, with T_{SX}, T_{SY}, T_{SZ} given in (20a,b,c) and T_{EX}, T_{EY}, T_{EZ} given in (27a,b,c).

III. FREE-MOLECULAR TORQUE ON FLAT PANELS

The vehicle under consideration here has the geometric configuration of a finite cylinder fitted longitudinally with two flat panels, with the plane of the panels inclined to the X-axis by an angle α (see Figure 3). Let us now take only the two panels (of equal sizes), which have a width D_p and a length L_p. The geometric centers of the panels, denoted by O_1 and O_2, are at a radial distance R_p from the axis of the cylinder, so that their position vectors are written in the O-XYZ coordinate system as

$$\begin{aligned} \overrightarrow{OO}_1 &= R_p\cos\alpha\hat{X} + R_p\sin\alpha\hat{Y} + Z_p\hat{Z} \\ \overrightarrow{OO}_2 &= -R_p\cos\alpha\hat{X} - R_p\sin\alpha\hat{Y} + Z_p\hat{Z} \end{aligned} \qquad (28)$$

We denote the local coordinate systems of the panels by $O_1-x_1y_1z_1$ and $O_2-x_2y_2z_2$, and the unit vectors are given as

$$\begin{aligned} \hat{x}_j &= \cos\alpha\hat{X} + \sin\alpha\hat{Y} \\ \hat{y}_j &= -\sin\alpha\hat{X} + \cos\alpha\hat{Y} \quad , \quad j=1,2 \\ \hat{z}_j &= \hat{Z} \end{aligned} \qquad (29)$$

The components of $\vec{S}$ in the local coordinate systems are

$$S_{x_j} = -S\sin\phi\cos\alpha \quad , \quad S_{y_j} = S\sin\phi\sin\alpha \quad , \quad S_{z_j} = -S\cos\phi \quad ,$$
$$j=1,2 \qquad (30)$$

In the same way as in the foregoing sections, we obtain the force (per unit area) tensor on the front surface of panel 1 as

$$\vec{P}_1^{(f)} = \tau_{x1}^{(f)}\hat{x}_1 + p_{y1}^{(f)}\hat{y}_1 + \tau_{z1}^{(f)}\hat{z}_1 \qquad (31)$$

where

$$\tau_{x1}^{(f)} = p_\infty \sigma S_{x1}\left\{S_{y1}[1+\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|)] + \pi^{-1/2}\exp(-S_{y1}^2)\right\} \quad (31a)$$

$$p_{y1}^{(f)} = p_\infty\left\{(2-\sigma')S_{y1}[S_{y1}(1+\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|)) + \pi^{-1/2}\exp(-S_{y1}^2)] + \frac{(2-\sigma')}{2}[1+\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|)] + \frac{\sigma'}{2}(T_w/T_\infty)^{1/2}[\exp(-S_{y1}^2) + \pi^{1/2}S_{y1}(1+\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|))]\right\} \quad (31b)$$

$$\tau_{z1}^{(f)} = p_\infty \sigma S_{z1}\left\{S_{y1}[1+\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|)] + \pi^{-1/2}\exp(-S_{y1}^2)\right\} \quad (31c)$$

and on the back surface of panel 1

$$\vec{P}_1^{(b)} = \tau_{x1}^{(b)}\hat{x}_1 + p_{y1}^{(b)}\hat{y}_1 + \tau_{z1}^{(b)}\hat{z}_1 \quad (32)$$

where

$$\tau_{x1}^{(b)} = -p_\infty \sigma S_{x1}\left\{S_{y1}[1-\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|)] - \pi^{-1/2}\exp(-S_{y1}^2)\right\} \quad (32a)$$

$$p_{y1}^{(b)} = -p_\infty\left\{(2-\sigma')S_{y1}[S_{y1}(1-\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|)) - \pi^{-1/2}\exp(-S_{y1}^2)] + \frac{(2-\sigma')}{2}[1-\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|)] + \frac{\sigma'}{2}(T_w/T_\infty)^{1/2} \cdot [\exp(-S_{y1}^2) - \pi^{-1/2}S_{y1}(1-\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|))]\right\} \quad (32b)$$

$$\tau_{z1}^{(b)} = -p_\infty \sigma S_{z1}\left\{S_{y1}[1-\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|)] - \pi^{-1/2}\exp(-S_{y1}^2)\right\} \tag{32c}$$

The expressions of the forces on panel 2 are exactly the same as (31) and (32). Let us define the sum of the forces on the front and back surfaces as

$$\vec{F}_1^{(+)} = \vec{F}_1^{(f)} + \vec{F}_1^{(b)} = \tau_{x1}^{(+)}\hat{x}_1 + p_{y1}^{(+)}\hat{y}_1 + \tau_{z1}^{(+)}\hat{z}_1 \tag{33}$$

where

$$\tau_{x1}^{(+)} = \tau_{x1}^{(f)} + \tau_{x1}^{(b)} = 2\sigma p_\infty S_{x1}\left\{S_{y1}\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|) + \pi^{-1/2}\exp(-S_{y1}^2)\right\} \tag{33a}$$

$$p_{y1}^{(+)} = p_{y1}^{(f)} + p_{y1}^{(b)} = p_\infty\left\{2(2-\sigma')S_{y1}[S_{y1}\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|) + \pi^{-1/2}\exp(-S_{y1}^2)] + (2-\sigma')\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|) + \sigma'(\pi T_W/T_\infty)^{1/2}S_{y1}\right\} \tag{33b}$$

$$\tau_{z1}^{(+)} = \tau_{z1}^{(f)} + \tau_{z1}^{(b)} = 2\sigma p_\infty S_{z1}\left\{S_{y1}\mathrm{sgn}(S_{y1})\mathrm{erf}(|S_{y1}|) + \pi^{-1/2}\exp(-S_{y1}^2)\right\} \tag{33c}$$

The expressions of $\vec{F}_2^{(+)}$ for panel 2 are the same as (33). The torque due to both panels is then

$$\vec{T}_{PML} = L_p D_p [O\vec{\delta}_1 - \vec{R}_m] \times \vec{\beta}_1^{(+)} + L_p D_p [O\vec{\delta}_2 - \vec{R}_m] \times \vec{\beta}_2^{(+)}$$

$$= T_{PX}\dot{X} + T_{PY}\dot{Y} + T_{PZ}\dot{Z} \qquad (34)$$

where

$$T_{PX}/2L_p D_p p_\infty = [-Y_m \tau_{z1}^{(+)} + (Z_m - Z_p)(\tau_{x1}^{(+)}\sin\alpha + p_{y1}^{(+)}\cos\alpha)]/p_\infty$$

$$= 2\sigma[-Y_m S_{z1} + (Z_m - Z_p) S_{x1}\sin\alpha] B_1(\phi)$$

$$+ (Z_m - Z_p)\cos\alpha \; B_2(\phi) \qquad (34a)$$

$$T_{PY}/2L_p D_p p_\infty = [X_m \tau_{z1}^{(+)} - (Z_m - Z_p)(\tau_{x1}^{(+)}\cos\alpha - p_{y1}^{(+)}\sin\alpha)]/p_\infty$$

$$= 2\sigma[X_m S_{z1} - (Z_m - Z_p) S_{x1}\cos\alpha] B_1(\phi)$$

$$+ (Z_m - Z_p)\sin\alpha \; B_2(\phi) \qquad (34b)$$

$$T_{PZ}/2L_p D_p p_\infty = [(-X_m\sin\alpha + Y_m\cos\alpha)\tau_{x1}^{(+)} - (X_m\cos\alpha + Y_m\sin\alpha) p_{y1}^{(+)}]/p_\infty$$

$$= (-X_m\sin\alpha + Y_m\cos\alpha) 2\sigma S_{x1} B_1(\phi)$$

$$- (X_m\cos\alpha + Y_m\sin\alpha) B_2(\phi) \qquad (34c)$$

where

$$h_1(s) = [S_{y1}\,\mathrm{sgn}(S_{y1})\,\mathrm{erf}(|S_{y1}|) + \pi^{-1/2}\exp(-S_{y1}^2)] \qquad (34d)$$

$$h_2(s) = [\text{[illegible]}(2S_{y1}^2 + 1)\,\mathrm{sgn}(S_{y1})\,\mathrm{erf}(|S_{y1}|)$$

$$+ \text{[illegible]}\,\pi^{-1/2}S_{y1}\exp(-S_{y1}^2)] + \text{[illegible]}(T_w/T_\infty)^{1/2}S_{y1} \qquad (34e)$$

and (S_{x1}, S_{y1}, S_{z1}) are as defined in (30).

IV. FREE-MOLECULAR TORQUE ON THE SPACE VEHICLE

The overall free-molecular torque acting on a vehicle, the geometry of which is approximated as a single finite cylinder fitted with two flat panels (see Figure 3), may be given as the sum of the torque on the finite cylinder given in (28) and that on the panels given in (34), namely,

$$\vec{T} = T_X\hat{X} + T_Y\hat{Y} + T_Z\hat{Z} \qquad (35)$$

Here,

$$T_X = T_{CX} + T_{PX} = T_{SX} + T_{EX} + T_{PX}$$

$$T_Y = T_{CY} + T_{PY} = T_{SY} + T_{EY} + T_{PY}$$

$$T_Z = T_{CZ} + T_{PZ} = T_{SZ} + T_{EZ} + T_{PZ}$$

where T_{SX}, T_{SY}, T_{SZ} are given in (20a, b,c), T_{EX}, T_{EY}, T_{EZ} in (27a,b,c) and T_{PX}, T_{PY}, T_{PZ} in (34a,b,c).

It should be pointed out that the foregoing representation of the torque on the cluster vehicle as the sum of the torques on its component units is only an approximation. This is because of the neglect of the effect of the molecules reflecting back and forth between some part of the cylindrical surface and the flat panels. Each component unit alone constitutes a concave surface with respect to the incident flow; but when the component units are put together, they constitute a convex surface as indicated by the dotted lines in Figure 3. Any surface element in the convex surface zone will see some other part of the surface, and thus will receive some molecules reemitted from there. However, for the present geometric configuration of the vehicle, it is believed that the contribution to the torque by the chain reflections of the molecules in the convex surface zone is small. This is due to the fact that some chain-reflection effects are cancelled out and that the solid angle subtended by the convex surface is much smaller than 4π. Hence, the expression of the overall torque on the vehicle given in (35) is believed to be a good approximation.

In the case when the center of mass M and the center of geometry O lie on the Z- or Z_0-axis, i.e., when $X_0 = Y_0 = 0$, the expressions for the components of aerodynamic torque are simplified to

$$T_X = 2L_P D_P p_\infty (Z_m - Z_P)[-2S^2 \sin\phi \sin\alpha \cos\alpha B_1 + \cos\alpha B_2] \qquad (37a)$$

$$T_Y = 2L_P D_P p_\infty (Z_m - Z_P)[2S^2 \sin\phi \cos^2\alpha B_1 + \sin\alpha B_2]$$

$$+ 2R_0 L Z_m p_\infty \Big\{ \frac{1}{\pi} \sigma' \pi^{3/2} (T_W/T_\infty)^{1/2} S \sin\phi$$

$$+ \pi^{1/2} S \sin\phi \exp(-S^2 \sin^2\phi/2)[(2-\sigma') + \sigma/2][(1 + \tfrac{2}{3} S^2 \sin^2\phi)$$

$$\cdot I_0(S^2 \sin^2\phi/2) + \tfrac{1}{3}(1 + 2S^2 \sin^2\phi) I_1(S^2 \sin^2\phi/2)]\Big\}$$

$$+ 2\pi R_0^2 Z_m p_\infty \sigma S \sin\phi [S \cos\phi \,\mathrm{sgn}(\cos\phi) \,\mathrm{erf}(S|\cos\phi|)$$

$$+ \pi^{-1/2} \exp(-S^2 \cos^2\phi)] \qquad (37b)$$

$$T_Z = 0 \qquad (37c)$$

If, furthermore, the speed ratio S is much larger than unity, and the angle ϕ is neither 0 nor $\pi/2$, so that $V = |S \sin\phi|$ or $|S \cos\phi| >> 1$, then we may asymptotically expand the error function and the modified Bessel Function as

$$\mathrm{erf}(V) \sim 1$$

$$I_0(V^2/2) \sim \frac{1}{\sqrt{\pi}\, V} \exp(V^2/2)[1 + \frac{1}{4V^2} + \frac{9}{32V^4} + \cdots] \qquad (38a)$$

$$I_1(V^2/2) \sim \frac{1}{\sqrt{\pi}\, V} \exp(V^2/2)[1 - \frac{3}{4V^2} - \frac{15}{32V^4} + \cdots] \qquad (38b)$$

The torque components in (37a,b) are further simplified to

$$T_X = 2L_pD_pp_\infty(Z_m-Z_p)[-2\sigma S \sin\phi\sin\alpha\cos\alpha B_1' + \cos\alpha B_2'] \qquad (39a)$$

$$T_Y = 2L_pD_pp_\infty(Z_m-Z_p)[2\sigma S \sin\phi\cos^2\alpha B_1' + \sin\alpha B_2'] + T_{CY} \qquad (39b)$$

where, as seen from (34d,e),

$$B_1' \sim \begin{cases} S|\sin\phi\sin\alpha| & \text{for } \alpha>0 \text{ and } S|\sin\phi\sin\alpha|>>1 \\ \pi^{-1/2} & \text{for } \alpha \tilde{\to} 0 \end{cases} \qquad (40a)$$

$$B_2' \sim \begin{cases} (2-\sigma')(1+2S^2\sin^2\phi\sin^2\alpha\ \mathrm{sgn}(\sin\phi\sin\alpha) \\ \quad + \sigma'(\pi T_W/T_\infty)^{1/2} S \sin\phi\sin\alpha \\ \quad\quad \text{for } \alpha>0 \text{ and } S|\sin\phi\sin\alpha|>>1 \\ 0 \quad \text{for } \alpha \tilde{\to} 0 \end{cases} \qquad (40b)$$

and T_{CY} is the torque due to the finite cylinder only, given as

$$T_{CY} = (2R_0 L)p_\infty Z_m \Big\{S^2[\tfrac{4}{3}(2-\sigma') + \sigma/2]\sin\Phi|\sin\Phi|$$

$$+ \sigma(R_0/L)\sin\Phi|\cos\Phi|] + \tfrac{1}{4}\pi^{3/2}\sigma'(T_W/T_\infty)^{1/2}S\sin\Phi$$

$$+ O(1) + O((S\sin\Phi)^{-2})\Big\} \qquad (41)$$

Meirovitch and Wallace [3] obtained the following result for the aerodynamic torque on a finite cylinder

$$T_{CYL} = q\epsilon DL\cos\theta_1[2.16\cos\theta_1 + 1.39(D/L)\sin\theta_1] \qquad (42)$$

which is analogous to the first term of (41)* where

$$q = \tfrac{1}{2}p_\infty U^2 = p_\infty S^2 \ , \ \epsilon = Z_m \ , \ D = 2R_0 \ , \text{ and } \theta_1 = \pi/2 - \Phi$$

They employed the formulas given in Reference [5] for the normal and tangential forces on a front differential surface area. Since these formulas are derived using a local coordinate system in which one component of $\vec{U}$ is taken to be zero, the integration over the entire body surface has to be performed numerically in order to yield the total torque, as has been pointed out previously. In fact, they computed the torque numerically for a number of discrete attitude angles θ_1 (or Φ) and approximated the resulting set of points by trigonometric functions. Their numerical value 2.16 of the coefficient of $\cos^2\theta_1$ ($\sin\Phi|\sin\Phi|$)*calculated with $\sigma = \sigma' = 0.885$, is very close to the exact value given here in (41), namely, $\frac{4}{3}[(2-\sigma') + \sigma/2] = 2.077$, and their value of the coefficient of $\cos\theta_1\sin\theta_1$ ($\sim|\cos\Phi|\sin\Phi$)*

---

*Meirovitch and Wallace omitted the magnitude signs on the $\sin\theta_1$ and $\cos\theta_1$ terms inside the bracket in (42), evidently an error.

checks exactly with the value $\pi\sigma/2 = 1.3$[illegible] here. However, they have not included in their result the second term in (41), which, as contributed by the reflected molecules from the body surface, is of the order of ($S \sin\psi$) and is not negligible compared with the leading term. At any rate, the analytical result of the aerodynamic torque given here will enable us to compute the torque magnitude for any specified values of the accommodation coefficients and the temperature ratio (T_w/T_∞).

V. CONCLUDING REMARKS

The analytical expression of the aerodynamic torque acting on a vehicle, configured as a finite cylinder fitted with two flat solar panels, is given in (35). The torque is expressed as a function of two angular parameters, θ and ϕ, which define the direction of the uniform incident air velocity, $\vec{U}$, with respect to the vehicle. For a given inertial orientation of the vehicle (as in the case of the AAP cluster configuration), we can integrate the torque $T(\theta,\phi)$ around an orbit (i.e., integrate with respect to θ and ϕ) to obtain the net angular momentum per orbit that would have to be counteracted by the active attitude control system of the vehicle. It should be noted that in the integration the air density, ρ_∞, or the pressure, p_∞, is a variable even in a circular orbit. This is due to the fact that the earth atmosphere is not spherically symmetric and that the atmospheric density varies from day to night.

The analytical expression for $T(\theta,\phi)$ obtained here for the cluster vehicle is quite lengthy in the general case of arbitrary location of the center of mass, M, with respect to the center of geometry, O. If, however, M lies on the axis of symmetry, the torque expressions can be simplified (to those given in (37)). Furthermore, if the speed ratio, $S(=U/(2RT_\infty)^{1/2})$, is large and if the velocity, $\vec{U}$, is neither normal nor parallel to the symmetry axis for a certain attitude motion of the vehicle, the torque expressions can be further simplified (to those given in (39) and (41)). In any case, these analytical expressions of $T(\theta,\phi)$ may be used to study the equilibrium and stability of the attitude motion of the vehicle in the earth's gravitational field. Such an investigation remains to be done.

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1022-LYY-jdc E. Y. Yu

Attachments
Appendix
Figures 1 thru 3

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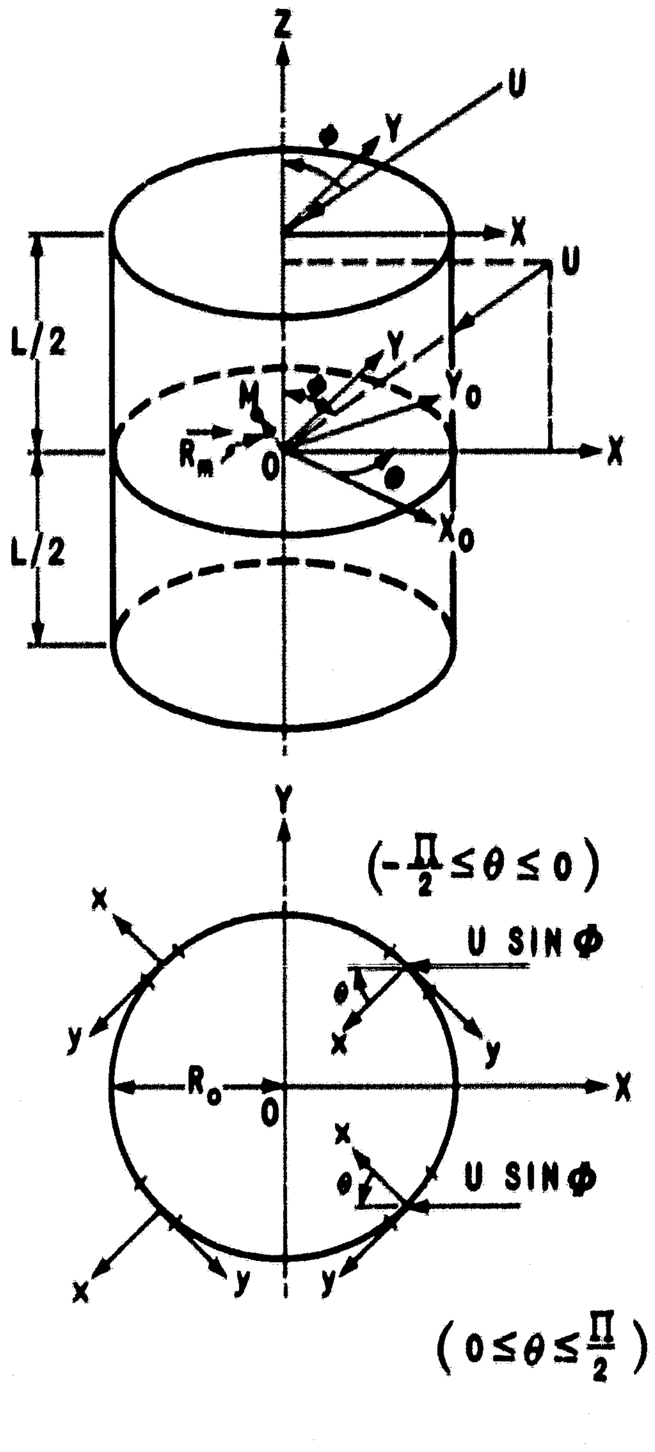


Z
U
Y
X
U
L/2
Y
Y_0
M
R_m
O
X
X_0
L/2
Y
$(-\frac{\Pi}{2} \leq \theta \leq 0)$
U SIN ϕ
x
y
R_o
O
X
U SIN ϕ
$(0 \leq \theta \leq \frac{\Pi}{2})$

FIGURE 1

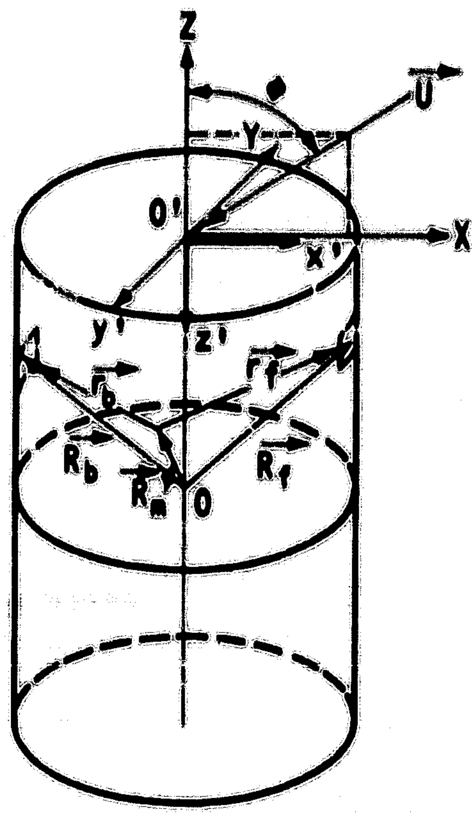
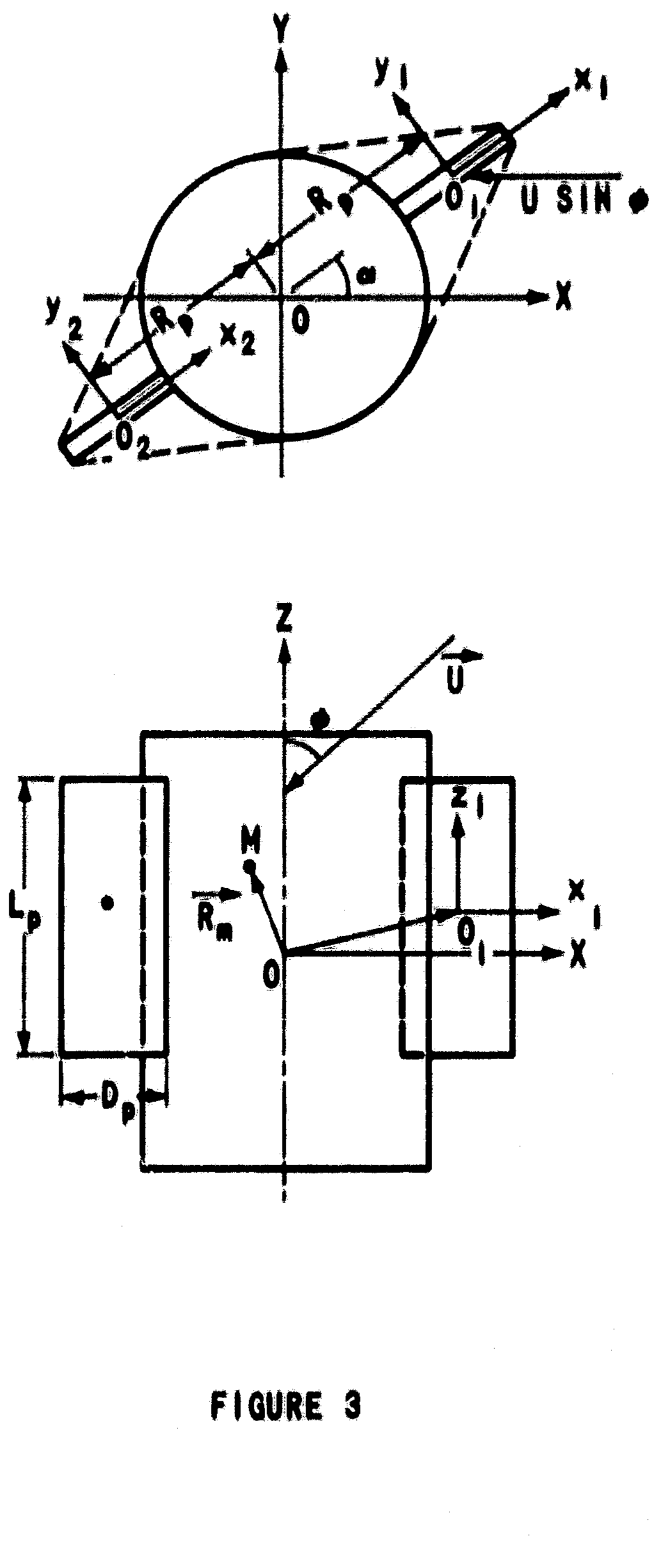


FIGURE 2



Y
y_1
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R_ϕ
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$\vec{U}$
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M
L_p
$\vec{R}_m$
x_1
O_1
X
O
D_p

FIGURE 3

APPENDIX A

Formulas Used in the Integration of (19)

A. On Exponential Functions:

1. $$\int_{-\pi/2}^{+\pi/2} \exp(-x^2\cos^2\theta)d\theta = +\pi \exp(-x^2/2)I_0(x^2/2)$$

2. $$\int_{-\pi/2}^{+\pi/2} \sin\theta \exp(-x^2\cos^2\theta)d\theta = 0$$

3. $$\int_{-\pi/2}^{+\pi/2} \cos^2\theta \exp(-x^2\cos^2\theta)d\theta$$

$$= +\frac{\pi}{2} \exp(-x^2/2)[I_0(x^2/2) - I_1(x^2/2)]$$

4. $$\int_{-\pi/2}^{+\pi/2} \sin^2\theta \exp(-x^2\cos^2\theta)d\theta$$

$$= +\frac{\pi}{2} \exp(-x^2/2)[I_0(x^2/2) + I_1(x^2/2)]$$

5. $$\int_{-\pi/2}^{+\pi/2} \sin\theta\cos\theta \exp(-x^2\cos^2\theta)d\theta = 0$$

B. On Error Functions:*

6. $$\int_{-\pi/2}^{+\pi/2} \cos\theta \operatorname{erf}(x\cos\theta)d\theta$$

$$= +\pi^{1/2}x \exp(-x^2/2)[I_0(x^2/2) + I_1(x^2/2)]$$

---

*In all cases x is considered to be positive.

Appendix A (Contd.)

7. $\int_{-\pi/2}^{+\pi/2} \cos^3\theta \ \mathrm{erf}(x\cos\theta)d\theta$

$$= +\frac{1}{3}\ \pi^{1/2}x\ \exp(-x^2/2)[2I_0(x^2/2)$$

$$+\ (2+x^{-2})I_1(x^2/2)]$$

8. $\int_{-\pi/2}^{+\pi/2} \sin\theta\cos\theta \ \mathrm{erf}(x\cos\theta)d\theta = 0$

9. $\int_{-\pi/2}^{+\pi/2} \sin\theta\cos^2\theta \ \mathrm{erf}(x\cos\theta)d\theta = 0$

10. $\int_{-\pi/2}^{+\pi/2} \cos\theta\sin^2\theta \ \mathrm{erf}(x\cos\theta)d\theta$

$$= +\frac{1}{3}\ \pi^{1/2}\exp(-x^2/2)[I_0(x^2/2)$$

$$+\ (1-x^{-2})I_1(x^2/2)]$$

11. $\int_{-\pi/2}^{+\pi/2} \sin\theta \ \mathrm{erf}(x\cos\theta)d\theta = 0$