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TITLE- Diversity Performance of the Wideband FSK System in a Two-Path/Multipath Channel with a Planetary Landing Probe

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AUTHOR(S)- L. Schuchman

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(NASA-CR-96103) DIVERSITY PERFORMANCE OF THE WIDEBAND FSK SYSTEM IN A TWO-PATH/MULTIPATH CHANNEL WITH A PLANETARY LANDING PROBE (Bellcomm, Inc.) 65 p

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Multipath channels arise in space communications when a landing probe, descending upon a celestial body, communicates with a spacecraft relay orbiting about or flying by the celestial body. Two paths between the landing probe and the spacecraft arise from the fact that some of the signal energy transmitted by the probe will be reflected off the surface of the celestial body. The reflected wave is delayed in time with respect to the direct path transmission and in addition can be distorted by the reflection boundary. The net effect is to cause multipath distortion (primarily signal fading) of the received signal.

Diversity techniques have commonly been used to reduce the effects of a fading channel. Thus, the particular problem of evaluating the diversity performance of FSK wideband communication systems in a two-path/multipath channel was taken as the subject for this paper. In this paper, a detailed mathematical model of a fading channel is obtained. This allows one to study the effects of variations of such important parameters as doppler rate, of fade, and the degree of correlation between the direct path and reflected path transmissions. The model is used to derive performance equations for this system for an arbitrary number (D) of diversity channels. In turn, these equations have been programmed and a set of performance curves are presented which describe the system performance for a wide range of parameter variation.

The results obtained do not lend themselves to any simple generalized conclusions. It was found that the amount of a priori knowledge one assumes as to the nature of the fade determines in part whether diversity should be considered. For example, in nearly all cases, reflections from extremely smooth boundaries (specular reflections) show degradations in performance with the use of diversity while for most diffuse reflection cases, an improvement can be achieved through the use of diversity.

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BELLCOMM, INC.
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SUBJECT: Diversity Performance of the Wideband FSK System in a Two-Path/Multipath Channel with a Planetary Landing Probe Case 730

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FROM: L. Schuchman

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TECHNICAL MEMORANDUM

1.0 INTRODUCTION

1.1 Background

In a typical Mars mission, a landing probe will descend upon Mars, while a spacecraft orbits around or flys by the planet. In order to properly evaluate the performance of a communication link between the spacecraft and the landing probe, the effects of multipath must be considered. The multipath effect is generated since a direct transmission path from the capsule to the spacecraft exists together with a reflected path off the surface of Mars. The reflected signal will be modeled as a Ricean signal allowing us to investigate the effects of a reflection whose composition can vary from totally specular to totally diffuse.* For the remainder of this paper we restrict our attention to the detection of an FSK transmission for such a multipath channel.

Several researchers(1,2,3) have been concerned with the problem of detecting an FSK transmission when doppler and/or oscillator instabilities cause significant frequency uncertainty. Because of this uncertainty, an optimum non-coherent matched filter detection system cannot be used. The receiver used for this situation is illustrated in Figure 1-1. The received signal, z(t), is passed through both mark and space detectors each consisting of a bandpass filter (B Hz wide), followed by a square-law envelope detector and then through a finite time integrator (T-second integration time) to form two energy measures $\chi_1^2(t)$ and $\chi_2^2(t)$. The difference $\chi_1^2(t) - \chi_2^2(t)$ is compared with respect to a threshold (the threshold value being taken as zero if the a priori probability of mark and space transmission are equal), and a decision is made as to the transmitted bit information.

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*As proposed for such missions as Voyager, there will be little antenna directivity gain between the spacecraft and the landing craft. Therefore, the reflected signal is significant and must be accounted for.

The author has completed a prior paper in which such parameters as the detection bandwidth, rate of fade, depth of fade, reflected path delay doppler and composition of the multipath were studied.[4] In reference (4) a set of parametric curves were produced which describes the performance of the wideband FSK system as the several parameters are varied over their possible ranges. The set of performance curves are quite extensive since the problem does not lend itself to any simple overall conclusion. In this paper the work is extended by studying how and if diversity techniques can be used to reduce the effects of the fading. By a diversity technique we mean a redundant transmission technique in which the several repeats of the transmitted information are fading independent of one another. The redundant messages may for example be transmitted over separate frequency channels, repeated in time, transmitted by more than one antenna separated in space, etc. The number of such repetitions is defined as the order of the diversity. An FSK wideband receiver with D redundant, but independent, frequency subchannels, is shown in Figure 1-2. As can be seen from this figure the compared energy measures $x_1^2(t)$ and $x_2^2(t)$ are given by

$$x_1^2(t) = \sum_{i=1}^{D} x_{1i}^2(t)$$

$$x_2^2(t) = \sum_{i=1}^{D} x_{2i}^2(t)$$

There are two channel models which will be studied. The first model assumes that the delay of the reflected signal (due to its longer path length) relative to the data bit is small so that except for producing a phase variation such a delay can be ignored. This model is applicable for the final portion of the capsule's descent to Mars. The second model assumes this delay relative to the data rate is greater than one bit so that it is reasonable to assume the direct and reflected path transmissions are uncorrelated. This model is valid during the initial portion of the capsule's descent.

Evaluation of the performance for both channels bounds the performance of the real channel for the total time that the relay is in operation.

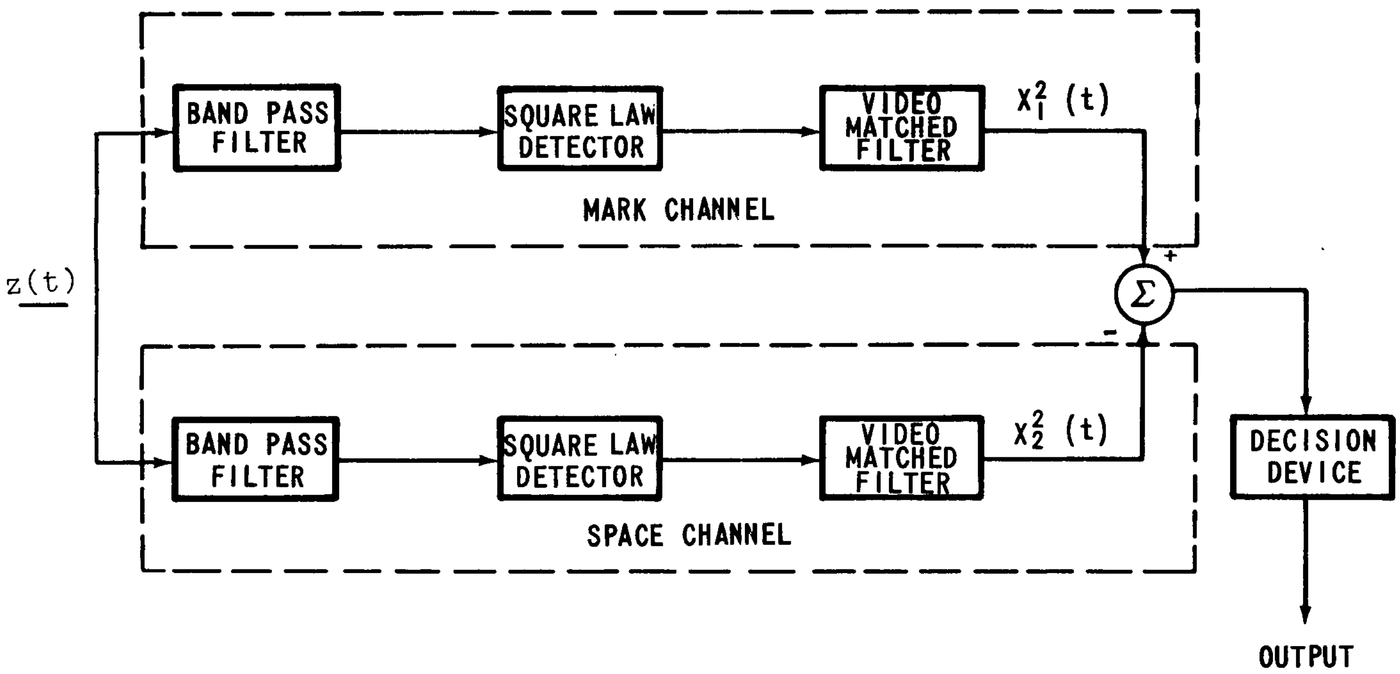


FIGURE 1-1 - MODEL OF FSK RECEIVER

1.2 Correlated Direct and Reflected Path Transmission - D Independent Frequency Diversity Subchannels* (Negligibly small delay between direct and reflected path transmissions.)

We assume that the transmitted signal is given by

$$\sum_{i=1}^{D} k \sqrt{\frac{2\varepsilon}{DT}} \cos\left(\omega_{ji} t + \theta_{ji}\right)$$

where θ_{ji} is an independent uniformly distributed random variable, k is a constant equal to the reciprocal of the propagation loss factor, and ω_{ji} $(j = 1,2,3,\ldots,M)$ describes a random binary pattern comprised of bits of duration T. The number of diversity subchannels used is D and ε is the received signal energy per bit.

The received signal z(t) is distorted by a reflected transmission off the surface of Mars, together with additive gaussian noise at the spacecraft receiver, so that it can be described by

$$\begin{aligned} z(t) = \sum_{i=1}^{D} \Bigg(& \sqrt{\frac{2\varepsilon}{TD}} \cos\left(\omega_{ji} t + \Delta\omega_{di} t + \theta_{ji}\right) \\ & + \rho \sqrt{\frac{2\varepsilon}{TD}} \cos\left(\omega_{ji} t + \Delta\omega_{ri} t\right) \\ & + \alpha_{1i}(t,\beta \sqrt{\frac{2\varepsilon}{TD}}) \cos\left(\omega_{ji} t + \Delta\omega_{ri} t\right) \\ & - \alpha_{2i}(t,\beta \sqrt{\frac{2\varepsilon}{TD}}) \sin\left(\omega_{ji} t + \Delta\omega_{ri} t\right) \\ & + N_{1i}(t) \cos \omega_{ji} t - N_{2i}(t) \text{ sn } \omega_{ji} t \end{aligned} \qquad (1)$$

*Results will be applicable to a wideband FSK system with D diversity subchannels independent of how this diversity is obtained.

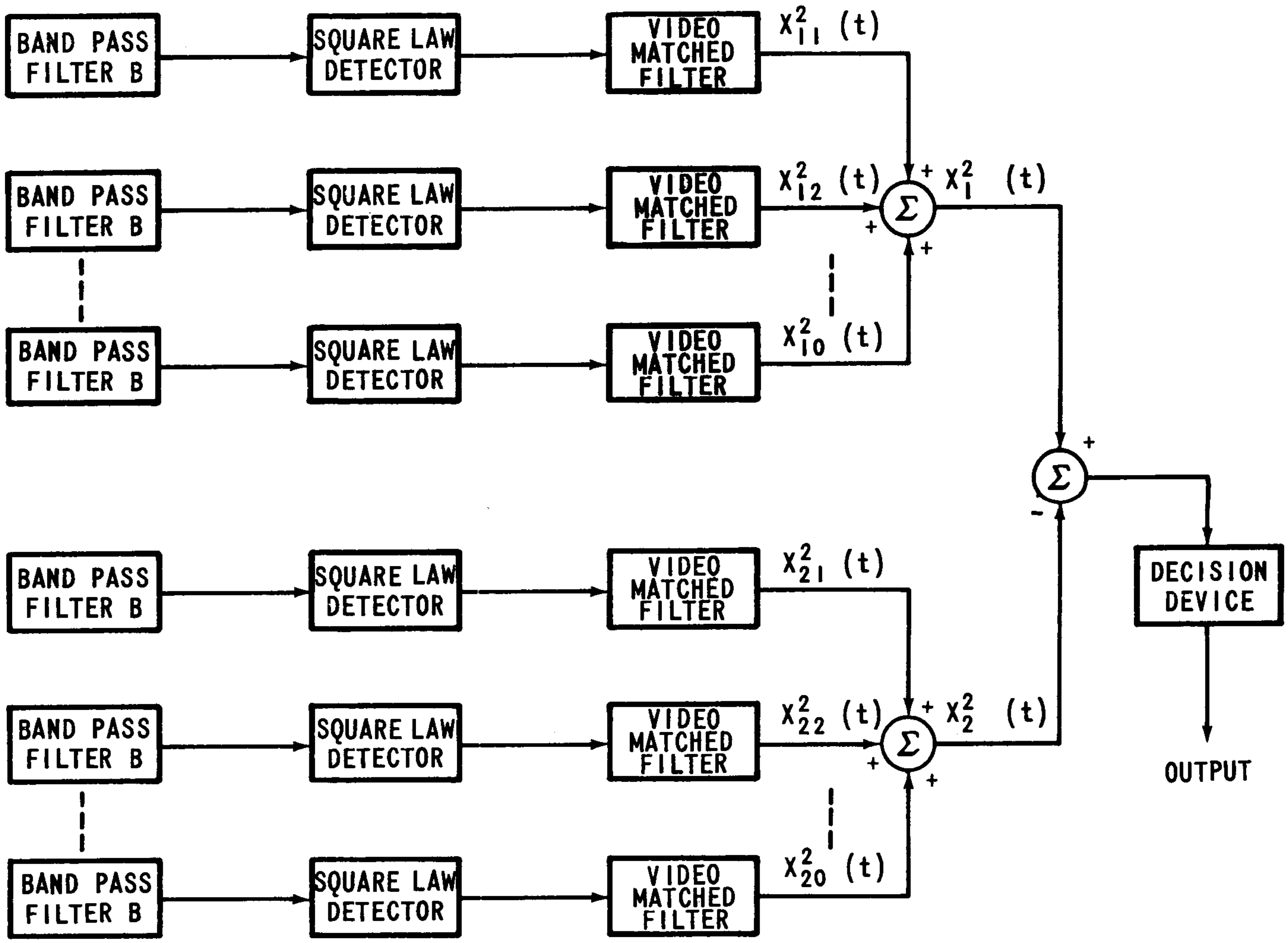


FIGURE I-2 - MODEL OF WIDEBAND FSK - DIVERSITY RECEIVER where: $\Delta\omega_{di}$ is the doppler shift on the direct path in the $i^{\underline{th}}$ diversity subchannel.

$\Delta\omega_{ri}$ is the doppler shift on the specular component of the reflection in the $i^{\underline{th}}$ diversity subchannel.

ρ is the reflection coefficient of the specular component of the reflection and is related to the total average reflected power ξ^2 by

$$\xi^2 = \rho^2 + 2\beta^2 \tag{2}$$

where β^2 is the average power in each of the quadrature components of the diffuse portion of the reflection.

$$\alpha_{1i}\left(t,\beta\sqrt{\frac{2\varepsilon}{TD}}\right) \text{ and } \alpha_{2i}\left(t,\beta\sqrt{\frac{2\varepsilon}{TD}}\right)$$

are independent gaussian random processes and are the quadrature components of a signal which characterize the diffuse portion of the reflection in the $i^{\underline{th}}$ diversity subchannel. These gaussian processes are obtained from assuming a very large number of nearly independent reflections from the surface which are again nearly identically distributed so that the central limit theorem can be applied to the composite reflected signal. $N_{1i}(t)$ and $N_{2i}(t)$ are also gaussian random variables and represent the quadrature components of the additive thermal noise found at the receiver in the $i^{\underline{th}}$ diversity channel.

Notice that we have assumed that the reflected signals in the several diversity subchannels can be characterized as independent but identically distributed random processes.

1.3 <u>Uncorrelated Direct and Reflected Path Transmissions</u> (Delay between direct and reflected path transmissions greater than a transmission bit interval.)

This channel model differs from the previous one only in the characterization of the multipath signal, r(t). The description of r(t) for a FSK modulation is given in the following:

$$r(t) = -\sqrt{\frac{2\varepsilon}{TD}} \left\{ \sum_{i=1}^{D} V(t) \cos (\omega_{ji}t + \Delta\omega_{ji}t + \theta_{ji}) \right.$$

$$+ \rho\, \nu_1(t) \cos (\omega_{ki}t + \Delta\omega_{ki}t) + \rho\, \nu_2(t) \cos (\omega_{li}t + \Delta\omega_{li}t)$$

$$+ \nu_1(t)\, \alpha_{1i} \left(t, \beta \sqrt{\frac{2\varepsilon}{DT}}\right) \cos \omega_{ki}t - \nu_1(t)\, \alpha_{2i} \left(t, \beta \sqrt{\frac{2\varepsilon}{DT}}\right) \sin \omega_{ki}t$$

$$\left. + \nu_2(t)\, \alpha_{1i} \left(t, \beta \sqrt{\frac{2\varepsilon}{DT}}\right) \cos \omega_{li}t - \nu_2(t)\, \alpha_{2i} \left(t, \beta \sqrt{\frac{2\varepsilon}{DT}}\right) \sin \omega_{ki}t \right\} \qquad (3)$$

where

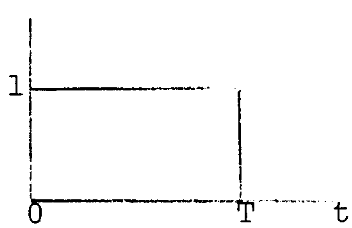
$$j = 1,2; \; k = 1,2; \; \ell = 1,2;$$

$$p(j=1) = p(j=2) = p(k=1 = p(k=2) = p(1=1) = p(k=2) = 1/2$$

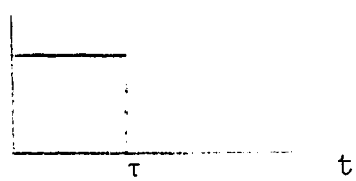
γ_{1i} and γ_{2i} are independent but identically distributed as α_{1i}.

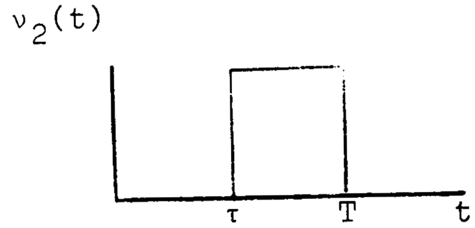
Over a bit interval, the wave forms $V(t)$, $\nu_1(t)$, and $\nu_2(t)$ have the following characterizations.

$V(t)$

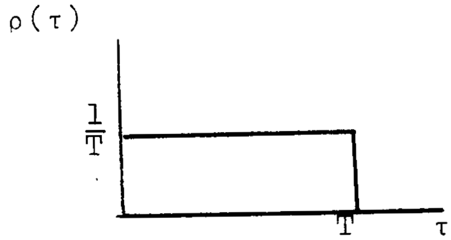


$\nu_1(t)$





and where τ is independent and uniformly distributed such that



What we have done is to assume the delay of the reflected path relative to the data rate will be such that the direct path reception of say, the j^{th} bit of information, will be received at the same time as the k^{th} bit is being received from the reflected path.* However, since the delay between paths will be variable, the bit reception from the two paths will not be synchronous. Therefore, portions of two bits, k and l, will be received during the T time interval that it takes to detect bit j. The fraction of time that k is interfering with the reception of j is an unknown variable, so we have used the variable τ, to describe this randomness. In addition, since we are transmitting FSK and since we assume j, k, and l are independent, some of the time the energy from k and l will fall in the same filter band as j causing fading while at other times the reflection will increase the amount of noise falling the the no-signal filter band increasing the probability of false detection. This randomness is accounted for by the parameters $\nu_1(t)$ and $\nu_2(t)$.

1.4 Organization

A graphical presentation of computed performance curves obtained from the derivations given in Appendix A is the subject of section 2. The overall conclusions based on the previous sections are given in section 3. Since these in effect extend those obtained in Reference 4 the conclusions from this referenced paper are given in Appendix B.

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*The j, k, and l notation does not stand for a sequence of three successive bits. The notation merely conforms to the labeling of the different bits as described by equation 3.

2.0 DIVERSITY PERFORMANCE OF THE WIDEBAND FSK RECEIVER

In this section, performance curves are presented which are computed from the formulas derived in Appendix A. Section 2.1 describes the diversity performance of the optimum non-coherent FSK receiver (BT=1) in a slow fading 2 path/multipath channel for both correlated and uncorrelated reflections (short and long delays between the direct and reflected path transmissions). The performance of the wideband FSK receiver (BT>1) in the slow fading 2 path/multipath channel is the subject of section 2.2, while the effects of changing fade patterns occurring within a transmission bit interval are studied in section 2.3.

Two bit error rates are defined when discussing errors arising from uncorrelated reflections. Type one errors are generated when both the reflected and direct path transmissions are received in the same mark or space subchannel of the FSK receiver. Such errors lead to a $P_1(e)$ bit error rate. A type 2 error arises when the reflected and direct path transmissions simultaneously appear in separate subchannels of the FSK receiver. $P_2(e)$ is the bit error rate generated by such errors. By selecting $\max(P_1(e), P_2(e))$, the worst case bound for uncorrelated reflection (no delay) error rate can be determined and, in addition, $P_1(e)$ can also be interpreted as the correlated (long delay) error rate (as has been shown in Appendix A). In the several subsections to follow $P_1(e)$ and $P_2(e)$ are studied to see how the significant system parameters affect the system performance in this 2 path/multipath channel.

2.1 Diversity Performance of the Optimum Non-Coherent FSK Receiver in a Slow Fading Channel (BT=1)

Although the optimum non-coherent FSK receiver is not the primary concern of this paper it is useful to study for the following reasons. The mathematical solution is the degenerate case of the wideband FSK solution and as such is much simpler. Therefore, programming and computer time is less expensive. Nearly all of the significant parameters can be studied and generalized conclusions obtained which hold true even for values of BT>1.

To determine $P_1(e)$ equations A-16 and A-19 were programmed for BT=1 and in addition equations A-26 and A-28 were programmed to obtain $P_2(e)$.* These equations were used to generate the graphical results presented in Figure 2-1 through Figure 2-6. As can be seen from the graphs, a range of values of total average reflected power are given and for each the cases of totally diffuse (ρ^2=0) and totally specular ($\rho^2=\zeta^2$) reflections are presented. The graphs demonstrate that when type one errors are caused by totally specular reflections, diversity techniques can be used to improve performance. The degree of improvement is dependent upon the amount of power in the reflection. Thus, as the reflected power increases, so does the diversity improvement. On the other hand, the $P_2(e)$ error rate due to type two errors generated by specular reflections is not helped by diversity techniques. In fact, the error rate is seen to increase as the degree of diversity increases.

When the reflection is diffuse (ρ^2=0), diversity techniques are seen to significantly improve performance independent of type of error (1 or 2). In comparing type 1 errors with those of type 2, it can be seen that the latter generate higher error rates for a given received signal to noise ratio. In addition, we note that nearly all of the time a totally diffuse reflection causes greater error rates than a totally specular reflection of the same average power and the same degree of diversity.

To illustrate the use of the curves we hypothesize the following situation. It is estimated that on the landing probes initial phase of its descent that the relative reflection power is .05 and the fade rate is slow. On the final phase of the descent, the fade rate is still taken to be slow but, since the path length of the reflection wave is nearly that of the direct path together with the knowledge that the reflecting surface gets smaller as the probe nears its destination, the relative reflected power is estimated to be .1. We assume that when ζ^2 is taken to be .05 the reflective path delay is very long while during the final phase of the probes descent the delay is negligible. In addition, the BT product design is taken to be 1 while the maximum allowable bit error rate is

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*The receiver described in Figure 1-2 does not reduce to the optimum receiver. However, the mathematics is not bounded in this manner. Thus, the results for this paper in the degenerate case of BT=1 is equivalent to that which is computed directly for the optimum non-coherent FSK receiver.

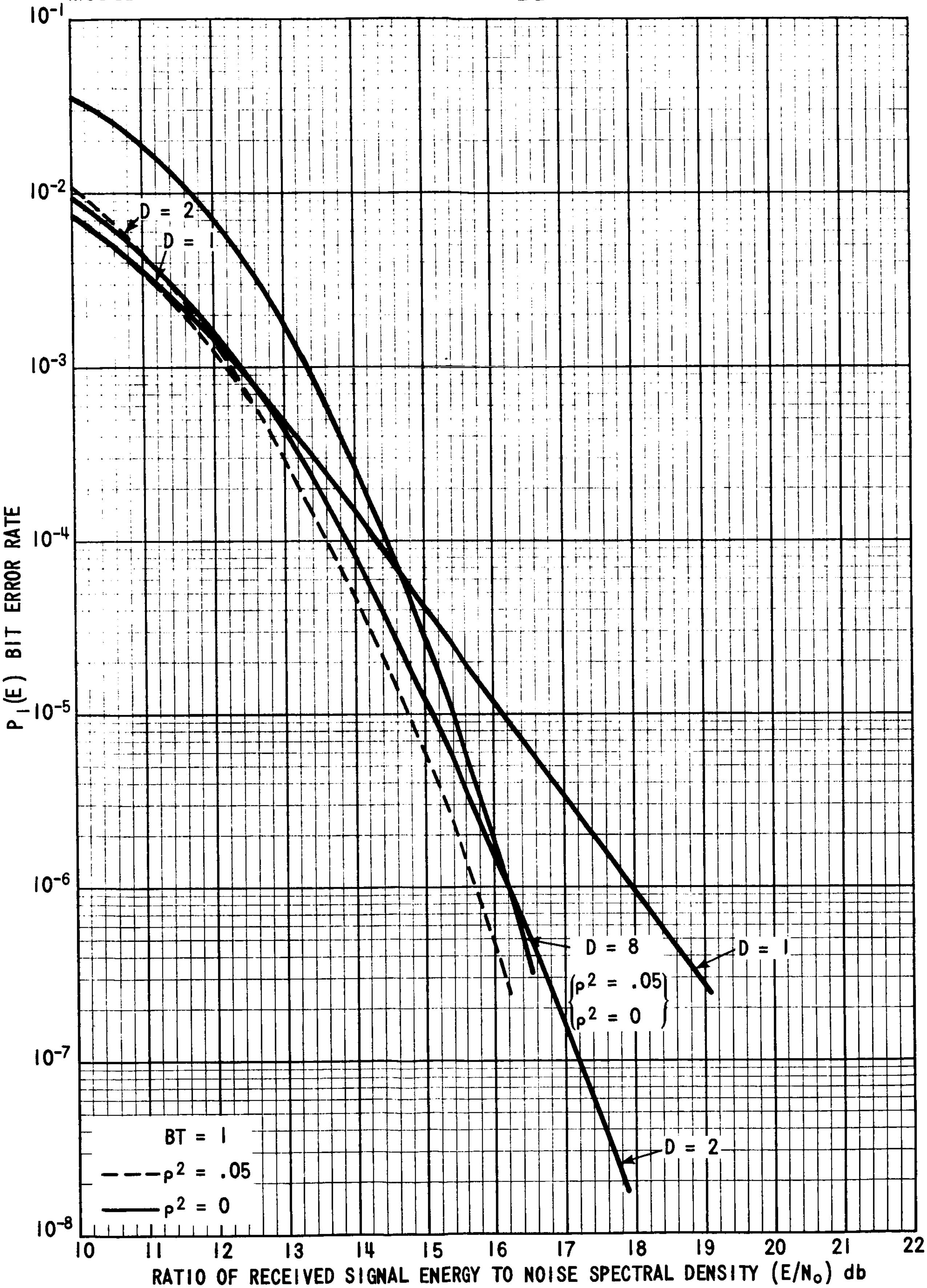

FIGURE 2-1. DIVERSITY PERFORMANCE OF THE OPTIMUM FSK RECEIVER IN A SLOW 2 PATH/MULTIPATH CHANNEL (TYPE ONE ERRORS ζ^2 = .05)

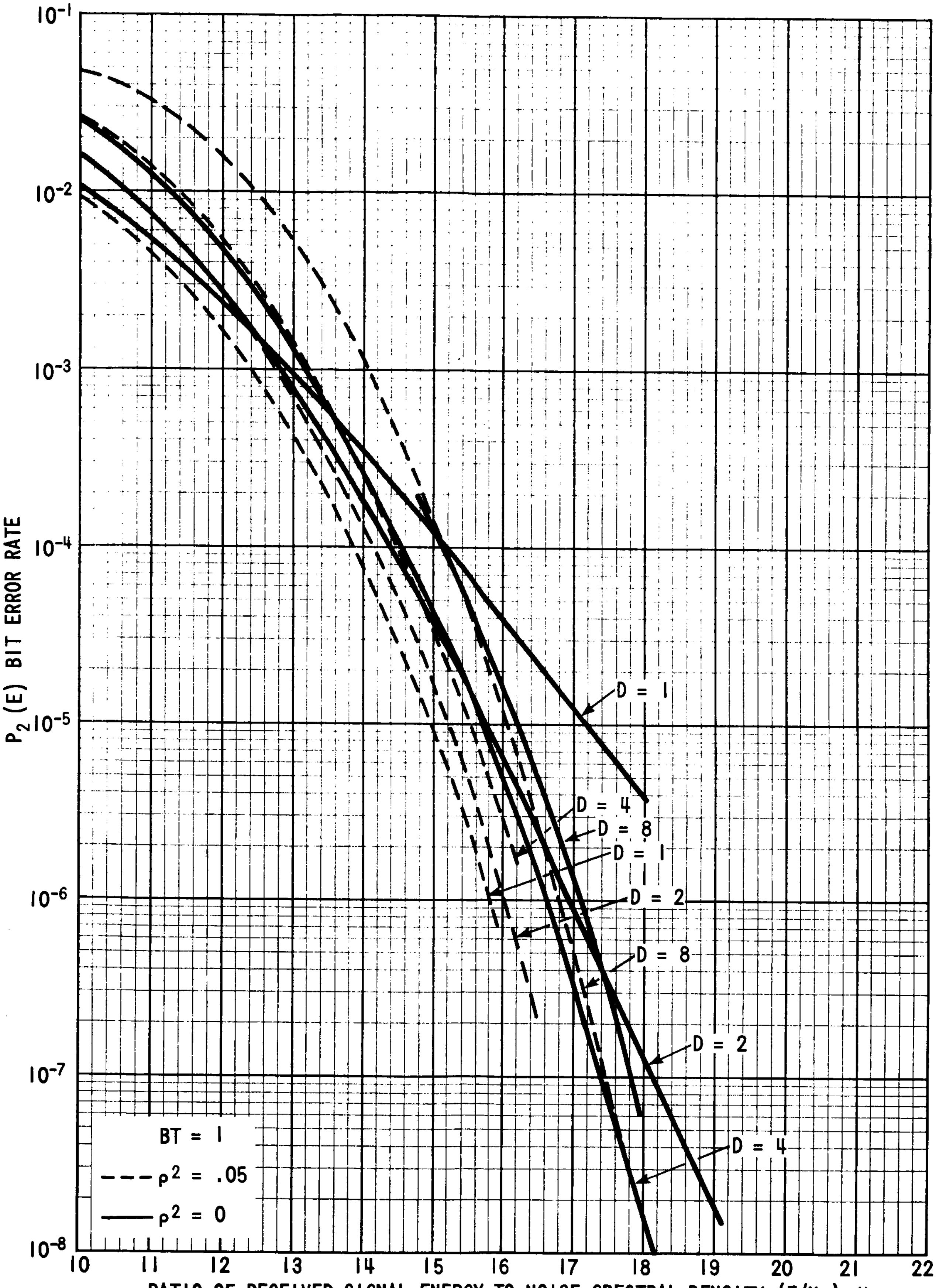

FIGURE 2-2. DIVERSITY PERFORMANCE OF THE OPTIMUM FSK RECEIVER IN A SLOW 2 PATH/MULTIPATH CHANNEL.
(TYPE TWO ERRORS ζ^2 = .05)

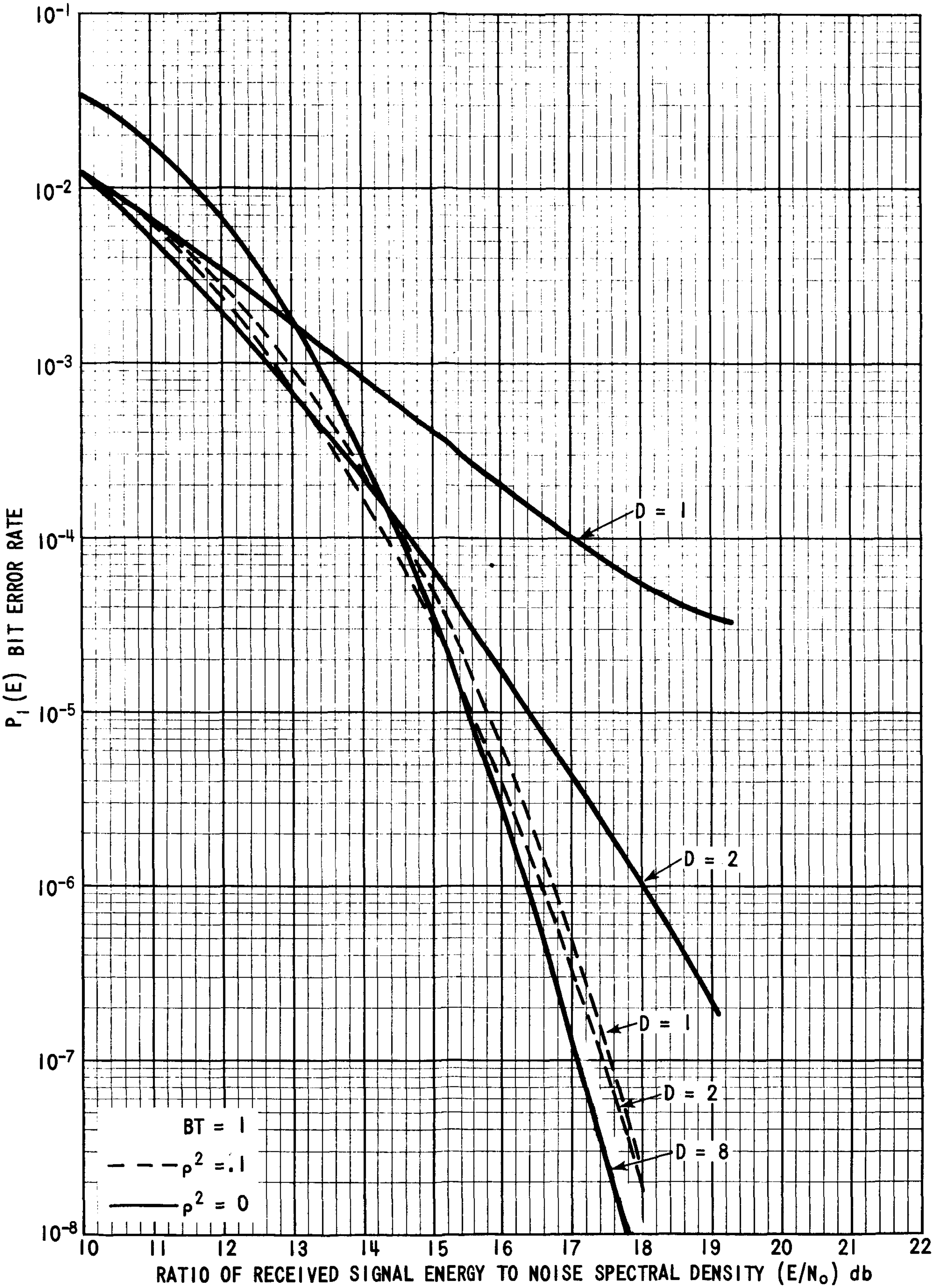

FIGURE 2-3. DIVERSITY PERFORMANCE OF THE OPTIMUM FSK RECEIVER IN A SLOW 2 PATH/MULTIPATH CHANNEL (TYPE ONE ERRORS $\zeta^2 = .1$)

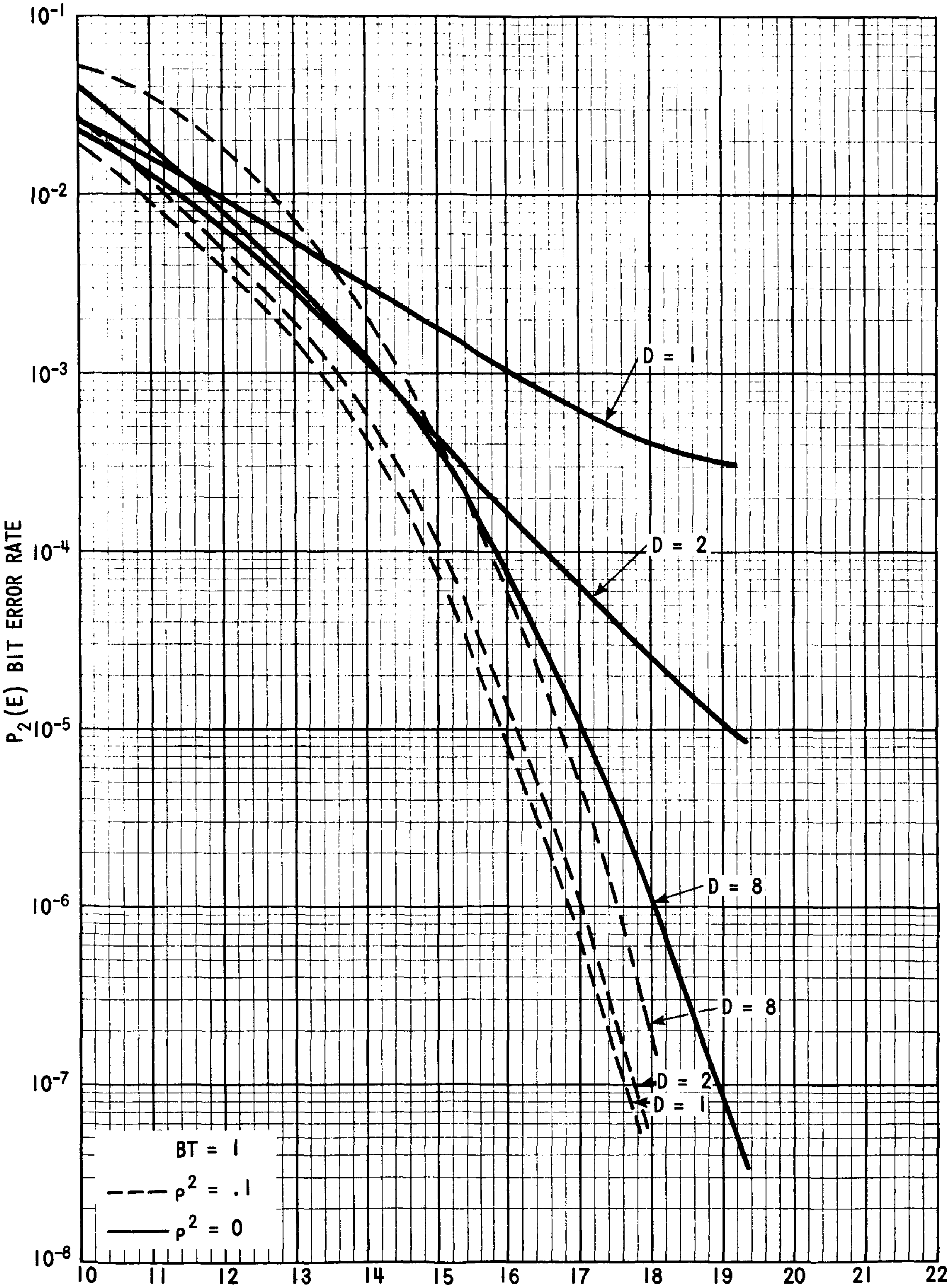

FIGURE 2-4. DIVERSITY PERFORMANCE OF THE OPTIMUM FSK RECEIVER IN A SLOW 2 PATH/MULTIPATH CHANNEL (TYPE TWO ERRORS ζ^2 = .1)

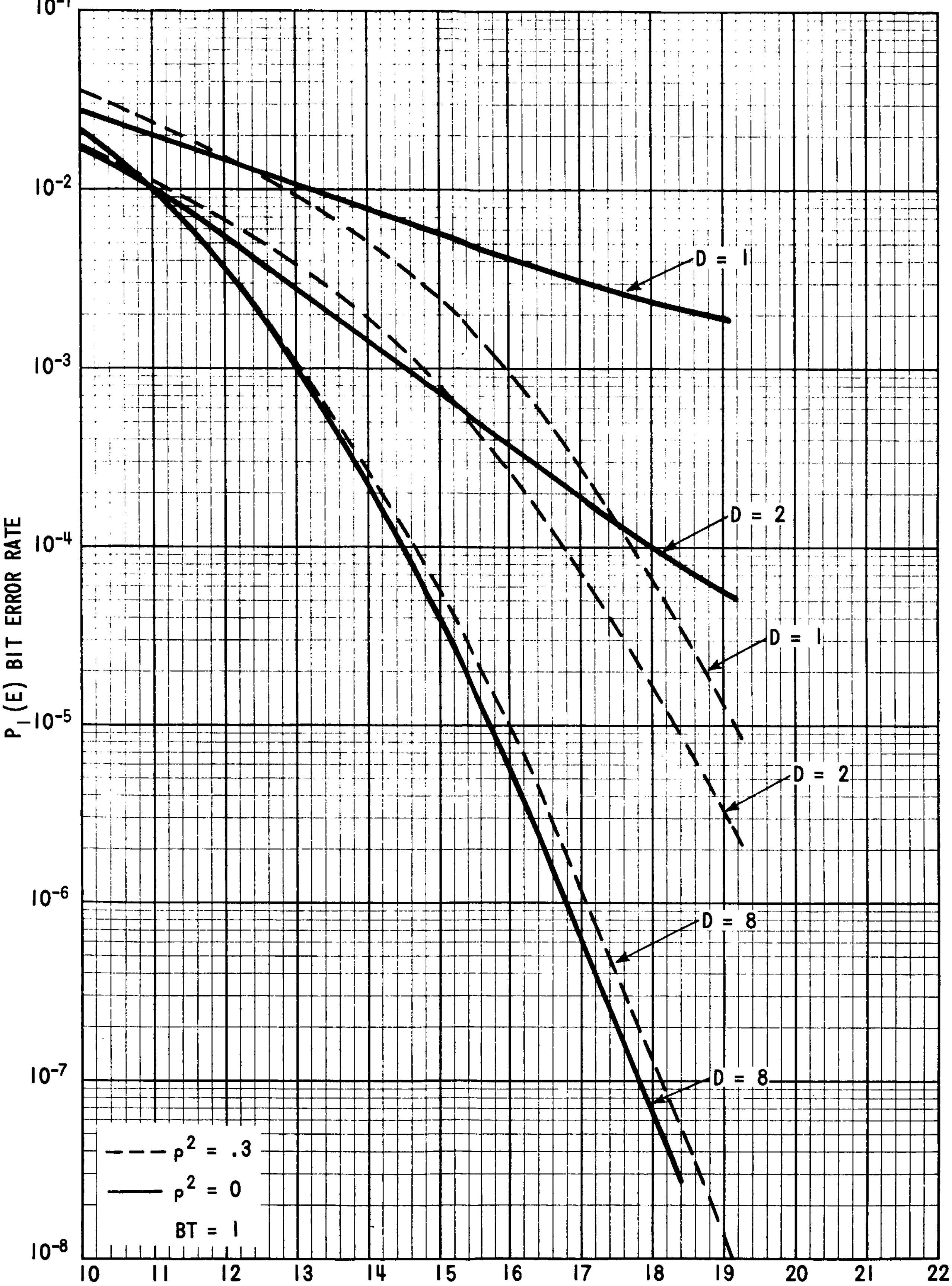

FIGURE 2-5. DIVERSITY PERFORMANCE OF THE OPTIMUM FSK RECEIVER IN A SLOW 2 PATH/MULTIPATH CHANNEL (TYPE ONE ERRORS $\zeta^2 = .3$)

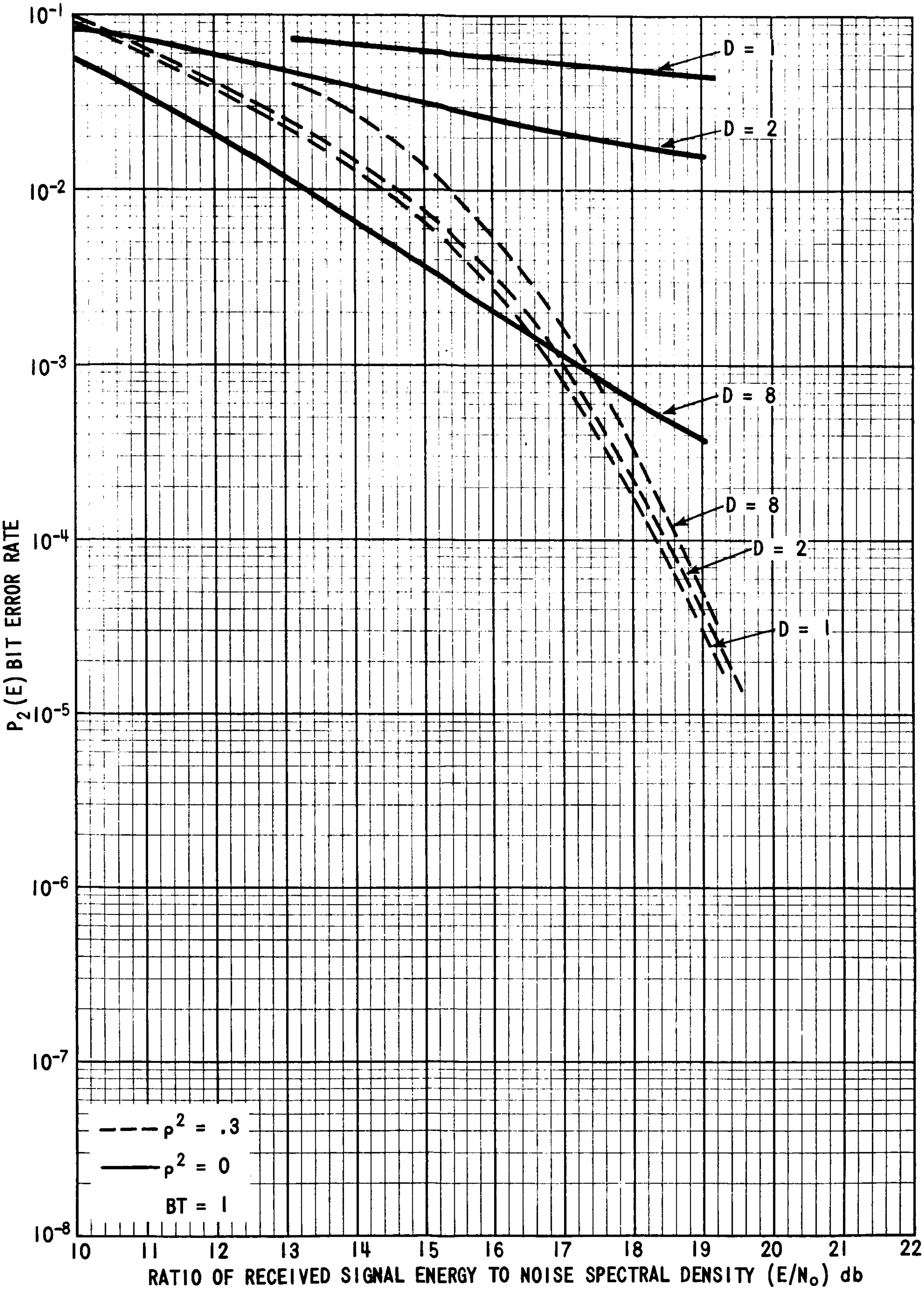

FIGURE 2-6. DIVERSITY PERFORMANCE OF THE OPTIMUM FSK RECEIVER IN A SLOW 2 PATH/MULTIPATH CHANNEL (TYPE TWO ERRORS ζ^2 = .3)

10^{-5}. Under these conditions an optimum wideband FSK system design is required.*

Since initially the delay is long, the reflection will be uncorrelated with the direct path transmission. In Table 2-1 we have extracted from Figures 2-1 and 2-2 the pertinent data for this portion of the flight. The final phase with

Table 2-1

E/N_0 requirements (db) for a 10^{-5} error rate (ζ^2=.05 BT=1)(Uncorrelated reflections)

Reflection Composition	Type One Error			Type Two Error		
	D=1	2	8	D=1	2	8
ρ^2 = .05	14.8	14.8	15.1	15	15.2	16.0
ρ^2 = 0	16.1	15.4	15.1	17.2	15.8	16.2

no delay between the reflected and direct paths leads to Table 2-2. To obtain Table 2-2 we have used Figure 2-3.

Table 2-2

E/N_0 requirements (db) for a 10^{-5} error rate (ζ^2=.1 BT=1)(Correlated reflections)

Reflection Composition	Type One Error		
	D=1	2	8
ρ^2 = .05	15.8	15.6	--
ρ^2 = 0	Greater than 19	16.4	15.5

*We shall assume that a study of the initial and final stages of the landing probe's flight is sufficient to determine the optimum design criteria. This simplification merely makes our example more compact without detracting from the key points which we wish to illustrate.

To provide the required communications link for all phases of the probes descent, we must design for a given selected diversity the maximum (E/N_0) db requirement in Table 2-3. It can be seen that the optimum performance following a sort of minimax criteria is to use a diversity of 8. However since the gain in going from a diversity of 2 to 8 is only .2 db while that from 1 to 2 is significant and since the size and complexity of an 8 diversity system as opposed to a 2 diversity system is substantial we would pragmatically select a 2 diversity system for our optimum design. In addition, we note once more that if we knew that a specular reflection $\rho^2 = \zeta^2$ would occur, then from Tables 2-1 and 2-2, diversity is of no help and indeed may cause poorer performance.

Table 2-3

Minimum E/N_0 (db) to satisfy probes mission requirements as a function of diversity

D	E/N_0
1	greater than 19 db
2	16.4
8	16.2

2.2 Wideband FSK Diversity Performance in a Slow Fading Channel

In this section, the same programs used to evaluate performance for the optimum FSK receiver are used to determine the performance of the Wideband FSK receiver. Thus, the restriction for BT=1 is removed and in particular BT values of 5 and 10 were used in the actual computer runs.

2.2.1 Type One Errors

When the BT product is greater than one and for the space applications of interest a relative doppler variation between the reflected and direct transmission paths will normally exist. Thus, in addition to those parameters studied in section 2.1, the effect of doppler variation is considered in this section. The doppler is accounted for by assuming the direct path transmission to fall in the center of the detection bandwidth (y=0) while the reflection bandwidth varies uniformly throughout the band ($z_{max}=72^\circ$ for BT=10).* In Figure 2-7 through

*In those graphs where we have assumed all doppler to be zero we denote this by z=0. The parameters z and y we defined in Appendix A.

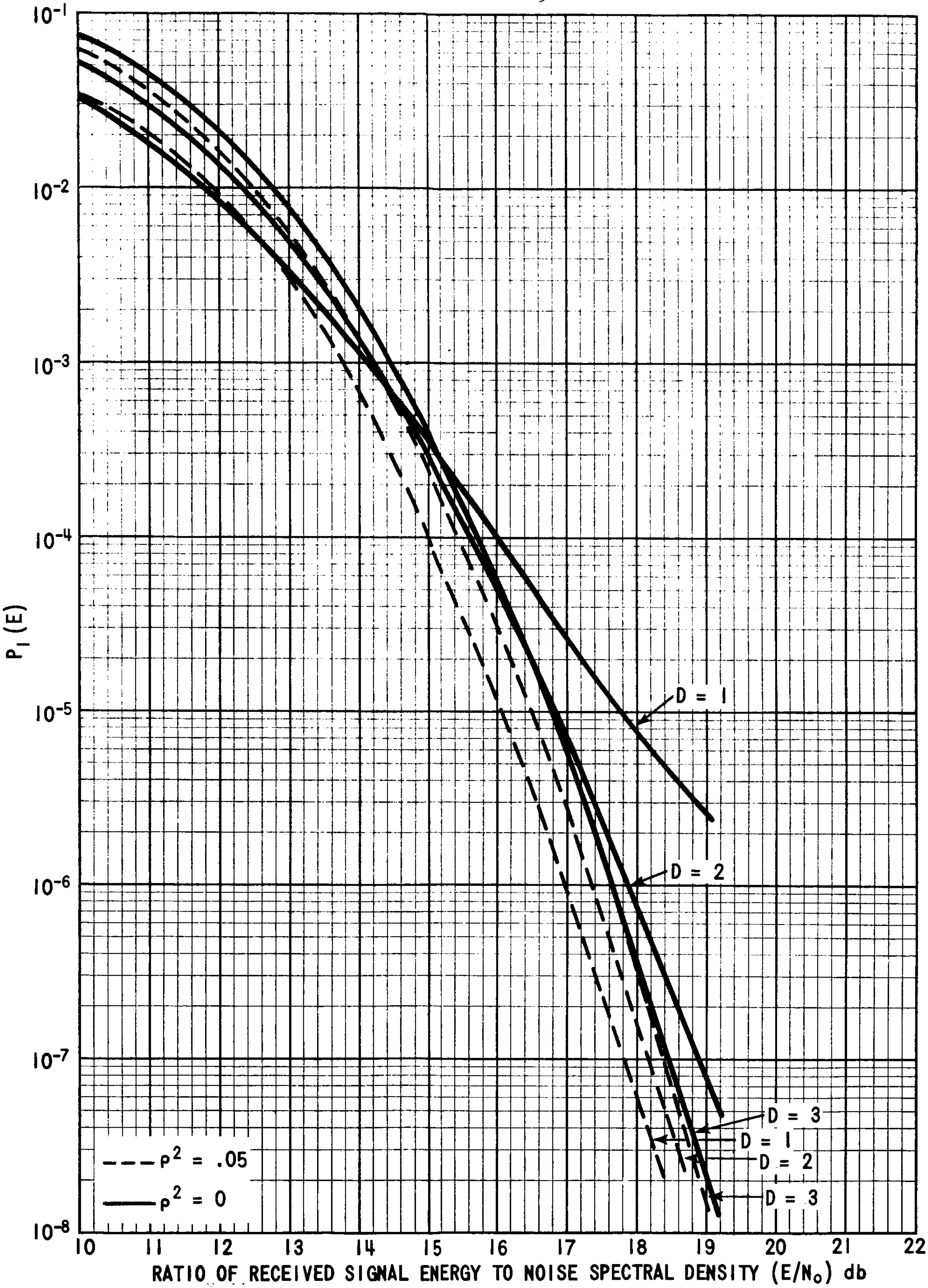

FIGURE 2-7. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE ONE ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL (ζ^2 = .0, z = 0, BT = 5)

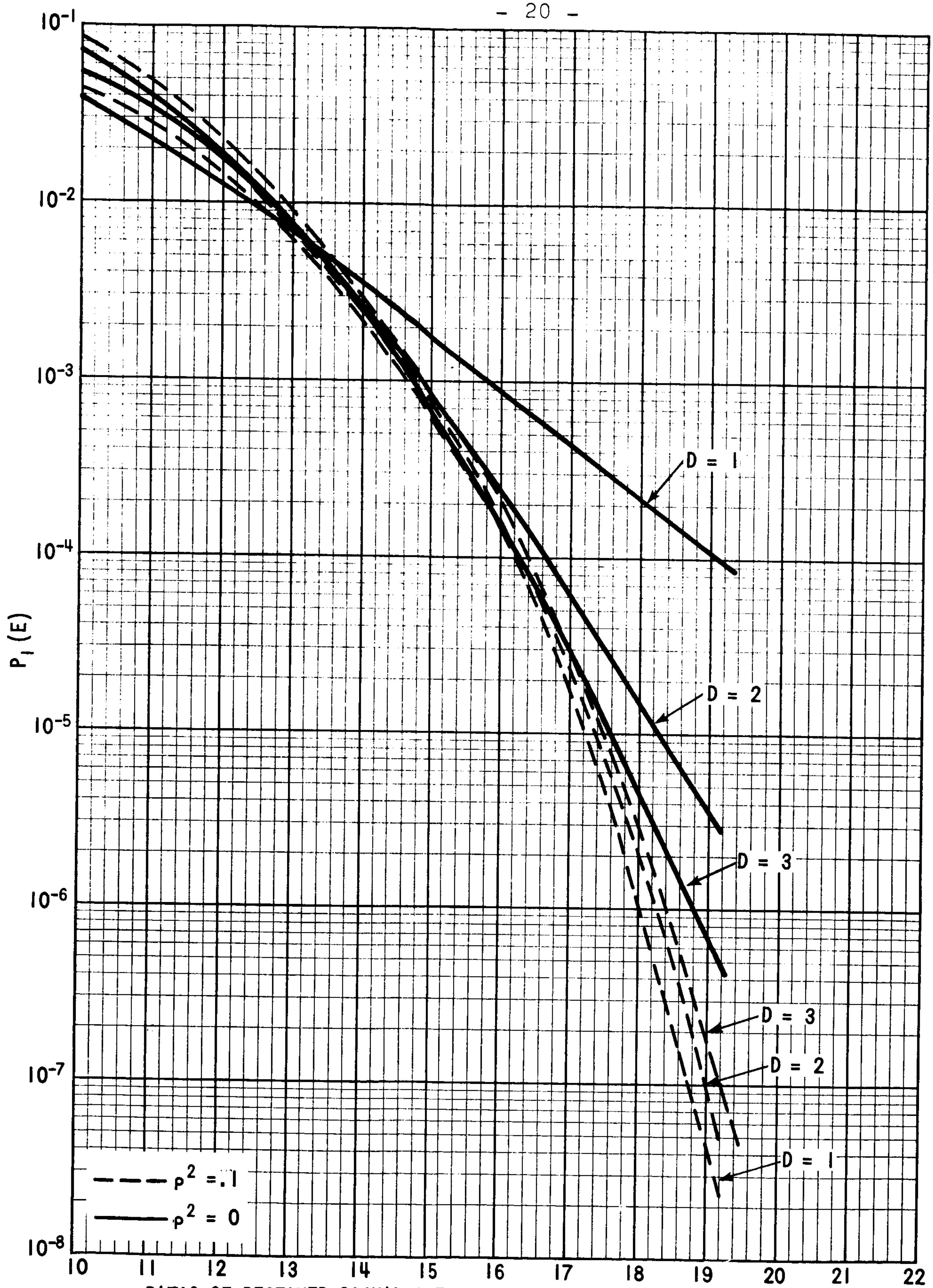

FIGURE 2-8. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE ONE ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL ($\zeta^2 = .1$, $z = 0$, $BT = 5$)

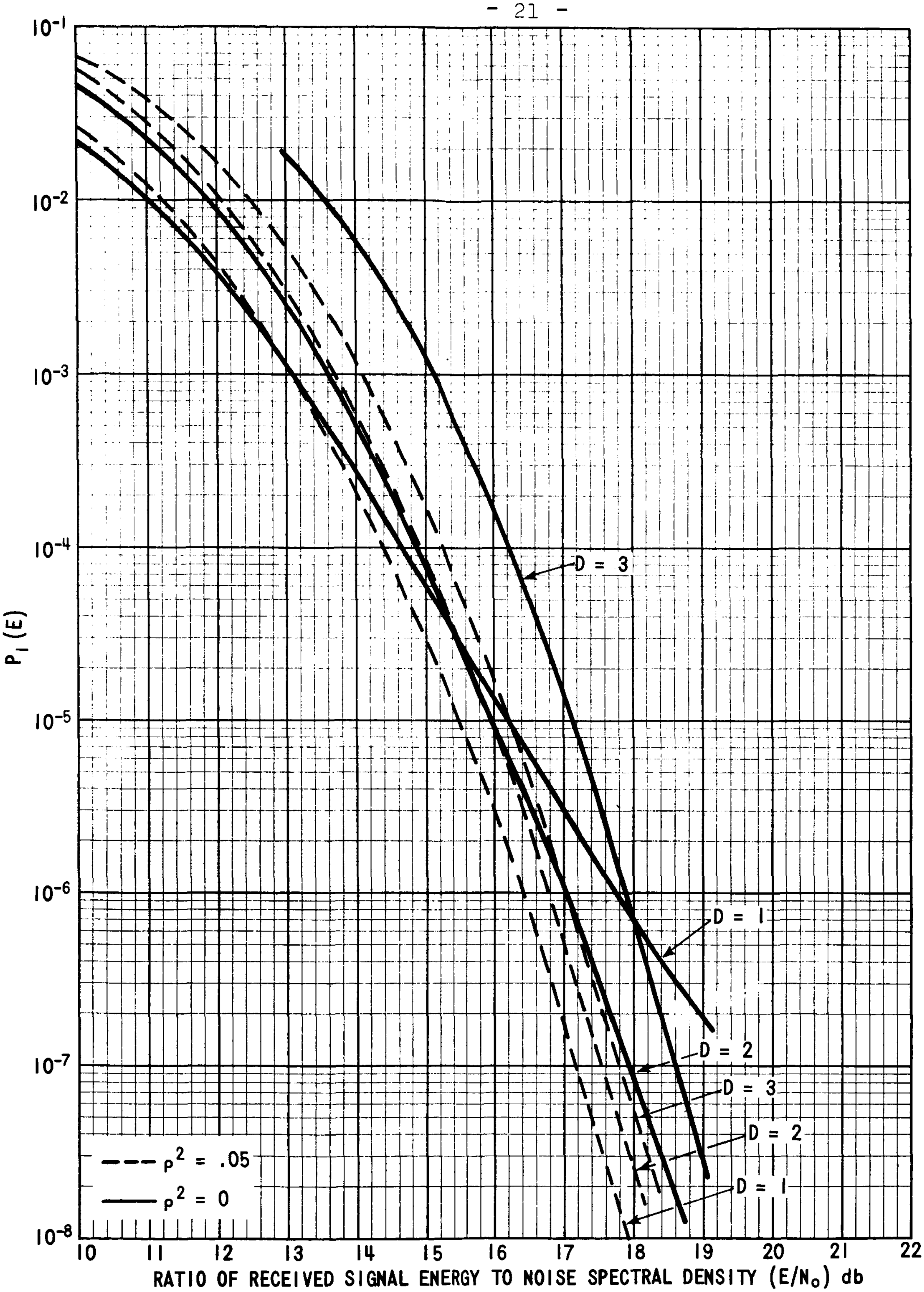

FIGURE 2-9. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE ONE ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL. (ζ^2 = .05) z_{max} = 54°, BT = 5)

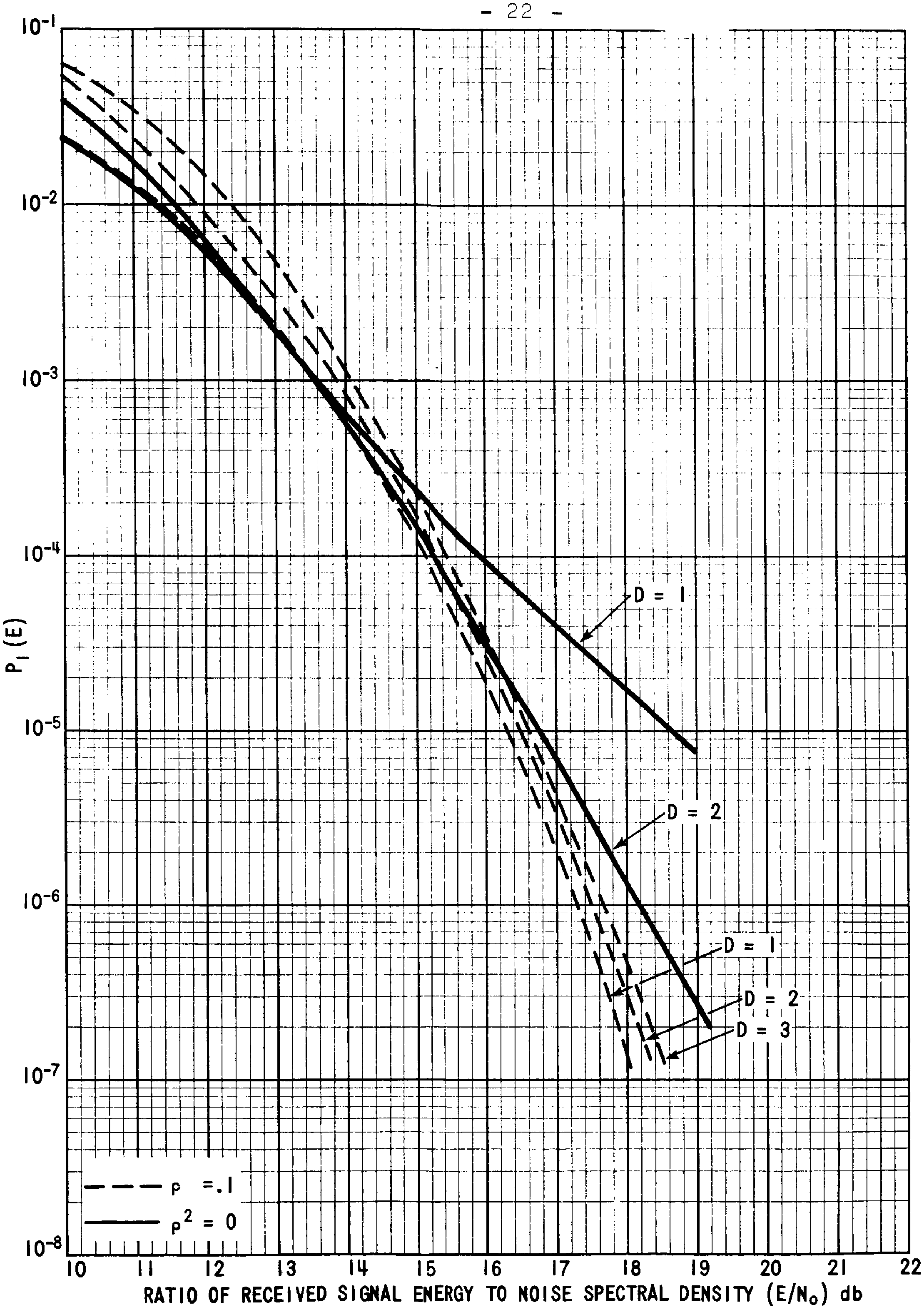

FIGURE 2-10. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE ONE ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL ($\zeta^2 = .1$, $z = 54°$, $BT = 5$)

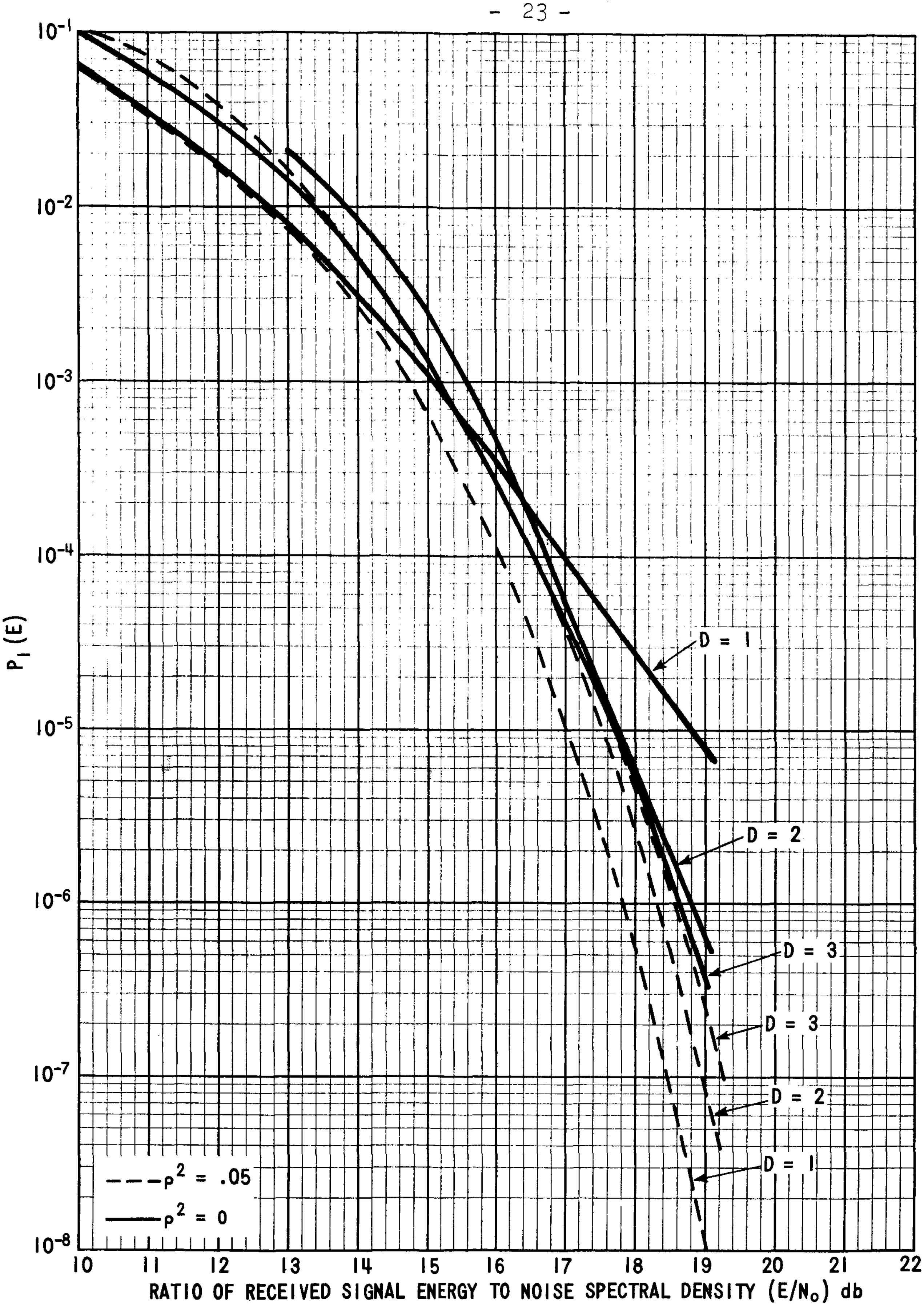

FIGURE 2-11. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE ONE ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL (ζ^2 = .05, z = 0, BT = 10)

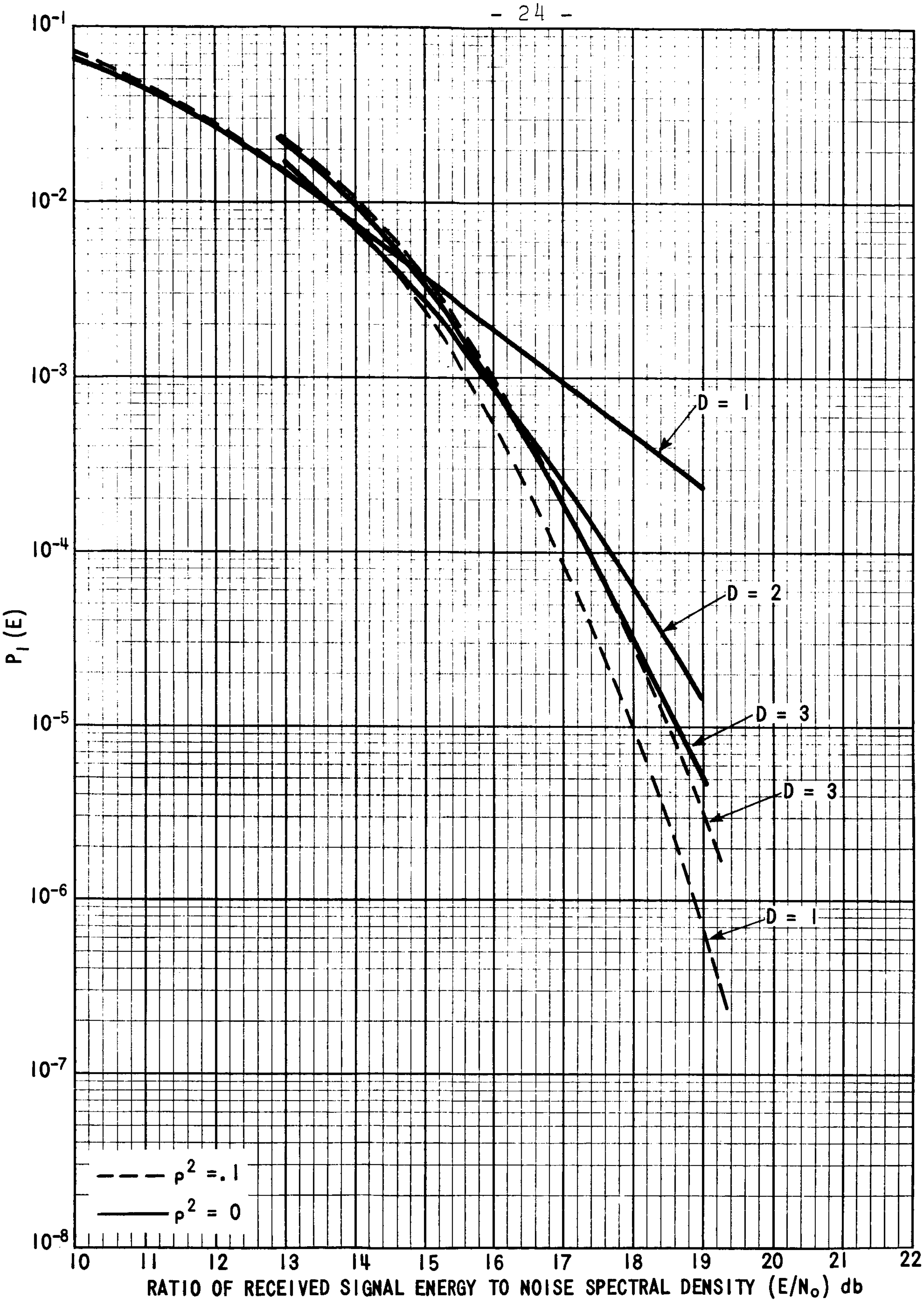

FIGURE 2-12. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE ONE ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL ($\zeta^2 = .1$, $z = 0$, $BT = 10$)

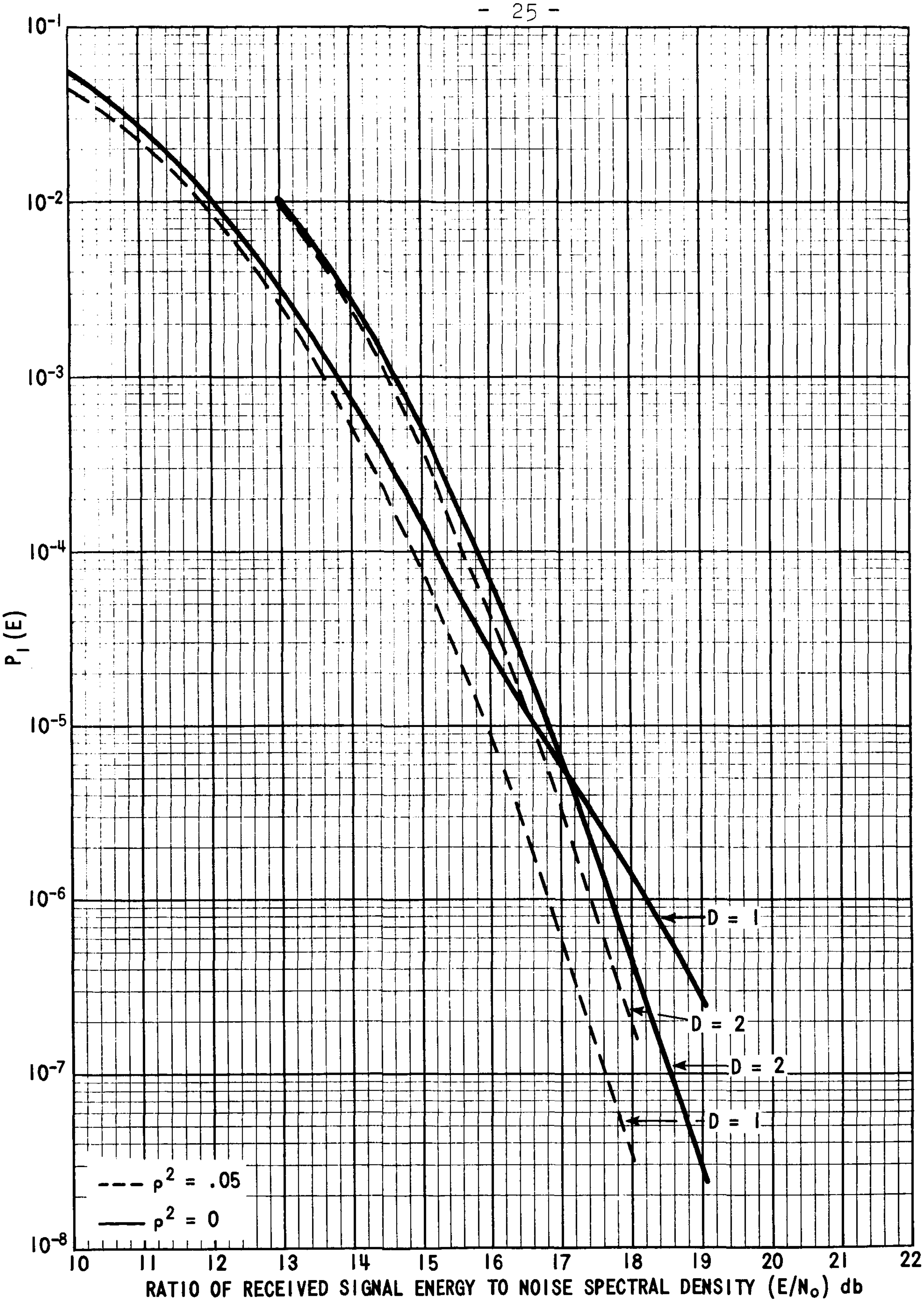

FIGURE 2-13. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE ONE ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL (ζ^2 = .05, z_{max} = 72°, BT = 10)

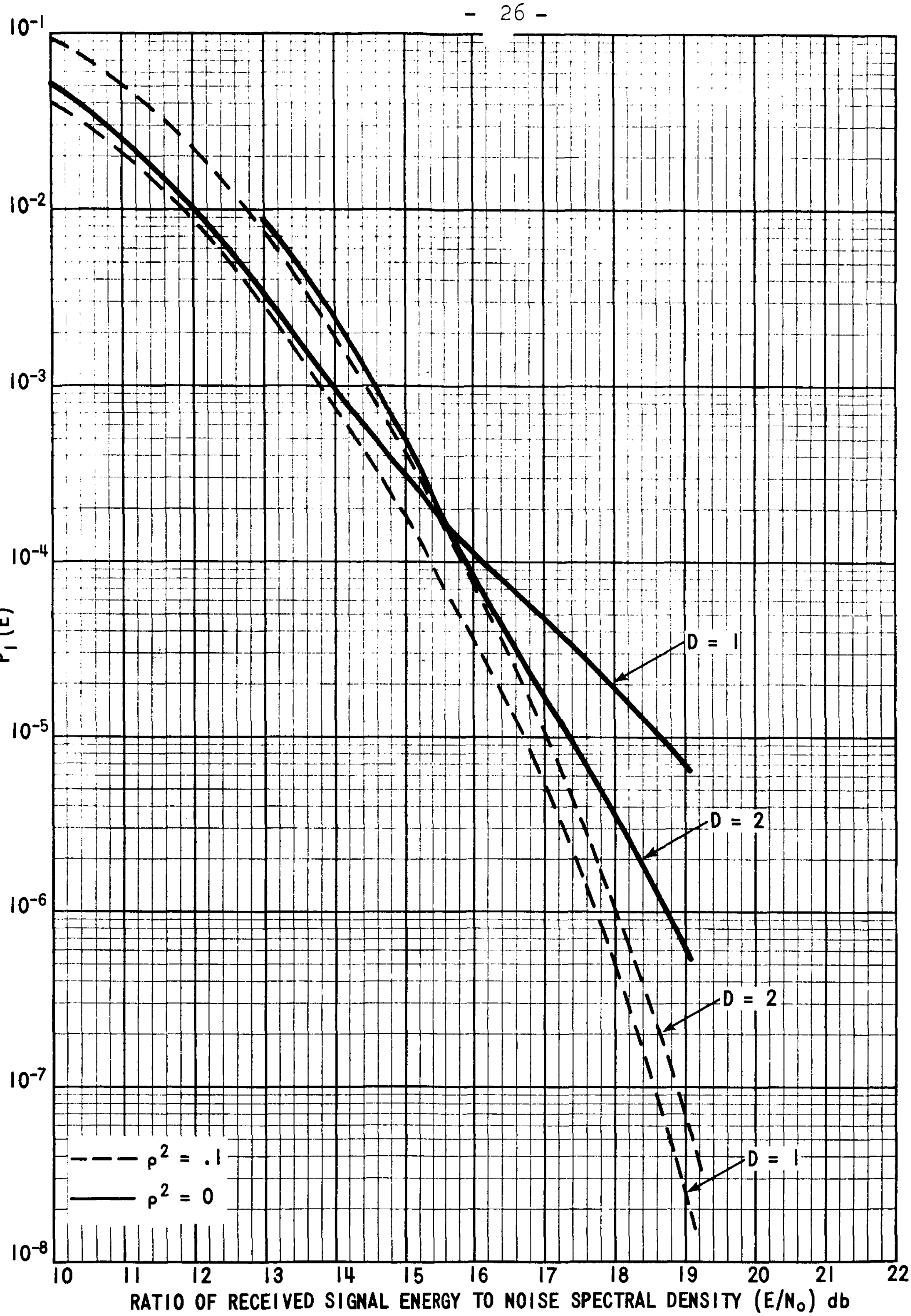

FIGURE 2-14. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE ONE ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL (ζ^2 = .1; z_{max} = 72°, BT = 10)

3-14, our results are presented for the performance of the wideband receiver when type one errors are made. Two values of varying reflected power were used $\{\zeta^2 = .05$ and $\zeta^2 = .1\}$ and for each, the cases of totally diffuse and totally specular reflected signals are plotted. The case of no doppler variation is studied and one in which the relative doppler varies uniformly from one extreme edge of the detection subchannel passband to the other.

In general, the results demonstrate the following: Independent of the BT value, diversity nearly always leads to a greater error rate when the reflection is specular. On the other hand, the effects of a diffuse reflection may be significantly reduced by increasing the diversity. Thus, by Figure 2-7, if a 10^{-6} bit error rate is desired nearly 1.8 db in signal power can be saved by using a diversity of 2 if the reflection is diffuse. On the other hand, if a 10^{-3} error rate is required increasing the diversity does not lead to an improved system performance. A realistic design is one in which the doppler offset is accounted for. It can be seen from the graphical results that when not accounted for (z=0) a conservative design results. That is proper design results in a signal energy savings which is greater than a db. Finally, we note that as the diffuse reflection power increases, the maximum order of diversity for which improvement can be realized in the normal communication bit error rate range of 10^{-3} to 10^{-6}, also increases.

Type Two Errors

In Figures 2-15 through 2-18, we have plotted our results for the performance of the wideband FSK receiver when type two errors are made. We note that the observations made for type one errors are also valid for discussion in this subsection. In addition, we note that type two errors generally cause poorer performance than type one errors.

2.2 Time-Varying Fading (Correlated reflections)

A specific case of a time varying fading channel (as discussed in Appendix 2) in which the effective bandwidth of the diffuse reflection is equal to the information bandwidth and whose spectrum shape is white was assumed. Two approaches were taken to determine the bit error rate. A chi square approximation which results in equation A-21 and a Karhumen-Loweve expansion of the $\alpha_1(t)$ parameters which leads to equation A-22. Both equation A-21 and equation A-22 were programmed.

It can be seen from equation A-22 that the eigenvalues λ_i and the integral of the eigen functions $\frac{1}{\sqrt{T}} \int_0^T \phi_i(t)dt = V_i$ are required. In general, there are an infinite number of eigenvalues and V_i's to consider. However as can be seen by Tables A-1 and A-2, their values decrease rapidly. Using the first four V_i, solutions were compared (for a variety of variations on the essential parameters) where, the first four eigenvalues were used with that in which the first nine were used. In all cases tested, the results were essentially identical and it was concluded that we could thus assume that we had four non zero eigenvalues. As discussed in reference 4, only the first four V_i values could easily be determined (it was found that for i even $V_i = 0$). To determine the significance of the V_i, a programmed run in which V_i $(i \neq 1)$ were set equal to zero was compared with results obtained in which the correct values for V_1 and V_3 were used. The results of this comparison are demonstrated in Figure 2-19.

It can be seen that less than a .1 db error is made when setting $V_3 = 0$. Although this is no rigorous proof, it does imply together with the decreasing values of the higher order eigenvalues that little is lost in setting the remainder of the V_i $(i>3)$ equal to zero.

The results of our programming equations A-21 and A-22 are illustrated in Figures 2-20 to 2-23. The bit error is plotted for values of BT = 10, 20, and 30. The two values of average reflected power were $\zeta^2 = .05$ and $\zeta^2 = .1$ and for each two cases were run. In the first, the specular component was set equal to zero $(\rho^2=0)$ while in the second the average specular power was set equal to the average diffuse power $(\rho^2 = 2\beta^2)$.

In all cases, it was found that when using the Karhunen-Loweve expansion, performance was worsened as the diversity increased. On the other hand, the use of the chi-square approximation led to just the opposite conclusion.

Since the K-L expansion was the more rigorous approach we conclude that when the fading is not slow, diversity is of little help in improving performance. In addition, this section points up the dangers of using crude approximates to arrive at meaningful results.

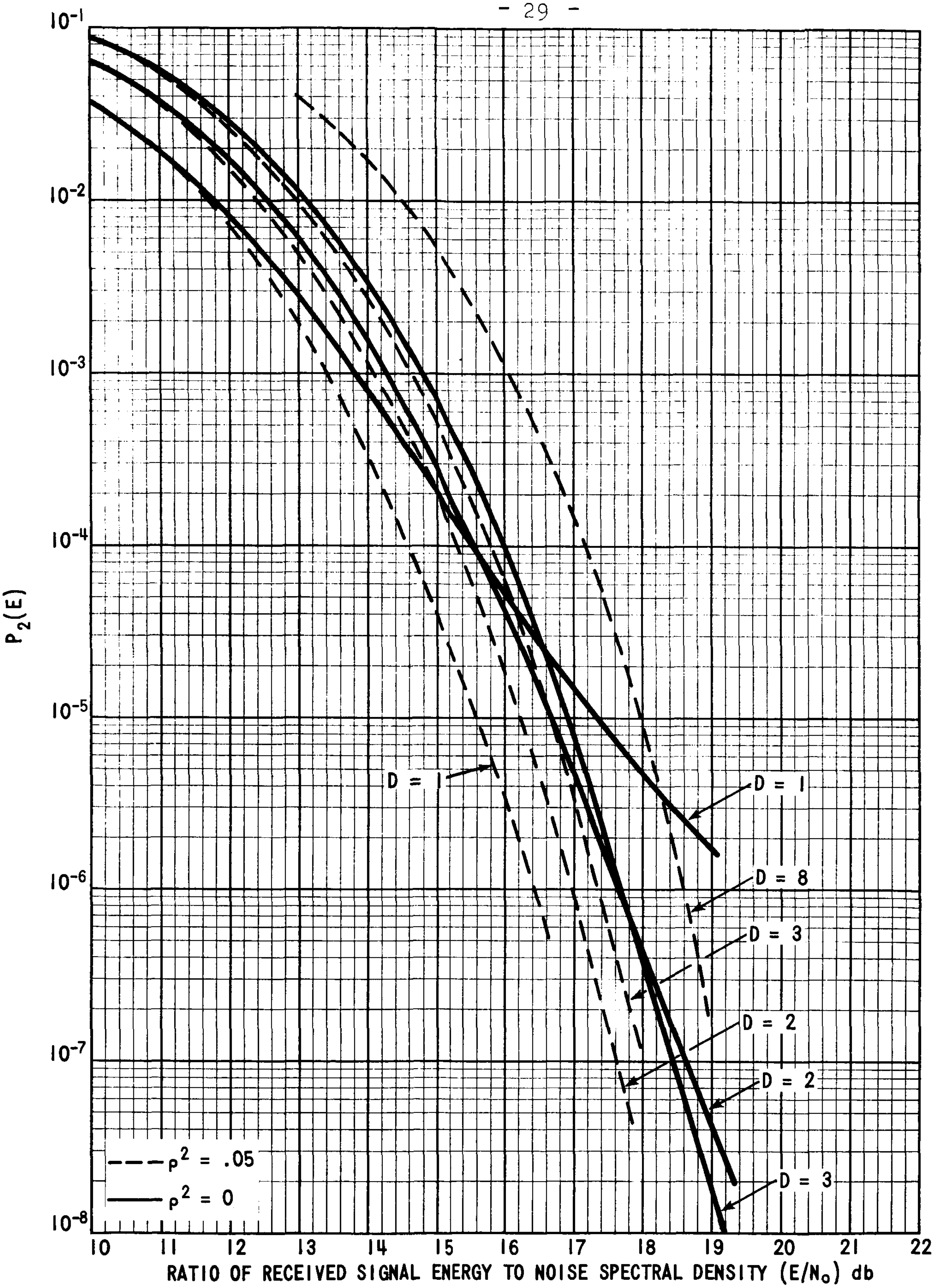

FIGURE 2-15. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE TWO ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL (ζ^2 = .05, z = 0, BT = 5)

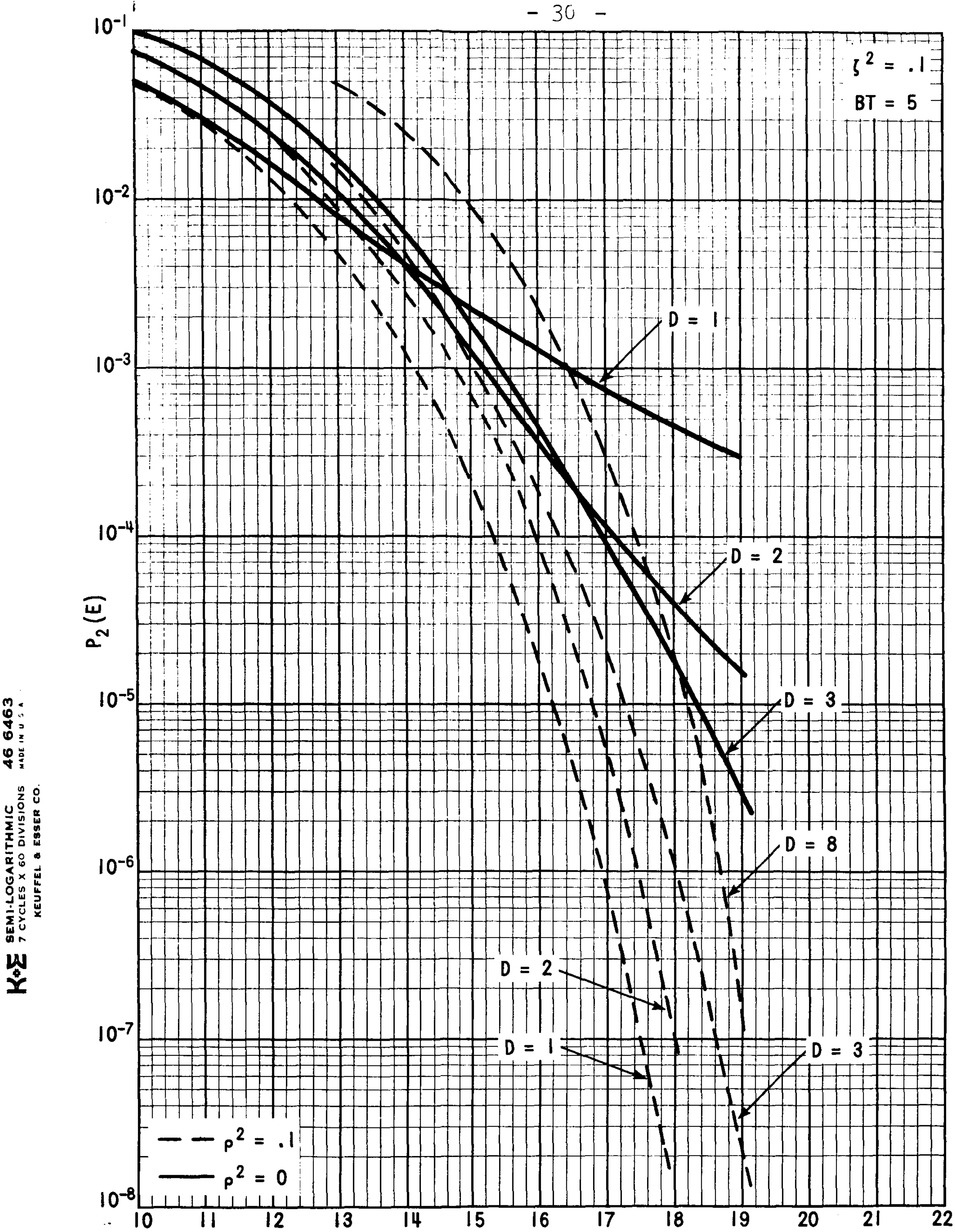

FIGURE 2-16. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE TWO ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL ($\zeta^2 = .1$, z = 0, BT = 5)

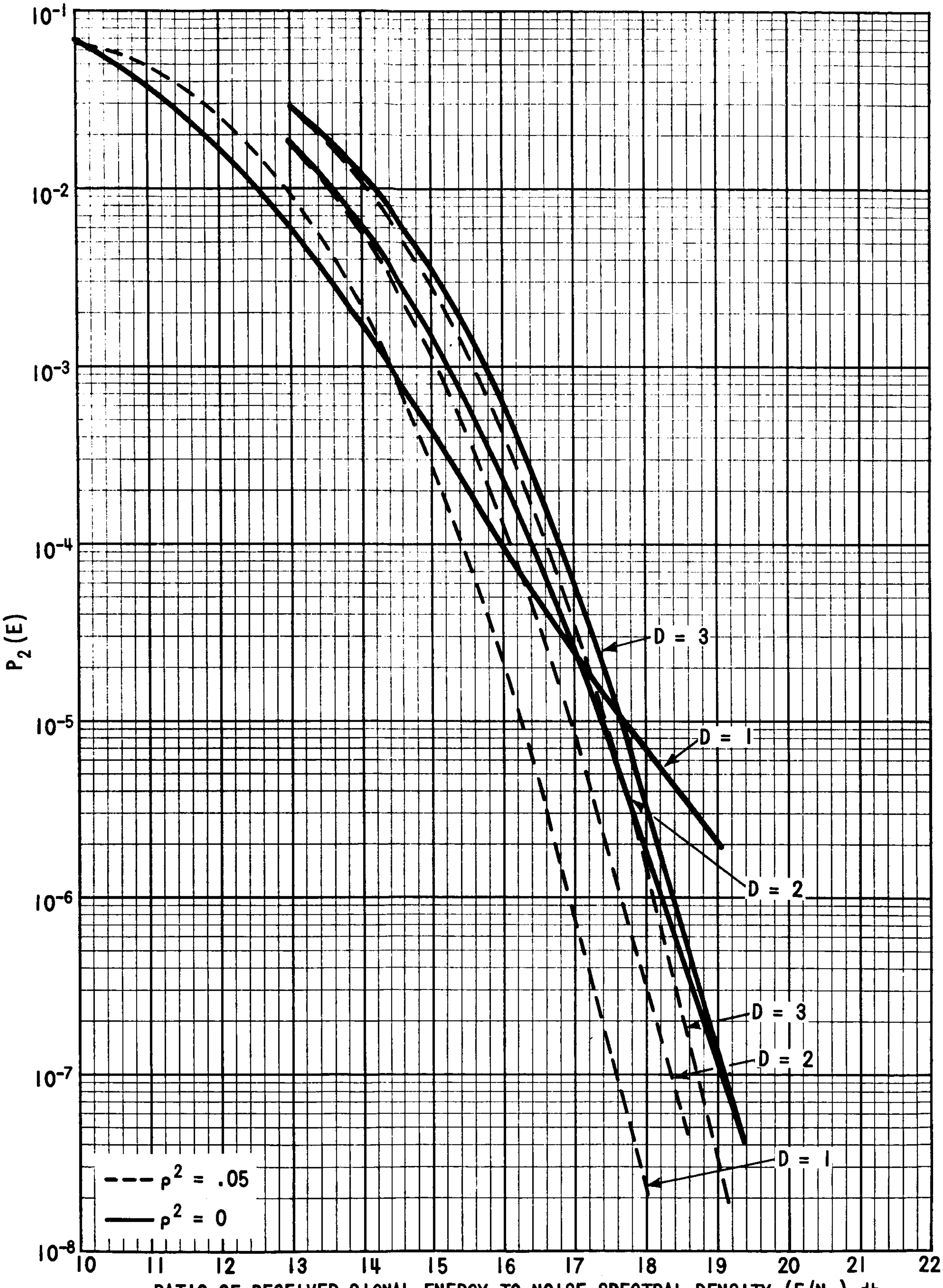

FIGURE 2-17. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE TWO ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL (ζ^2 = .05, z = 0, BT = 10)

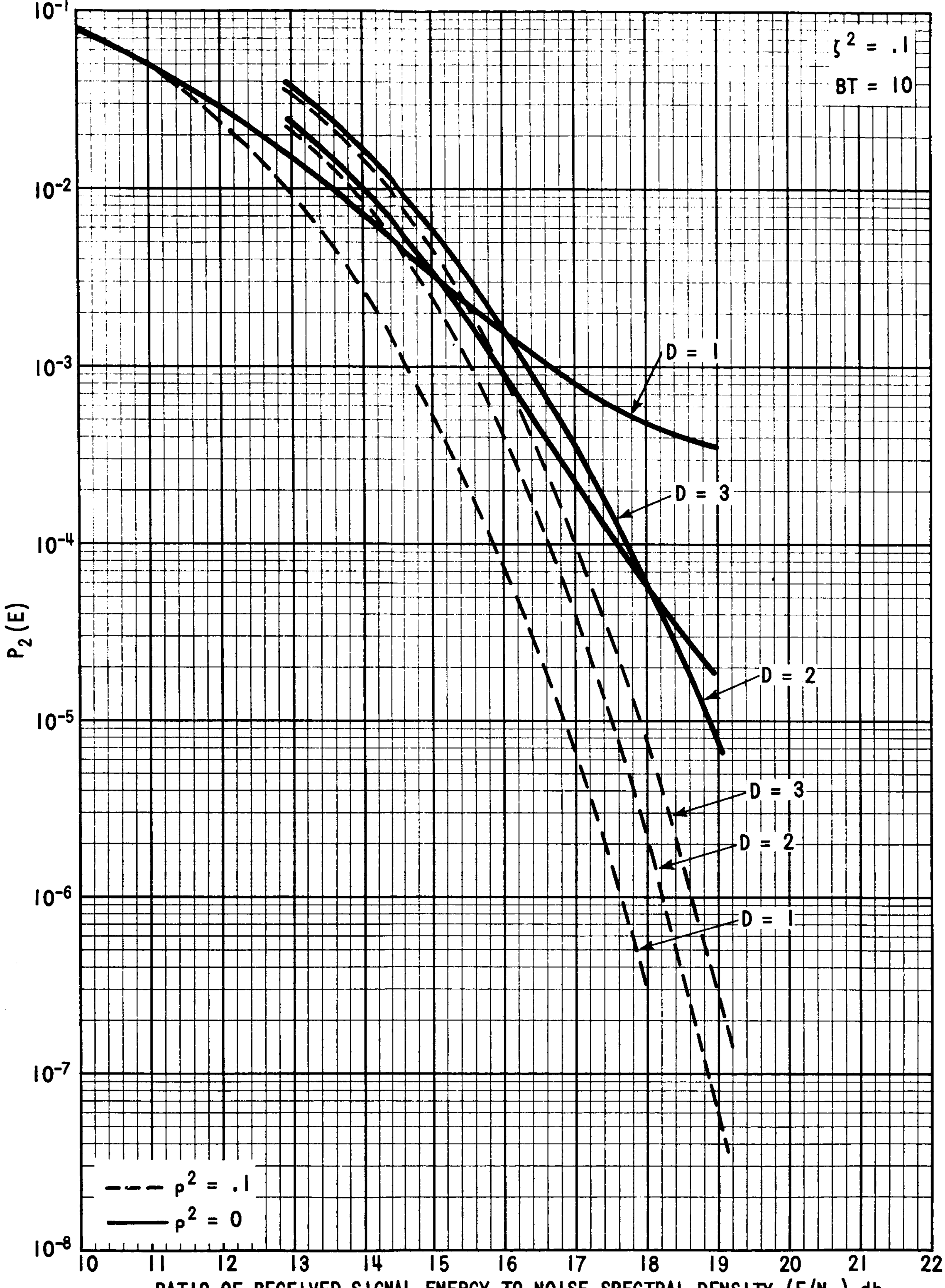

FIGURE 2-18. WIDEBAND FSK DIVERSITY PERFORMANCE FOR TYPE TWO ERRORS IN A SLOW 2 PATH/MULTIPATH CHANNEL ($\zeta^2 = .1$, $z = 0$, $BT = 10$)

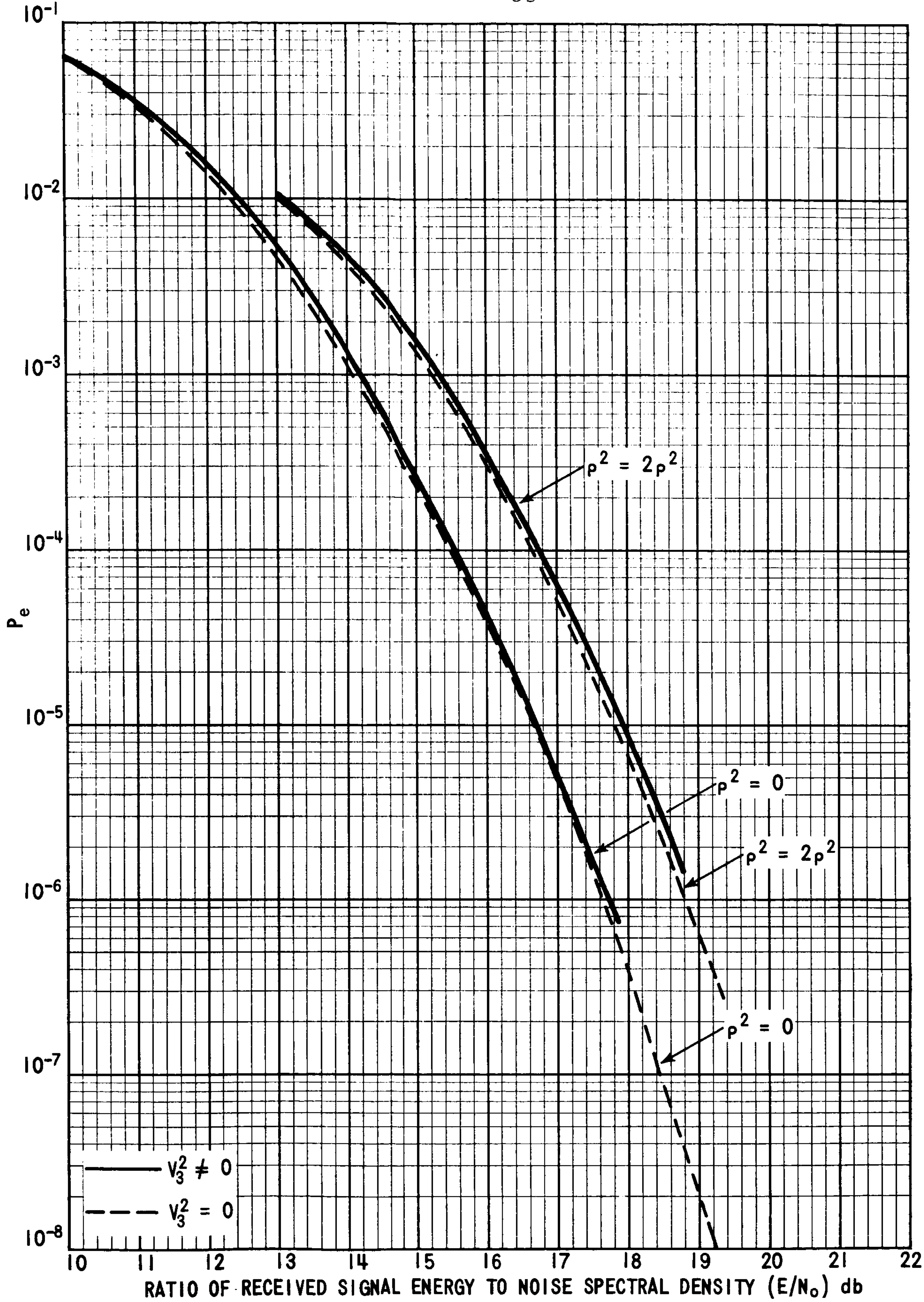

FIGURE 2-19. RESULTS USING THE KARHUNEN-LOEVE EXPANSION WHEN THE FADING IS NOT SLOW AND THE V_i PARAMETERS FOR $i \neq 1$ ARE SET EQUAL TO ZERO.

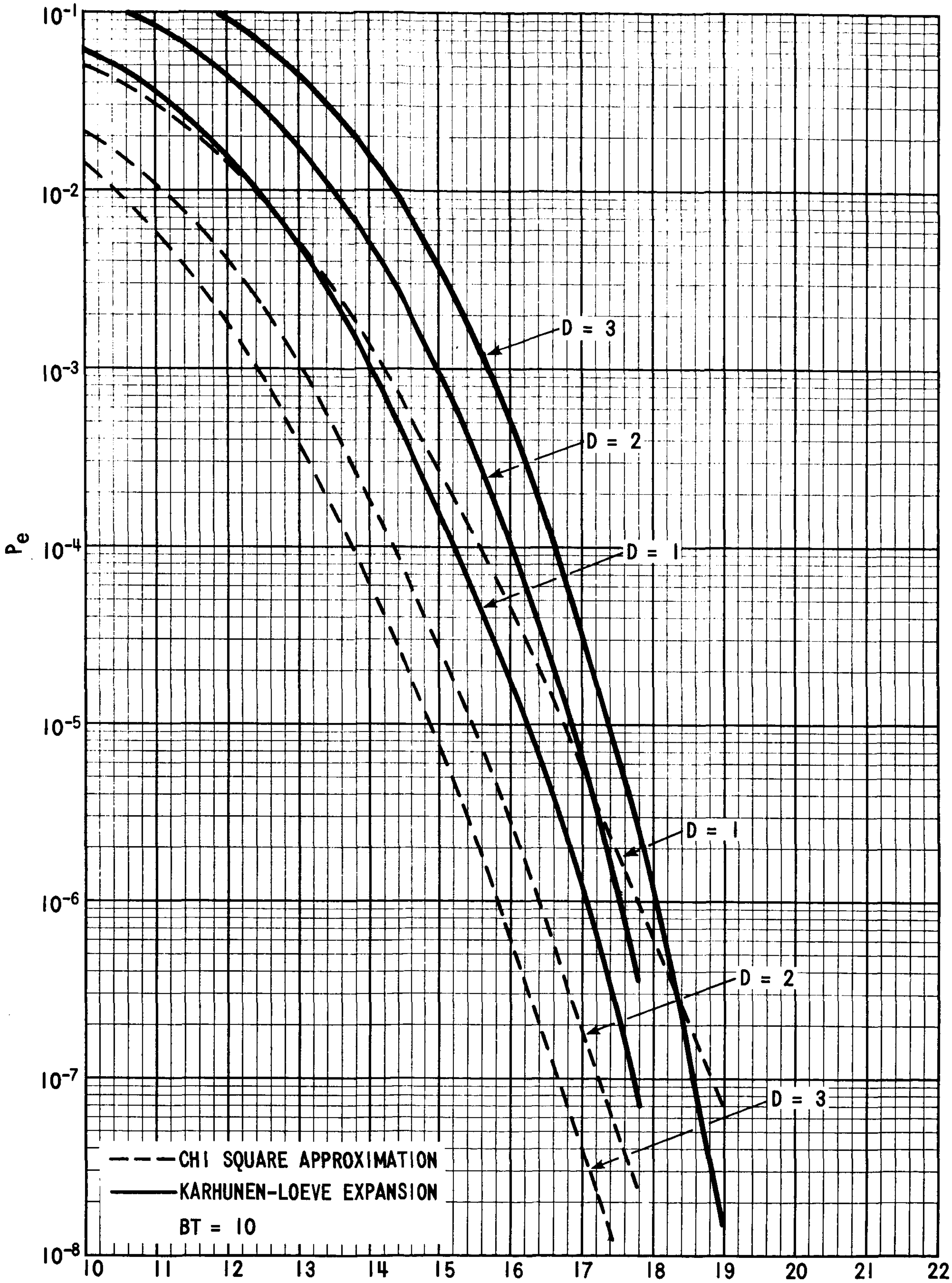

FIGURE 2-20. DIVERSITY PERFORMANCE OF THE WIDEBAND FSK RECEIVING WHEN THE FADE RATE IS NOT SLOW (ζ^2 = .05, ρ^2 = 0)

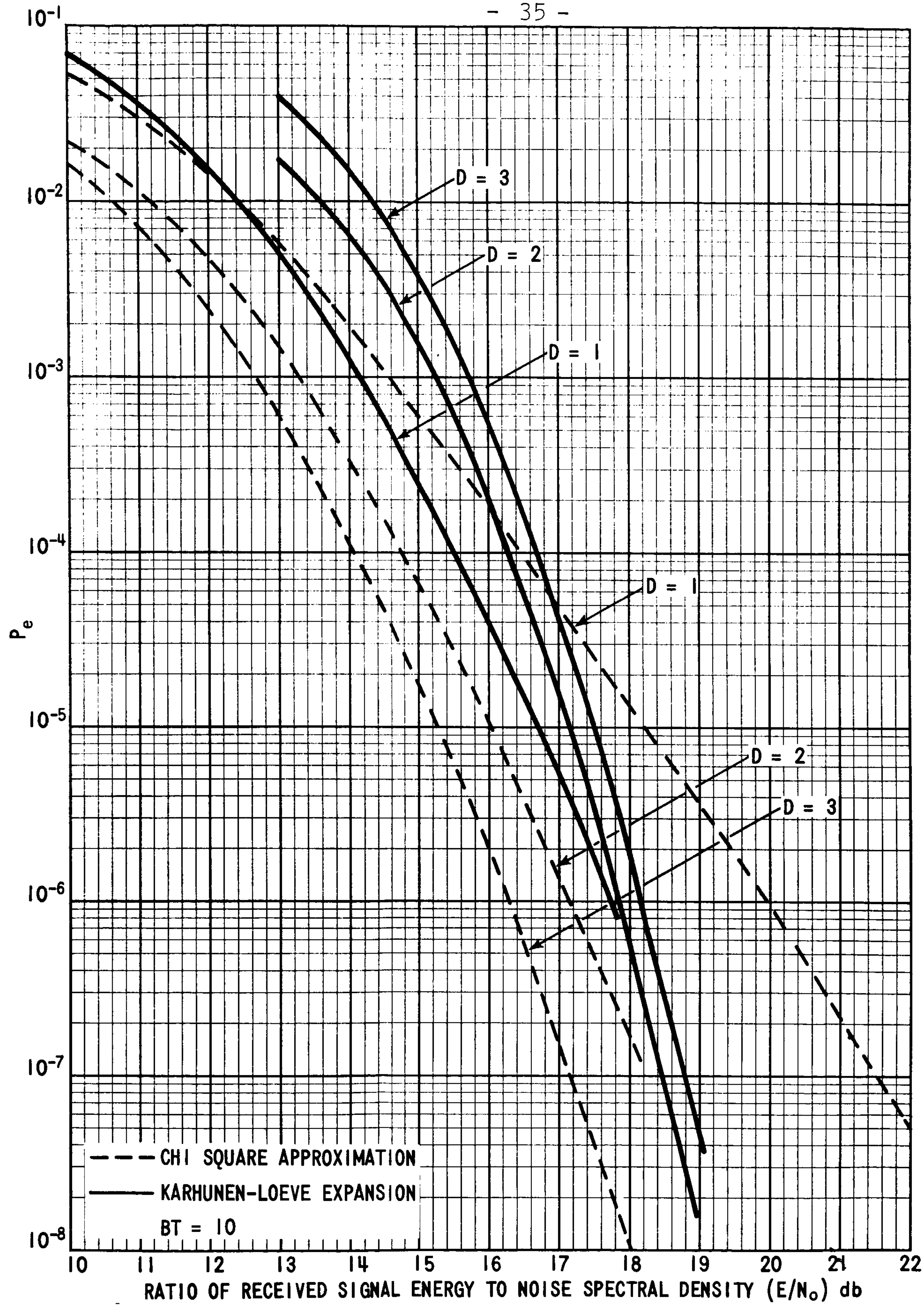

FIGURE 2-21. DIVERSITY PERFORMANCE OF THE WIDE BAND FSK RECEIVER WHEN THE FADE RATE IS NOT SLOW ($\zeta^2 = .1$, $\rho^2 = 0$)

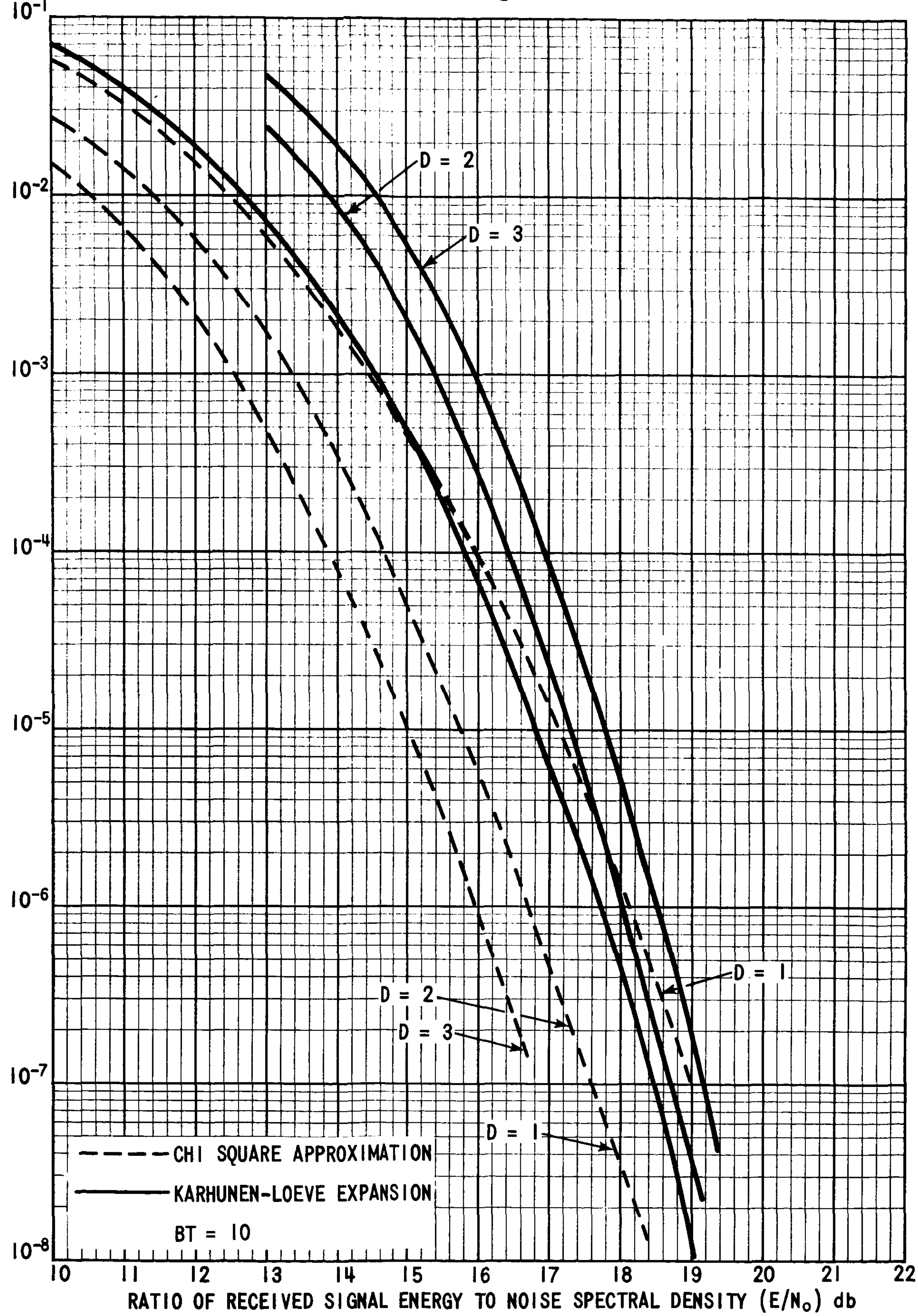

FIGURE 2-22. DIVERSITY PERFORMANCE OF THE WIDEBAND FSK RECEIVING WHEN THE FADE RATE IS NOT SLOW ($\zeta^2 = .05$, $\rho^2 = 2\rho^2$)

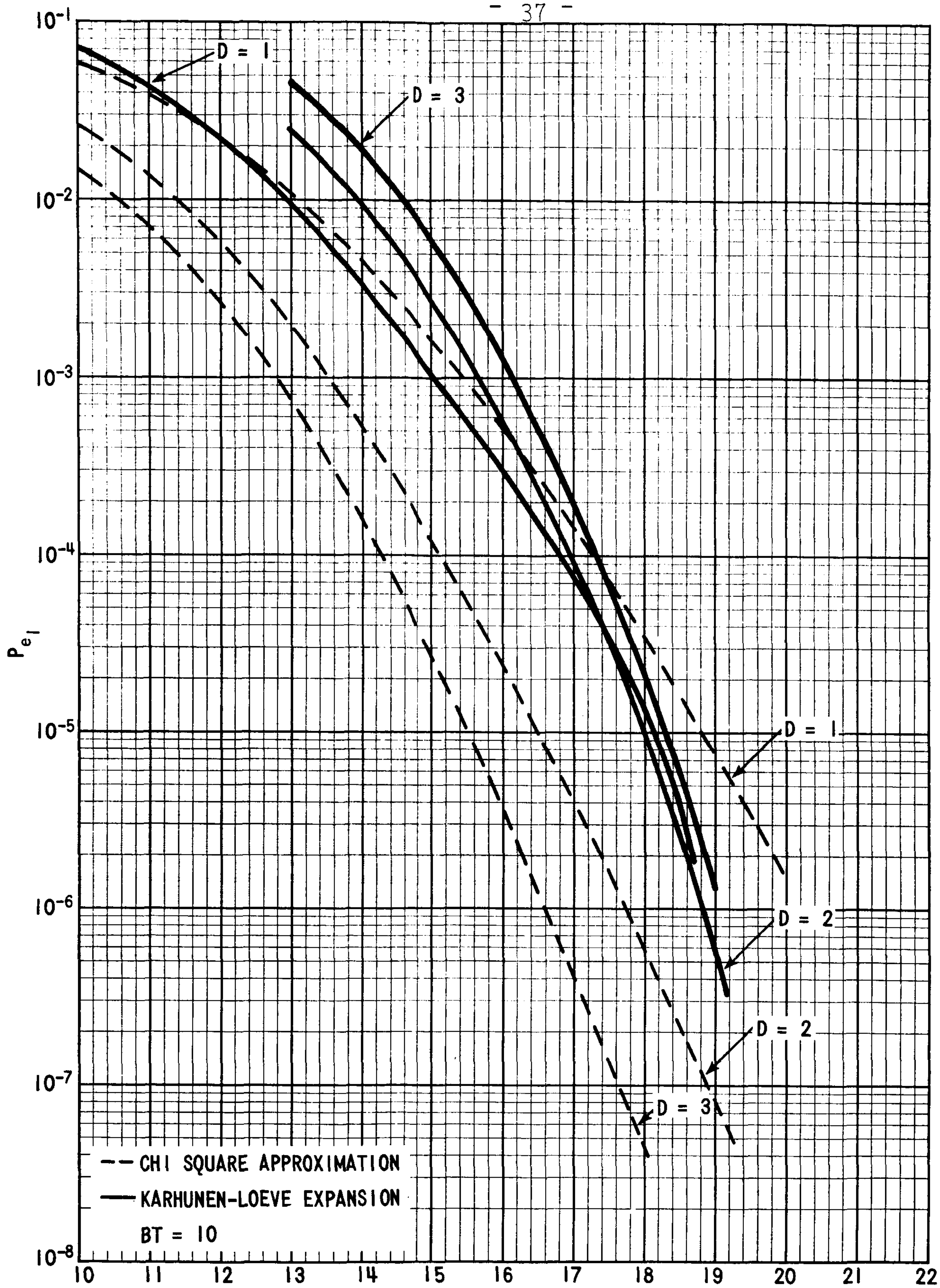

FIGURE 2-23. DIVERSITY PERFORMANCE OF THE WIDEBAND FSK RECEIVER WHEN THE FADE RATE IS NOW SLOW ($\zeta^2 = .05$, $\rho^2 = 2\rho^2$)

CONCLUSIONS

This paper demonstrates once more the importance of properly describing the multipath found in a relay space link. It is shown that diversity improvement depends upon the composition of the reflection together with the fade rate. Indeed, it is shown that specular reflections nearly always lead to system degradation in performance when the system uses diversity while the effects of diffuse reflections can be significantly reduced for most cases with the use of diversity.

More specifically, we have generated programs which can be used for any space relay link which has a two-path/multipath problem to predict system performance. Such predictions can in turn be used to determine system sizing requirements.

In addition the graphs given in section 2 have been plotted for a wide range of parameters of interest and illustrate more specifically the effect of using diversity techniques.

ACKNOWLEDGEMENTS

An integral part of this work is the programs generated to obtain the results shown in section 2. Therefore, I would like to thank Carol Friend, who did all of the programming, for her invaluable assistance.

2034-LS-jr L. Schuchman

Attachments
References
Appendices A, B and C

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APPENDIX A

A.0 DERIVATION OF CHANNEL PERFORMANCE METRICS D DIVERSITY WIDEBAND FSK

A.1 Correlated Direct and Reflected Transmission FSK Modulation

Case 1 Slow Fading

$$\alpha_{ki}(t_1, \beta \sqrt{\frac{A}{D}}) = \alpha_{ki}(t_2, \beta \sqrt{\frac{A}{D}})$$

where

$$A = (\frac{2\varepsilon}{T})^{1/2}$$

$$k \varepsilon \{1,2\}$$

$$i \varepsilon \{1,2,\cdots,D\}$$

It has been shown that the energy measures χ_1^2 and χ_2^2 for large BT* products can be described conditionally as chi-square variates (given α_1, α_2 and θ) each with 2BT degrees of freedom.[4] Similarly, since each of the diversity subchannels can be considered as a no diversity system the χ_{1i}^2 and χ_{2i}^2 can also be described conditionally as chi-square variates (given $\{\alpha_1\}$, $\{\alpha_2\}$, and $\{\theta\}$)** each with 2BT degrees of freedom.

Therefore in this paper we will only derive that which is additionally necessary to give the more generalized D diversity solution. Thus we may write the χ_{1i}^2 variate in the following manner.

$$\chi_{1i}^2 = \frac{2\varepsilon}{D} K_{3i}(\theta_i) + n_i^2 \qquad \text{(A-1)}$$

where

---

*BT is the product of the rf detection bandwidth (B) and the information bit duration (T).

**The notation {U} is used to define the set of random variables U_1 U_2 ... .

$$n_i^2 = \frac{1}{B} \sum_{j=1}^{BT} \left[n_{1ij} + \frac{A}{\sqrt{D}} \operatorname{sinc} y_i \cos (2y_i j + \theta_i - y_i) \right.$$

$$+ (\rho \frac{A}{\sqrt{D}} + \alpha_{1i}) \operatorname{sinc} z_i \cos (2z_i j - z_i)$$

$$\left. - \alpha_{2i} \frac{A}{\sqrt{D}} \operatorname{sinc} z_i \sin (2z_i j - z_i) \right]^2$$

$$+ \frac{1}{B} \sum_{j=1}^{BT} \left[n_{2ij} + \frac{A}{\sqrt{D}} \operatorname{sinc} y_i \sin (2y_i j + \theta_i - y_i) \right.$$

$$+ (\rho \frac{A}{\sqrt{D}} + \alpha_{1i}) \operatorname{sinc} z_i \sin (2z_i j - z_i)$$

$$\left. - \alpha_{2i} \frac{A}{\sqrt{D}} \operatorname{sinc} z_i \cos (2z_i j - z_i) \right]^2 \qquad \text{(A-2)}$$

ε is the total received energy of the direct path transmission

$$\operatorname{sinc} y_i = \frac{\sin y_i}{y_i}, \quad y_i = \frac{\pi \Delta f_d(i)}{B}, \quad z_i = \frac{\pi \Delta f_r(i)}{B}$$

$$k_{3i}(\theta_i) = k_{1i}(\theta_i) - k_{2i}(\theta_i) \qquad \text{(A-3)}$$

$$k_{1i}(\theta_i) = 1 + \left(\frac{\alpha_{2i}}{\frac{A}{\sqrt{D}}}\right)^2 + \left(\rho + \frac{\alpha_{1i}}{\frac{A}{\sqrt{D}}}\right)$$

$$+ 2\left(\rho + \frac{\alpha_{1i}}{\frac{A}{\sqrt{D}}}\right) \text{ sinc } BT\ (y_i - z_i) \cos\left[(y_i - z_i)BT + \theta_i\right]$$

$$+ 2 \frac{\alpha_{2i}}{\frac{A}{\sqrt{D}}} \text{ sinc } BT\ (y_i - z_i) \sin\left[BT(y_i - z_i) + \theta_i\right] \qquad \text{(A-4)}$$

$$k_{2i}(\theta_i) = \sin c^2 y_i + \left[\left(\frac{\alpha_{2i}}{\frac{A}{\sqrt{D}}}\right)^2 + \left(\rho + \frac{\alpha_{1i}}{\frac{A}{\sqrt{D}}}\right)^2\right] \sin c^2 z_i \qquad \text{(A-5)}$$

$$+ 2 \sin y_i \text{ sinc } z_i \left[\rho + \frac{\alpha_{1i}}{\frac{A}{\sqrt{D}}}\right] \frac{\sin BT(y_i - z_i)}{BT \sin(y_i - z_i)} \cos\ (BT(y_i - z_i) + \theta_i)$$

$$+ 2 \text{ sinc } y_i \text{ sinc } z_i \left[\frac{\alpha_{2i}}{\frac{A}{\sqrt{D}}}\right] \frac{\sin BT(y_i - z_i)}{BT \sin(y_i - z_i)} \cos\ (BT(y_i - z_i) + \theta_i)$$

We will assume that the doppler variations on the several diversity paths are all identical so that

$$y_i = y$$

$$z_i = z$$

and

$$k_{1i}(\theta_i) = k_1(\theta_i)$$

$$k_{2i}(\theta_i) = k_2(\theta_i)$$

Since the χ_{1i}^2 variates are independent, χ_1^2 is conditionally (given $\{\alpha_1\}$, $\{\alpha_2\}$, and $\{\theta\}$) a generalized non-central chi-square variate* with 2DBT degrees of freedom and can be written as

$$\chi_1^2 = \frac{2\varepsilon}{D} \sum_{i=1}^{D} k_3(\theta_i) + \sum_{i=1}^{D} n_i^2 \tag{A-6}$$

Similarly χ_2^2 is a chi-square variate with 2DBT degrees of freedom and is given by

$$\chi_2^2 = \sum_{i=1}^{D} \chi_{2i}^2$$

The conditional probability of error can therefore be written as

$$P(\varepsilon/\{\alpha_1\},\{\alpha_2\}, \{\theta\}) = P(\chi_1^2/\chi_2^2 < 1/\{\alpha_1\}, \{\alpha_2\} \{\theta\}) \tag{A-7}$$

Since χ_1^2/χ_2^2 is a generalized one sided non-central F variate, we can use the results of Appendix A of Reference 4 to write the conditional bit error rate as

$$P(\varepsilon/\{\alpha_1\}, \{\alpha_2\}, \{\theta\}) = \tag{A-8}$$

$$\left[\exp\left\{-\sum_{i=1}^{D} \frac{\varepsilon}{DN_o}\left[k_1(\theta_i) - \frac{k_2(\theta_i)}{2}\right]\right\}\right] \sum_{k=o}^{DBT-1} \frac{1}{k!}$$

$$\sum_{j=o}^{k} \binom{k}{j} \left[\frac{\varepsilon}{DN_o} \sum_{i=1}^{D} k_3(\theta_i)\right]^{k-j} \sum_{\ell=o}^{j} \binom{DBT + j-1}{j-\ell} \frac{j!}{\ell!} \left[\frac{\varepsilon}{N_o D} \sum_{i=1}^{D} k_2(\theta_i)\right]^{\ell} \frac{1}{2^{DBT + \ell+j}}$$

---

*A generalized non-central chi-square variate is defined as a non-central chi-square variate plus a constant.

Several special cases are of interest

1. No Multipath

When no multipath exists

$$k_1(\theta_i) = 1$$

$$k_2(\theta_i) = \text{sinc}^2 y$$

so that

$$\sum_{i=i}^{D} k_1(\theta_i) = D \tag{A-9}$$

$$\sum_{i=i}^{D} k_2(\theta_i) = D \text{ sinc}^2 y \tag{A-10}$$

Inserting equations (A-9), and (A-10) into equation (A-8) we see that performance gets worse as D increases as expected.* (Note that in this case the conditional bit error probability is equal to the bit error probability.)

2. Multipath - Specular Reflection Only

In this section we will obtain the average bit error probability when there is no diffuse component to the reflection. The sumed $k_j(\theta_i)$ parameters with $\{j=1,2\}$ are given by

$$\sum_{i=i}^{D} k_1(\theta_i) = (1 + \rho^2) D + 2\rho\, g_D(\{\theta_i\}) \text{ sinc}\left[(y-z) BT\right] \tag{A-11}$$

$$\sum_{i=i}^{D} k_2(\theta_i) = (\text{sinc}^2 y + \rho^2 \text{ sinc}^2 z) D$$

$$+ 2\rho\, g(\{\theta_i\}) \text{ sincy sincz } \frac{\sin BT(y-z)}{BT \sin(y-z)} \tag{A-12}$$

*If we assume that the transmitted signal energy is held constant, then in an additive gaussian noise environment, signal diversity reception does not improve performance and if not coherently summed at the receiver, actually degrades performance.

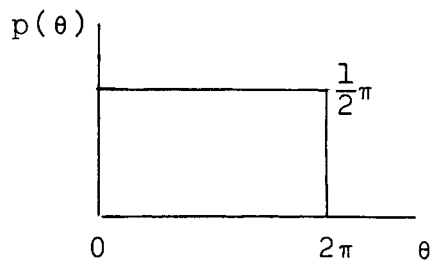
where

$$g(\{\theta_i\}) = \sum_{i=1}^{D} \cos\,(BT(y-z) + \theta_i) \qquad \text{(A-13)}$$

$g(\theta_i)$ is equal to the sum of D identically distributed random cosine waves. The probability density function of one cosine wave can easily be derived and is given by

$$P(g(\theta)) = \begin{cases} \dfrac{1}{\pi[1-g^2(\theta)]^{1/2}} & |g(\theta)|<| \\ 0 & \text{otherwise} \end{cases}$$

where $g(\theta) = \cos\theta$



while the characteristic function of $g(\theta)$ is given as

$$< e^{-g(\theta)t} >_{g(\theta)} = I_o(t) \qquad \text{(A-14)*}$$

where $I_o(t)$ is the modified Bessel function of the o^{th} order.

To obtain the characteristic function of $g(\{\theta_i\})$, we simply use equation (2-14) and transform theory to obtain

$$< e^{-g(\{\theta_i\})t} >_{g(\{\theta_i\})} = [I_o(t)]^D \qquad \text{(A-15)}$$

---

*$<\chi>_k$ is defined as the expectation of χ with respect to the variable k.

Equation (2-15) is all that is needed to generate the solution for

$$< [g(\{\theta_i\})]^{\alpha} \; e^{-g(\{\theta_i\})} >$$

Such a result allows one to express the bit error probability for the specular reflection case in closed form.

Unfortunately, the solution of the required expectation is in terms of higher order modified Bessel functions and although such a solution is mathematically satisfying it does not easily lend itself to the obtaining of numerical results.* Therefore we shall use the following alternative approach to obtain the desired results.

If we take the expectation of equation (A-8) with respect to $g(\{\theta\})$ we can write the bit error probability in the following integral form

$$P(E) = \sum_{k=o}^{DBT-1} \frac{1}{k!} \sum_{j=o}^{k} \sum_{\ell=o}^{k} \binom{k}{j} \binom{DBT + j-1}{j-\ell} \frac{j!}{\ell!} \frac{1}{2^{DBT + \ell+j}} \cdot$$

$$\int_{-D}^{D} \left[\exp - \left\{ \sum_{i=i}^{D} \frac{\varepsilon}{DN_o} [K_1(\theta_i) - \frac{k_2(\theta_i)}{2}] \right\} \right] \cdot$$

$$\left[\frac{\varepsilon}{N_o D} \sum_{i=i}^{D} K_3(\theta_i) \right]^{k-j} \left[\frac{\varepsilon}{N_o D} \sum_{i=i}^{D} k_2(\theta_i) \right]^{\ell} p(g\{\theta\})) \; d \; g(\{\theta\}) \quad \text{(A-16)}$$

where $p(g\{\theta\})$ is the probability density function of the random variable $g(\{\theta\})$.

The probability of the sum of D identically distributed random cosine waves has been derived by both Slack[5] and Leong[6]. Figure A-1 is reproduced from Slack's paper.

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*The reader is referred to Reference 4 where the evaluation of the expectation is obtained for the no diversity case.

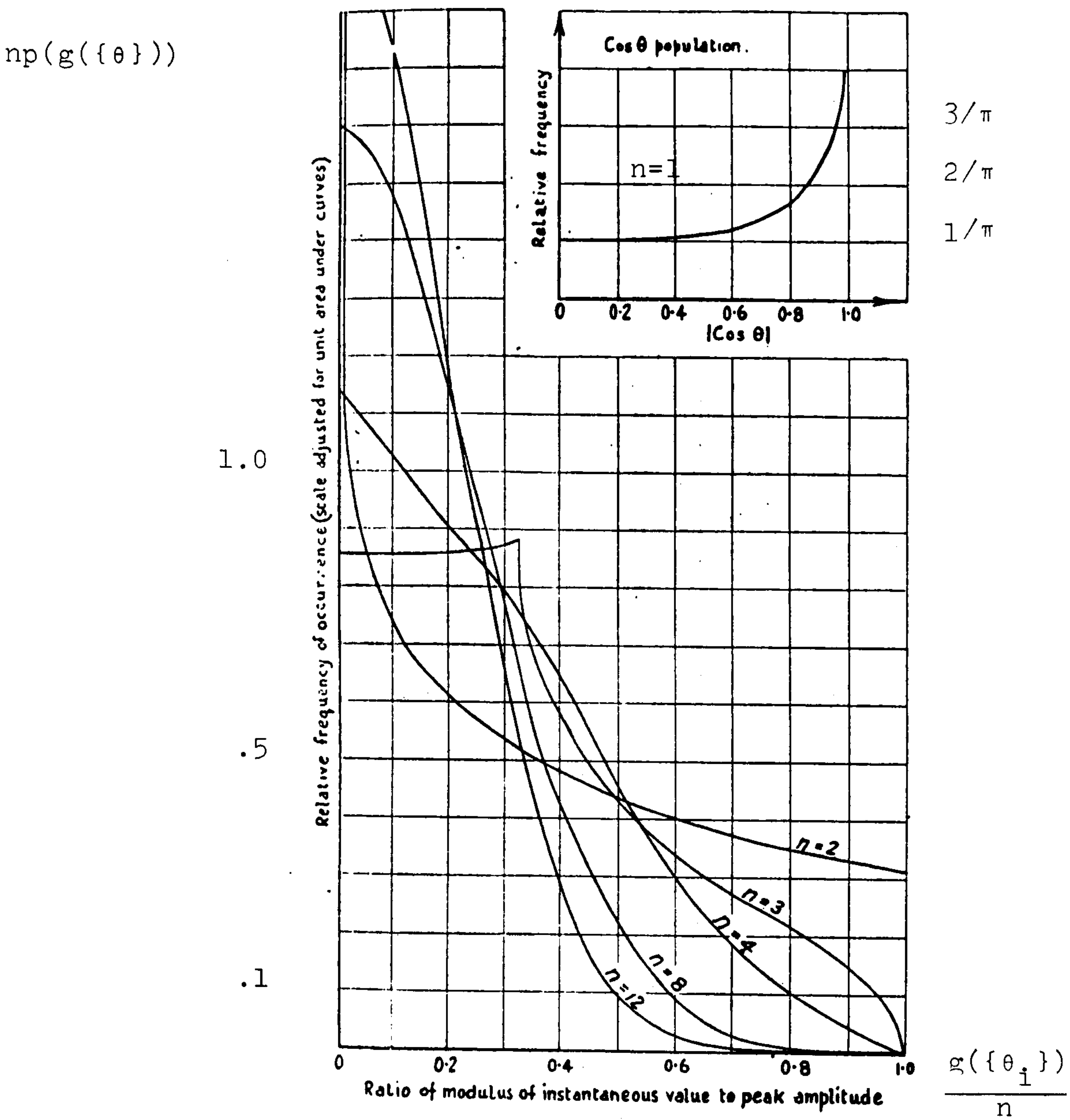

Figure A-1 Instantaneous value obtained from the combination of n cosine oscillations*

*Taken from Figure 3 of Reference 5.

In numerically integrating equation 2-16 we shall use the exact distributions given by Slack for $p(g\{\theta\})$ when $D \leq 4$.

3. Multipath - Slow Diffuse Fading

In this case both a specular and diffuse reflection are assumed to exist. However, the variations of $\{\alpha_1(t)\}$ and $\{\alpha_2(t)\}$ over a bit interval is assumed to be so slow that they can be taken as constant random variables over a bit time interval. We now proceed to derive the bit error probability when averaged over the $\{\alpha_{1i}(t)$, $\{\alpha_{2i}(t)\}$, and $\{\theta_i\}$ parameters for the special case of small doppler. Thus, we assume

$$K_1(\theta_i) = K_2(\theta_i) = 1 - \sin c^2\ BT(y-z) + \omega_i^2 \tag{A-17}$$

where

$$\omega_i^2 = \left[\frac{\alpha_{1i}\sqrt{D}}{A} + \rho + \mathrm{sinc}\ BT(y-z)\ \cos[(y-z)BT + \theta_i]\right]^2 + \left[\frac{\alpha_{2i}\sqrt{D}}{A} = \mathrm{sinc}\ BT(y-z)\ \sin[(y-z)BT + \theta_i]\right]^2$$

To compute the averaged bit error probability given $\{\theta\}$ $p_e[\varepsilon/No/_{\{\theta\}}]$, we can write

$$p_e\left[{}^{\varepsilon/}No/_{\{\theta\}}\right] = \exp\left\{-\frac{\varepsilon}{2N_o}[1 - \mathrm{sinc}^2\ BT(y-z)]\right\} \cdot$$

$$\sum_{k=o}^{DBT-1}\ \sum_{\alpha=o}^{k}\ \sum_{m=o}^{\alpha} \binom{DBT + k - 1}{k - \alpha} \binom{\alpha}{m} \frac{1}{\alpha!} \quad .$$

$$\frac{\left[\frac{\varepsilon}{2N_o}\left(1 - \mathrm{sinc}^2\, BT(y-z)\right)\right]^{\alpha-m}}{2^{DBT+k}} < \left(\frac{\varepsilon}{DN_o}\frac{\omega^2}{2}\right)^m e^{-(\varepsilon/DN_o \frac{\omega^2}{2})} >_{\{\alpha_1\},\{\alpha_2\}} \tag{A-18}$$

where

$$\omega^2 = \sum_{i=i}^{D} \omega_i^2$$

Noting that ω given θ is a Rician random variable with 2D degrees of freedom, the expectation in equation (A-18) can be evaluated using a special result from Appendix A of Reference 4. Thus we have

$$P_e\left[\frac{\varepsilon/N_o}{\{\theta_j\}}\right] = \exp\left\{-\frac{\varepsilon}{2N}\left[1 - \mathrm{sinc}^2\, BT(y-z)\right]\right\} \cdot$$

$$\sum_{k=o}^{DBT-1} \sum_{\alpha=o}^{k} \sum_{m=o}^{\alpha} \sum_{j=o}^{m} \binom{DBT+k-1}{k-\alpha} \frac{1}{\alpha!} \binom{\alpha}{m}$$

$$\frac{\left[\frac{\varepsilon}{2N_o}\left(1 - \mathrm{sinc}^2\, BT(y-z)\right)\right]^{\alpha-m}}{2^{DBT+k}} \frac{\binom{D-1+m}{m-j} \frac{m!}{j!} \left(\frac{\nu^2}{2\beta^2}\right)^j \left(\frac{\varepsilon}{N_oD}\beta^2\right)^m}{\left(1 + \frac{\varepsilon}{N_oD}\beta^2\right)^{D+m+j}} \cdot$$

$$\exp\left\{-\frac{\varepsilon}{2N_oD}\frac{\nu^2}{1 + \frac{\varepsilon}{N_oD}\beta^2}\right\} \tag{A-19}$$

where

$$\nu^2 = \sum_{i=i}^{D} \nu_i^{\,2}$$

$$\nu_i^{\,2} = \rho^2 + \text{sinc } BT(y-z) + 2\rho \text{ sinc } BT(y-z) \cos \; ((y-z)\; BT + \theta_i)$$

so that

$$\nu^2 = D\;[\rho^2 + \text{sinc}^2\; BT(y-z)] + 2\rho \text{ sinc } BT(y-z)\; g(\{\theta\})$$

Numerical integration will be used to integrate equation (2-19) with respect to $g(\{\theta\})$ in a manner analogous to that which was used to obtain the bit error probability for the no diffuse multipath case (equation A-16).

We note that if we use equation (A-19) when, in fact $K_2(\theta) < K_1(\theta)$, then the effect of the approximation $K_2(\theta) = K_1(\theta)$ is to give a pessimistic prediction of performance and so establishes an upper bound to performance.

<u>Case 2</u> $\alpha_{ki}\;(t,\; \beta\;\frac{2\varepsilon}{TD})$ time varying over the bit interval for the special case of doppler.

In this section we assume the $\alpha_{ki}(t)$ parameters to be time varying. We further assume the diffuse portion of the signal to have been generated from essentially an infinite number of reflected paths. Although the amplitude and phase distribution of the resultant reflected signal is delayed from that of the incident signal, it is assumed that no new frequency components are created in the reflection process so that the bandwidth of the diffuse reflection is equal to the information bandwidth and is assumed to have a white spectrum.

Proceeding in a manner analogous to that in Reference 4, for D=1 we have that the bit error probability given θ can be written as

$$P_e\left[\varepsilon/N_o,\ BT,\ \rho,\ D\right] = \left[\frac{1}{2}\right]^{DBT} \sum_{j=o}^{DBT-1} \sum_{\ell=j}^{DBT-1} \frac{1}{j!\ 2^{j+\ell}} \binom{DBT+\ell-1}{\ell-j}$$

$$\cdot < \left[\frac{\eta^2}{2N_o}\right]^2 \exp - \left(\eta^2/4N_o\right) >_{\alpha_{1i},\ \alpha_{2i}} \qquad \text{(A-20)}$$

where

$$\eta^2 = \sum_{i=1}^{D} \eta_i^2$$

$$\eta_i^2 = \frac{2}{D}\int_o^T \left[\left[\cos\theta_i + \rho + \sqrt{\frac{D}{A}}\,\alpha_{1i}\ (t)\right]^2 + \left[\sin\theta_i + \sqrt{\frac{D}{A}}\,\alpha_{2i}\ (t)\right]^2\right] dt$$

Two approaches will be used to obtain solutions to equation (A-20). In the first, a chi-square approximation is used, while in the second the more rigorous Karhunen-Loeve expansion of the $\alpha_i(t)$ random processes is used.

(2) Chi-Square Approximation - 4D degrees of freedom

If we can use the chi-square approximation then η^2 is given by

$$^2 = \frac{2}{T} \sum_{i=1}^{D} \sum_{j=1}^{2} \left[\left[\frac{A}{D}\ (\cos\theta_i + \rho) + \alpha_{1ij}\right]^2 + \left[\sqrt{\frac{A}{D}}\sin\ _i + \frac{\alpha_{2ij}}{\sqrt{A}}\right]^2\right]$$

Since η^2 is chi-square distributed with 4D degrees of freedom we can evaluate the expectation in equation (A-20) by use of Appendix A of Reference 4 with the result that the bit error probability given $\{\theta_i\}$ is

$$P_e\left(\xi/N_o \,\middle|\, \{\theta_i\}\right) = \left[\frac{\exp\left\{-\left[\frac{\xi}{2N_o}\,\frac{r^2(\{\theta_i\})}{\left(1+\frac{\xi\beta^2}{2DN_o}\right)}\right]\right\}}{2^{DBT}}\right]\left[\sum_{j=o}^{DBT-1}\ \sum_{\ell=j}^{DBT-1}\ \sum_{m=o}^{j}\ \frac{1}{2^{j+\ell}}\,\cdot\right.$$

$$\left.\binom{DBT+\ell-1}{\ell-j}\left(\frac{\xi}{D}\,\frac{\beta^2}{N_o}\right)^j\binom{2D+j-1}{j-m}\,\frac{\left[r^2(\{\theta_i\})\right]^m}{m!\left[1+\frac{\xi\beta^2}{2DN_o}\right]^{2D+j+m}}\ \frac{1}{\beta^{2m}}\right] \qquad \text{(A-21)}$$

where $r^2(\{\theta_i\}) = 1+\rho^2 + \frac{2\rho}{D}\, g(\{\theta_i\})$

b) Karhunen-Loeve Expansion of $\alpha_1(t)$ and $\alpha_2(t)$*

If we expand the α parameters in a Karhunen-Loeve Expansion we have

$$\eta^2 = \sum_{i=1}^{D} \eta_i^2$$

where

$$\eta_i^2 = \int_{-1/2}^{T/2}\left[\frac{A}{\sqrt{D}}\, r(\theta_i) + \sum_{j=1}^{N} \lambda_j^{1/2}\, \alpha_{1ji}\, \phi_i\ (t)\right]^2 dt + \sum_{j=1}^{N} \lambda_j \alpha_{2ji}^2$$

---

*For a detailed derivation see Reference 4.

where $r^2(\theta_i) = 1 + \rho^2 + 2\rho \cos \theta_i$

$$\alpha_{ki}(t) = \sum_{q=1}^{N} j_{li}^{1/2} \alpha_{kji} \phi_{ji}(t)$$

$$j_{gi}^{1/2} \alpha_{kji} = \int_{-T/2}^{T/2} \alpha_{ki}(t) \phi_{ji}(t) dt$$

$$\int_{-T/2}^{T/2} \phi_i(t) \phi_j(t) dt = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\langle \alpha_{kji} \alpha_{kit} \rangle = \begin{cases} 0 & i \neq t \text{ and/or } j \neq \ell \\ \frac{\xi\beta^2}{D} & i=t \text{ and } j=\ell \end{cases}$$

Thus we need find the expectation given in equation (A-20) where η^2 is the sum of 2ND independent non-central chi-square variables. Since each of the diversity channels can be expanded in an identical manner, there will be D groups of 2N variates, each of whose members will be identically distributed, except for different non-central parameters. Therefore, we can write the bit error rate given $\{\theta_i\}$ as

$$P_e\left[\xi/n\ /\{\theta_i\}\right] = \left[\frac{1}{2}\right]^{DBT} \exp\left\{-\frac{\xi r^2(\{\theta_i\})\ C}{2N_o}\right\} \sum_{j_1=o}^{DBT-1} \sum_{j_2=o}^{j_1} \sum_{j_3=o}^{j_2}$$

$$\frac{\binom{DBT+j_1-1}{j_1-j_2}\binom{j_2}{j_3}}{2^{j_1+j_2}\ j_2!} \left(\frac{\xi r^2(\{\theta_i\})\ C}{N_o}\right)^{j_2-j_3} \Phi_{j_3}[-1/2] \qquad \text{(A-22)}$$

where $C = \left[1 - \sum_{\gamma=1}^{N} V_\gamma^2\right]$. The V_γ^2 values are given in Table A-1

$$\Phi_{j_3}[-1/2] = \begin{cases} \sum_{j_4=o}^{j_3-1} \binom{j_3-1}{j_4} B_{j_4+1}^{(-1/2)}\ \Phi_{j_3-j_4-1}^{(-1/2)} & j_3 \geq 1 \\ \Phi_o(-1/2) & j_3 = 0 \end{cases} \qquad \text{(A-23)}$$

and

$$B_j[-1/2] = \sum_{\gamma=1}^{N} \left(\frac{(j-1)!\ D\left[\frac{\lambda_\gamma \xi\ \beta^2}{2N_oD}\right]^j + j!\left[\frac{\lambda_\gamma \xi\ \beta^2}{2N_oD}\right]^{j-1}\left[\frac{\varepsilon r^2(\{\theta_i\})\ V_\gamma^2}{N_o}\right]}{\left[1+\left[\frac{\lambda_\gamma \xi\ \beta^2}{4N_oD}\right]\right]^j} \right.$$

$$\qquad \text{(A-24)}$$

$$\left. - \frac{j!\left[\frac{\xi}{2N_o}\ r^2(\{\theta_i\})\ V_\gamma^2\right]\left[\frac{\lambda_\gamma \xi\ \beta^2}{2N_oD}\right]^j}{\left[1+\frac{\lambda_\gamma \xi\ \beta^2}{4N_oD}\right]^{j+1}} \right)$$

The λ_γ values are given in Table A-2. $\Phi_o(-1/2)$ is given by

$$\Phi_o[-1/2] = \exp\left\{-\sum_{\gamma=1}^{N} \frac{\xi\, r^2(\{\theta_i\})}{2N_o} V_\gamma^2 \left[1 + \frac{\lambda_\gamma \xi \beta^2}{4N_o D}\right]^{-1}\right\} \cdot$$

$$\prod_{\gamma=1}^{N} \left(1 + \frac{\lambda_\gamma \xi \beta^2}{4N_o D}\right)^{-D} \qquad \text{(A-25)}$$

TABLE A-1

V_γ^2 for the Flat Spectrum

γ	V_γ^2
1	.894
2	0
3	.106
4	0

TABLE A-2

Eigen Values for the Flat Spectrum

γ	λ_γ (BT=2)
1	9.8405×10^{-1}
2	7.4962×10^{-1}
3	2.4359×10^{-1}
4	2.4647×10^{-2}
5	1.0661×10^{-3}
6	2.7415×10^{-5}
7	4.8028×10^{-7}
8	6.1332×10^{-9}
9	5.9714×10^{-11}

.2 Uncorrelated Direct and Reflected Path Transmission FSK Modulation

The received signal for this case has been described in Section I. As discussed earlier, during the initial portion of the direct path i^{th} bit detection (0 to τ), there is simultaneous detection of the reflected path's j^{th} bit, while for the remainder of the bit detection period (τ to T) the reflected path's k^{th} bit is the multipath noise. Assuming the i^{th} bit to be a mark, several different multipath effects can be generated depending upon the mark or space characterization of the j^{th} and k^{th} bits. However if we take advantage of symmetry considerations, the bit error probability Pe can be written as the sum of three terms. Thus

$$Pe = \frac{Pe_1}{4} + \frac{Pe_2}{4} + \frac{Pe_3}{2}$$

where we have, with respect to equation (1-1) that

$$\begin{array}{ll} i = j = k & \text{for } Pe_1 \\ i \neq j = k & \text{for } Pe_2 \\ \left.\begin{array}{l} i = j \neq k \\ i = k \neq j \end{array}\right\} & \text{for } Pe_3 \end{array}$$

We now proceed to calculate Pe_1 and Pe_2. These results will be used to bound Pe_3.

When $i = j = k$ the detection problem is identical to the correlated case, so that we may use the results derived in Section A.1 to compute Pe_1. Thus Pe_1, is given by equations A.16 and A-19 when the fading is either specular or Rician.

To compute Pe_2 where $i \neq j = k$, the energy measure x_1^2 is given by

$$x_1^2 = \sum_{i=1}^{D} \int_0^T \left[\left[\frac{A}{\sqrt{D}} \cos \theta_i + n_{1i}(t) \right]^2 + \left[\frac{A}{\sqrt{D}} \sin \theta_i + n_{2i}(t) \right]^2 \right] dt$$

where we have assumed that i=1 while χ_2^2 is

$$\chi_2^2 = \sum_{i=1}^{D} \left[\int_0^T \left[\frac{A}{\sqrt{D}}\rho + \alpha_{1i} + n_{1i}(t)\right]^2 dt + \int_c^T \left[\alpha_{2i} + n_{2i}(t)\right]^2 dt \right]$$

Proceeding once more in a manner analogous to the one followed when BT=1 (reference 4), we have that the conditional bit error probability $Pe_2\ (\xi/N_o/\{\alpha_1\},\{\alpha_2\})$ is given by

$$Pe_2\ (\xi/N_o/\{\alpha_1\},\{\alpha_2\}) = Q\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$$

$$-\frac{1}{2}\exp\left\{-\left[\frac{a^2+b^2}{4}\right]\right\} I_o\left(\frac{ab}{2}\right)$$

$$+ \exp\left\{-\left[\frac{a^2+b^2}{4}\right]\right\} \sum_{j=o}^{DBT-1} C_j \left[\left(\frac{b}{a}\right)^j - \left(\frac{a}{b}\right)^j\right] I_j\left(\frac{ab}{2}\right)$$

where

$$Q\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right) = \int_{\frac{b}{\sqrt{2}}}^{a} t \exp\left[-\frac{t^2}{2} - \frac{a^2}{4}\right] I_o\left(\frac{at}{\sqrt{2}}\right) dt. \tag{A-26}$$

$$C_j = \sum_{\ell=j}^{DBT-1} \frac{(DBT-1+\ell)!}{(DBT-1+j)!} \frac{2^{-\ell-DBT}}{(\ell-j)!}$$

$$a^2 = \frac{2\xi}{N_o D} Z^2$$

$$Z^2 = \sum_{i=1}^{D} \left[\left(\rho + \frac{\alpha_{1i}}{\frac{A}{\sqrt{D}}}\right)^2 + \left(\frac{\alpha_{2i}}{\frac{A}{\sqrt{D}}}\right)^2\right]$$

$$b^2 = \frac{2\xi}{N_o}$$

The expectation of equation (A-26) for the general case is difficult to obtain. However, it can be solved for two important special cases. In the first, we assume the reflection to be totally specular, while in the second we assume the reflection to be completely diffuse.

1) Specular reflection $\{\alpha_1\} = \{\alpha_2\} = 0$

$$Pe_2(\varepsilon/N_o \mid \{\alpha_1\},\{\alpha_2\}) = Pe_2(\xi/N_o) \tag{A-27}$$

where

$$a^2 = \frac{2\rho}{N_o}\rho^2$$

2) Diffuse reflection $\rho^2 = 0$

In this case we have that a is a Rayleigh variate with 2D degrees of freedom. In Appendix B, the expectation of the conditional bit error with respect to the variate a, is carried out with the result that

$$Pe_2(\xi/N_o) = \left[\exp\left(-\frac{b^2}{2[\sigma^2+2]}\right)\frac{2^{D-1}}{[\sigma^2+2]^{D-1}}\left[\sum_{k=o}^{D-1}\frac{\binom{D-1}{k}}{k!}\frac{(\sigma^2)^k}{2^k}\left[\sum_{\ell=o}^{k} d_\ell\left(\frac{b^2}{2[\sigma^2+2]}\right)^{k-\ell}\right.\right.\right.$$

$$\left.\left.-\frac{b^{2k}}{[\sigma^2+2]^{k+1}}\frac{1}{2^k}\right] + \frac{2}{\sigma^2+2}\sum_{j=o}^{DBT-1}\frac{C_j}{j!}T_j\right] \tag{A-28}$$

where

$$d_\ell = \begin{cases} 1 & \ell=o \\ (k-\ell+1)d_{\ell-1} & \ell\neq o \end{cases} \tag{A-29}$$

$$T_j = \left(\frac{b}{2}\right)^{2j} {}_1F_1\left(j-D+1\ ;\ j+1;\ -\frac{b^2\sigma^2}{4(2+\sigma^2)}\right) -$$

$$\frac{(D+j-1)!}{(D-1)!}\left(\frac{\sigma^2}{2+\sigma^2}\right)^j {}_1F_1\left(1-D\ ;\ j+1\ ;\ -\frac{b^2\sigma^2}{4(2+\sigma^2)}\right) \tag{A-30}$$

${}_1F_1\ (a;b;-z)$ is the Confluent hypergeometric function defined in integral form by given by

$$_1F_1(a;b;-z) = \frac{\Gamma(b)}{\Gamma(a)\ \Gamma(b-a)} \int_0^1 e^{-zt}\ t^{a-1}(1-t)^{b-a-1}\ dt$$

$$\sigma^2 = \frac{2\xi}{N_o}\ \frac{\beta^2}{D}$$

For the special case $j \leq D$ and in particular when $BT=1$, T_j reduces to

$$T_j = \frac{D!}{(j+D)!}\ b^{2j}\ L^j_{D-j}(-z) - \frac{(D+j+1)!\ (D+j)!}{(D-1)!\ (D+2j+1)!}\ L^j_D(-z) \tag{A-31}$$

where $L^j_D\ (-z)$ is the generalized Laguerre polynomial given by

$$L^j_D\ (-z) = \sum_{m=o}^{D} \binom{j+D}{j-m} \frac{(z)^m}{m!}$$

To determine $Pe_3(\xi/N_o/\theta)$ and Pe_3 we assume that reasonable bounds can be obtained in the following manner:

$$Pe_3\ \text{(lower bound)} = \min(Pe_1,\ Pe_2)$$

$$Pe_3\ \text{(upper bound)} = \max(Pe_1,\ Pe_2)$$

APPENDIX B

Derivation of Bit Error when the Reflected Transmission Is Uncorrelated with the Direct Transmission when $\rho^2 = 0$.

For convenience we repeat equation A-26 of the text in the following form

$$Pe_2(\xi/N_o/a) = {}_1Pe_2(\xi/N_o/a) - {}_2Pe_2(\xi/N_o/a) + {}_3Pe_2(\xi/N_o/a) \tag{B-1}$$

where

$$ {}_1Pe_2(\xi/N_o/a) = Q\left(\frac{2}{\sqrt{2}} , \frac{b}{\sqrt{2}}\right)$$

$$ {}_2Pe_2(\xi/N_o/a) = \frac{1}{2} \exp\left(-\left[\frac{a^2 + b^2}{4}\right]\right) I_o\left(\frac{ab}{2}\right)$$

$$ {}_3Pe_2(\xi/N_o/a) = \exp\left(-\left[\frac{a^2 + b^2}{4}\right]\right) \sum_{j=1}^{DBT-1} C_j \left[\left(\frac{b}{a}\right)^j - \left(\frac{a}{b}\right)^j\right] I_j\left(\frac{ab}{2}\right)$$

As noted in the text when $p^2=o$, the parameter a is a Rayleigh variate with 2D degrees of freedom. Thus we have that the bit error probability is given by

$$Pe_2(\xi/N_o) = \int_o^\infty Pe_2(\xi/N_o/a)\ p(a)da \tag{B-2}$$

where

$$p(a) = \frac{a^{2D-1}}{2^{D-1}\sigma^{2D}(D-1)!} \quad \exp(-a^2/2\sigma^2) \tag{B-3}$$

Substituting equation (B-1) for $Pe_2(\xi/N_o/a)$

we have

$$Pe_2(\xi/N_o) = \int_o^\infty {}_1Pe_2(\xi/N_o/a)p(a)da + \int_o^\infty {}_2Pe_2(\xi/N_o/a)p(a)da$$

$$+ \int_o^\infty {}_3Pe_2(\xi/N_o/a)p(a)da \tag{B-4}$$

Each of the integrals on the right hand side of equation (B-4) is evaluated separately however for each we make use of the following result*

$$\Gamma(\mu+1) \int_0^\infty J_\mu(ct)\, e^{-\alpha^2 t^2} t^{\rho-1} dt = \left[\exp(-1/4\; c^2\alpha^{-2})\right] \tfrac{1}{2}\alpha^{-\rho}$$
$$\Gamma\left(\tfrac{1}{2}\mu + \tfrac{1}{2}\rho\right)\left(\tfrac{1}{2}\;\tfrac{c}{\alpha}\right)^\mu \;{}_1F_1\left(\tfrac{1}{2}\mu - \tfrac{1}{2}\rho + 1; \mu+1; \tfrac{1}{4}c^2\alpha^{-2}\right) \tag{B-5}$$

$$\mathrm{Re}\ \alpha^2 > 0 \quad \mathrm{Re}(\mu+\rho) > 0$$

where $J_\mu(ct)$ is a Bessel function of the first kind. Note the identity that

$$I_\eta(\chi) = i^{-\eta} J_\eta(i\chi)$$

where $I_\eta(\chi)$ is a modified Bessel function of the first kind.

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*"Higher Transcendental Functions", Volume 2 of the Bateman Manuscript Project, p. 50, equation (22) McGraw-Hill Book Company, Inc., 1953.

To evaluate $\int_0^\infty {}_1Pe_2(\xi/N_o/a)p(a)da = {}_1Pe_2(\xi/N_o)$

${}_1Pe_2(\xi/N_o)$ can be written as

$$\int_0^\infty Q\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right) p(a)da = \int_0^\infty \int_{\frac{b}{\sqrt{2}}}^\infty t\ e^{-\frac{t^2+\frac{a^2}{2}}{2}} I_o\left(\frac{at}{\sqrt{2}}\right)dt\ \ p(a)da$$

Interchanging order of integration and using equation (B-5) leads to

$${}_1Pe_2(\xi/N_o) = \int_{\frac{b}{\sqrt{2}}}^\infty t\ e^{-\frac{t^2}{2}(\sigma^2+2)} \frac{2^D}{(2+\sigma^2)^D}\ {}_1F_1\left(1-D;1;\ \frac{-t^2\sigma^2}{2(\sigma^2+2)}\right) dt \tag{B-6}$$

But ${}_1F_1\left(1-D;1;\ \frac{-t^2\sigma^2}{2(\sigma^2+2)}\right)$ is equal to the following Laguerre polynomial*

$${}_1F_1\left(1-D;1;\ \frac{-t^2\sigma^2}{2(\sigma^2+2)}\right) = L_{D-1}\left(\frac{t^2\sigma^2}{2(\sigma^2+2)}\right) \tag{B-7}$$

where

$$L_{D-1}(-X) = \sum_{m=o}^{D-1} \binom{D-1}{m} \frac{(X)^m}{m!} \tag{B-8}$$

Substituting equations (B-7) and (B-8) into equation (B-6), interchanging the order of summation and integration and rearranging terms leads to

$${}_1Pe_2(\xi/N_o) = \frac{2^{D-1}}{(2+\sigma^2)^{D-1}} \sum_{m=o}^{D-1} \left(\frac{\sigma^2}{2}\right)^m \binom{D-1}{m} \frac{1}{m} \int_{\frac{b^2}{2(\sigma^2+2)}}^\infty u^m e^{-u} du \tag{B-9}$$

*"Handbook of Mathematical Functions", p. 509 relation 13.6.9, National Bureau of Standards Applied Mathematical Series 55, June 1964.

The integral on the right hand side of equation (B-9) can be evaluated in the following manner

Assume $$\int_a^\infty u^m \exp -u \, du = e^{-a} \sum_{i=o}^{m} k_i a^{m-i} \qquad \text{(B-10)}$$

where $$k_i = \begin{cases} 1 & i=o \\ (m-i+1)k_{i-1} & i \neq o \end{cases}$$

Proof by induction:

It is easy to show true for m=0, 1, and 2 assume true for m=n then integrating by parts we have

$$\int_a^\infty u^{n+1} \exp -u \, du = -u^{n+1} \left[\exp -u\right]\Big|_a^\infty + (n+1) \int_a^\infty u^n \exp -u \, du \qquad \text{(B-11)}$$

the right hand side of equation (B-11) reduces to

$$\int_a^\infty u^{n+1} \exp -u \, du = \left[a^{n+1} + \sum_{i=o}^{n} k_i \; a^{n-1}(n+1)\right] e^{-a}$$

$$\text{but } a^{n+1} + \sum_{i=o}^{n} k_i \; a^{n-1} n+1 = \sum_{i=o}^{n+1} k_i \; a^{n-1}$$

$\therefore$equation (B-10) is true

Thus we substitute equation (B-10) into equation (9) with the result that

$$_1P_2(\xi/N_o) = \frac{\left[e^{-\frac{b^2}{2(\sigma^2+2)}}\right] 2^{D-1}}{(2+\sigma^2)^{D-1}} \sum_{m=o}^{D-1} \left(\frac{\sigma^2}{2}\right)^m \binom{D-1}{m} \frac{1}{m!} \sum_{i=o}^{m} k_i \left(\frac{b^2}{2(\sigma^2+2)}\right)^{m-i}$$

(B-11)

In a similar fashion we can evaluate ${}_2P_2(\xi/N_o)$ and ${}_3P_2(\xi/N_o)$ with the result that $P_2(\xi/N_o)$ is given by equation (2-28) of the text.