SUBJECT: A Linear Model of Atmospheric   DATE: January 31, 1969
Circulation - Case 103-7

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TECHNICAL MEMORANDUM

I. Introduction

In a recent investigation Gale, Liwshitz, and Sinclair[1] examine the available radar, microwave, Mariner V, and Venera 4 data pertaining to the atmosphere of Venus. On the basis of this evaluation they note that the best fit to all observations obtains when the lower atmosphere of Venus, from about 10 km altitude down to the mean surface level, is essentially isothermal or possesses at most a small, subadiabatic temperature lapse rate.

The purpose of the present investigation is to construct a mathematical model of a two-dimensional circulation pattern compatible with the above proposed isothermal regime between the surface and a given altitude level. The term "isothermal regime" will, in the present paper, mean a layer in which the temperature does not vary with altitude below some specified height; this layer, however, will display a horizontal temperature variation between the subsolar and anti-solar points. Such a regime could prevail if the absorption of radiation had a sharp maximum at some altitude inside the atmosphere. This is the case, for example, in the Earth's atmosphere where the presence of ozone causes an almost total absorption of ultraviolet radiation. Strong localized absorption could also be caused by the presence of a cloud layer. We do not assume, however, that all of the incoming radiation energy is absorbed in the layer where the opacity peaks, but allow some of it to penetrate to the surface where it is absorbed and, like in the above described layer, supports a temperature distribution that decreases monotonically from the subsolar to the anti-solar point. This horizontal temperature gradient is the driving force for the atmospheric circulation.

Since by assumption the atmosphere between the absorbing layer and the ground is isothermal, its optical properties have little effect on the temperature which is governed mainly by conduction and convection. The condition for balance of radiative energy fluxes may be used to determine the distribution of the absorbing properties of the atmosphere. This will depend on the postulated vertical profile of the atmosphere undisturbed by the circulation. We defer the discussion of these considerations to a later report and proceed here to the mathematical formulation of the fluid dynamics of the problem.

II. Formulation of the Boundary Value Problem

If we assume that the variation of surface temperature is described by night cooling alone, our model consists of flow in any vertical plane passing through the subsolar point at x=0 and the anti-solar point at x=πR, where R is the radius of the planet; the vertical coordinate is denoted by z. The analysis can also determine circulation in a meridional plane between the equator and the pole at $x=\frac{1}{2}\pi R$ if it is postulated that the polar cooling is the dominant effect; the change of parameter necessary to affect this latter interpretation of the results will be indicated in the text.

The governing system of differential equations is obtained by applying two approximations to the usual equations of fluid dynamics, expressing the conservation of mass, momentum, and heat. These approximations are:

1. The Boussinesq approximation for compressible fluids as derived by Spiegel and Veronis[2]; it implies that changes of density are neglected except in the buoyancy force, and obtained subsequently from the temperature variation alone.

2. The neglect of convective transports of momentum and heat.

This latter simplification is suggested by the following reasoning. The driving force for the atmospheric circulation is the temperature variation, T_s, between the subsolar and anti-solar points (or between the pole and points on the equator) which is very small in comparison to the mean temperature, T_o. Therefore, we may think of expanding the solution in powers of T_s/T_o; in that expansion the convective transport terms, being proportional to the square of the temperature perturbation, become negligible. Of course, such a systematic procedure would be only formal and by itself does not guarantee that the convective terms will be smaller than the viscous or conduction terms; for with a specified finite driving temperature perturbation the coefficients of viscosity or conductivity, when properly nondimensionalized, may be smaller than the given temperature perturbation, thus reversing the relative magnitude of terms deduced from the limiting process $\frac{T_s}{T_o} \to 0$. Since, in the present problem, we do not have a reference velocity, we cannot properly estimate a priori the relative magnitudes of convection and conduction. Hence, we use the above argument only as a motivation. The consistency of these approximations will be verified a posteriori, in Sec. V.

We introduce the temperature perturbation $\tau(x,z)$ by

$$t = T_o - \gamma_o z + \tau(x,z) \tag{1}$$

where t is the total temperature and γ_o the actual lapse rate. Then, with the above mentioned two approximations, the following four equations describe our problem:

a. Heat conduction equation for the temperature perturbation with the linearized convective term.

$$\kappa \nabla^2 \tau = \gamma w \tag{2}$$

where κ is the eddy heat conductivity, w is the vertical velocity component, and $\gamma > o$ is the difference between the dry adiabatic Γ and the actual γ_o lapse rates, i.e. $\gamma = \Gamma - \gamma_o$.

b. Two equations of motion for a slow, viscous flow with buoyancy force:

$$\rho \nu \nabla^2 u = p_x \tag{3}$$

$$\rho \nu \nabla^2 w = p_z - \rho g a \tau \tag{4}$$

where ρ is the density, p-pressure, ν-eddy kinematic viscosity, g-gravitational acceleration, a-coefficient of thermal expansion, and u-the horizontal velocity.

c. The equation of continuity for incompressible flow:

$$u_x + w_z = 0. \tag{5}$$

In equations (2), (3), (4), and (5) u, w and p are considered to be small perturbations about the state of rest.

To complete the formulation of the problem we must now discuss the boundary conditions. Subscripts a and b will distinguish solutions that apply above and below the absorbing layer, located at z=h. With the assumption that the temperatures at z=0 and z=h are specified, so that the radiative energy

balance prevails, we seek a solution that satisfies the following conditions:

    a.   at z=0 the temperature is prescribed and both velocity components vanish, as required for a viscous flow,

    b.   at z=h the temperature is prescribed and all variables are continuous functions of z,

    c.   all perturbations tend to zero as z increases beyond all bounds.

Expressed in mathematical terms these requirements are:

    a.   at z=0

$$\tau_b\,(x,o) = T_s\,\cos\frac{x}{R}$$

$$u_b\,(x,0) = 0 \tag{6}$$

$$w_b\,(x,0) = 0;$$

    b.   at z=h

$$\tau_b\,(x,h) = \tau_a\,(x,h) = \theta\,T_s\,\cos\frac{x}{R}$$

$$u_b\,(x,h) = u_a\,(x,h)$$

$$w_b\,(x,h) = w_a\,(x,h) = 0 \tag{7}$$

$$p_b\,(x,h) = p_a\,(x,h);$$

    c.   as $z \to \infty$

$$\tau_a\,(x,z) \to 0$$

$$u_a\,(x,z) \to 0 \tag{8}$$

$$w_a\,(x,z) \to 0.$$

In these equations T_s is the amplitude of the temperature variation, which, as mentioned previously, is the perturbation parameter

of the problem, and θ is an arbitrary constant. By specifying
which fractions of the solar flux are absorbed at z=0 and z=h,
θ characterizes, in some approximate sense, the optical properties
of the atmosphere.

All except one of the above conditions are dictated by
the differential equations and the assumption of radiative energy
balance which describe the model. The exception is the vanishing
of the vertical velocity at z=h; this requirement is suggested by
the hypothesis of a stable and stationary absorbing layer.

The specification of the cosine temperature variation
between the subsolar and anti-solar points is not restrictive at
all. Any general temperature distribution may be represented as
a Fourier series and, since the governing system of equations is
linear, the solution is determined for each mode separately. We
present the calculations for the first mode; since it corresponds
to the monotone temperature variation along the boundaries it is
sufficient to characterize the essential dominant features of the
problem.

The origin of the above described formulation and the
nature of the approximations used in the analysis will be dis-
cussed in more detail in Sec. V.

III. Representation of the Solution

The system of Equations (2), (3), (4), and (5) is
linear with constant coefficients and therefore it is natural
to try the following form for the solution:

$$\tau = Te^{\alpha x}e^{\beta z}$$
$$u = Ue^{\alpha x}e^{\beta z} \tag{9}$$
$$w = We^{\alpha x}e^{\beta z}$$

Then from Eq. (3)

$$p = \frac{\rho \nu}{\alpha} U(\alpha^2 + \beta^2)e^{\alpha x}e^{\beta z} \tag{10}$$

where a constant of integration (which may depend on z) is not
needed because it can be incorporated into the unperturbed
hydrostatic pressure.

Substituting (9) and (10) into the remaining three
equations we obtain:

$$
\begin{bmatrix}
\frac{\kappa}{\gamma}(\alpha^2 + \beta^2) & 0 & -1 \\[2em]
\frac{ga}{\nu} & -\frac{\beta}{\alpha}(\alpha^2 + \beta^2) & (\alpha^2 + \beta^2) \\[2em]
0 & \frac{\alpha}{\beta} & 1
\end{bmatrix}
\begin{bmatrix}
T \\[2em] U \\[2em] W
\end{bmatrix}
=
\begin{bmatrix}
0 \\[2em] 0 \\[2em] 0
\end{bmatrix}
\qquad (11)
$$

Existence of a nontrivial solution of the above system is assured by the vanishing of the determinant, which requires:

$$
(\beta^2 + \alpha^2)^3 + C\alpha^2 = 0 \qquad\qquad (12)
$$

where the constant C, an analogue to the Rayleigh number, is given by:

$$
C = \frac{ag\gamma}{\nu\kappa} \qquad\qquad (13)
$$

For our problem the Rayleigh number is zero because we seek the solution in an isothermal layer.

Since α will be determined from the boundary conditions, (12) is an equation for β.

Subject to (12), equation (11) may be solved to yield:

$$
U = -\frac{\beta\kappa}{\alpha\gamma}(\alpha^2 + \beta^2)T \qquad\qquad (14)
$$

$$
W = \frac{\kappa}{\gamma}(\alpha^2 + \beta^2)T \qquad\qquad (15)
$$

In this way all variables are determined in terms of temperature and thus the solution is expressed explicitly in terms of the boundary conditions and other parameters of the problem.

For the planet Venus we use the following values of the parameters obtained from the work of Ohring, Tang, and Mariano[3]:

$$a = 2 \times 10^{-3} \text{ per degree}$$

$$g = 8.8 \text{ m/sec}^2$$

$$\gamma = 2 \times 10^{-3} \text{ deg/m}$$

$$\nu = \kappa = 10^3 \text{ m}^2/\text{sec}$$

$$R = 6 \times 10^3 \text{ km.}$$

Not all of these are firm numbers but they are representative of the values quoted in the literature. Eddy transport coefficients are used because our analysis does not aspire to yield local wind velocities but to describe only the mean motions in the atmosphere.

For the fundamental mode compatible with the boundary conditions (6) and (7), corresponding to night cooling:

$$\alpha = \frac{i}{R} .$$

[If equator to pole circulation is considered instead the argument of the temperature variation in (6) and (7) is $\frac{2x}{R}$, and $\alpha = \frac{2i}{R}$; this is the only change required in the treatment.]

With the above listed values of the parameters:

$$c^{1/3}\alpha^{2/3} \approx 10^{-2} \text{ km}^{-2}$$

$$\alpha^2 \approx 2.78 \times 10^{-8} \text{ km}^{-2}$$

and therefore in (12) α^2 may be neglected in comparison with $c^{1/3}\alpha^{2/3}$ to yield for the six roots of β

$$\beta_j = be^{\pi i j/3} \tag{16}$$

where $j = 1, 2, \ldots 6$

$$b = c^{1/6} R^{-1/3} \approx 10^{-1} \text{ km}^{-1}$$

The solution, with the same approximations as in (16), is given by:

$$\tau = \sum_{j=1}^{6} T_j e^{\beta_j z} e^{\alpha x} \tag{17}$$

$$w = \frac{\kappa}{\gamma} \sum_{j=1}^{6} T_j \beta_j^2 e^{\beta_j z} e^{\alpha x} \tag{18}$$

$$u = -\frac{\kappa}{\gamma \alpha} \sum_{j=1}^{6} T_j \beta_j^3 e^{\beta_j z} e^{\alpha x} \tag{19}$$

$$p = -\frac{\rho \nu \kappa}{\gamma \alpha^2} \sum_{j=1}^{6} T_j \beta_j^5 e^{\beta_j z} e^{\alpha x} \tag{20}$$

Condition (8) requires $T_j^a = 0$ for those j for which $Re\beta_j > 0$; substitution of expressions (17) to (20) into the boundary conditions (6) and (7) results in nine linear equations for the remaining T_j's. The matrix of this system is too large for algebraic treatment; however, it depends on a single parameter h and can easily be inverted numerically for any particular value of this parameter.

For $h = 10km$ (i.e. for $bh = 1$) the result, in terms of the two parameters, T_s and θ, which specify the boundary conditions, is:

$$T_1^b = T_s \left[(.030 - .111\,\theta) + (.090 - .316\,\theta)i \right]$$

$$T_2^b = T_s \left[(.285 + .017\,\theta) + (.053 - .480\,\theta)i \right]$$

$$T_3^b = T_s\,(.440 + .223\,\theta)$$

$$T_4^b = T_s \left[(.285 + .017\,\theta) - (.053 - .480\,\theta)i \right] \tag{21}$$

$$T_5^b = T_s \left[(.030 - .111\,\theta) - (.090 - .316\,\theta)i \right]$$

$$T_6^b = -T_s\,(.070 + .033\,\theta)$$

$$T_2^a = -T_s \left[(.024 + .127\,\theta) + (.004 + .510\,\theta)i \right]$$

$$T_3^a = T_s \ (.040 + 1.71 \ \theta)$$

$$T_4^a = -T_s\left[(.024 + .127 \ \theta) - (.004 + .510 \ \theta)i\right]$$

IV. Presentation of the Results

Substituting (21) into (17) through (20) produces a solution of the problem which satisfies the prescribed boundary conditions. The physically significant real parts are shown in Figure 1 for θ = 1.5; this value of θ is chosen so that for $\gamma_o h$ = 20 and T_s = 40° the temperatures at z=0 and z=h are equal. As can easily be seen from Eqs. (1) and (7) that value of θ is given in general by:

$$\theta = 1 + \frac{\gamma_o h}{T_s} \qquad\qquad (22)$$

The most significant property of the solution shown in Figure 1 is the fact that, even in presence of nontrivial circulation, the temperature perturbation is a linear function of z and therefore, when θ satisfies Eq. (22), the total temperature will be constant between z = 0 and z = h. Similarly, if θ is given by

$$\theta = - \frac{\gamma h}{T_s} \qquad\qquad (23)$$

the atmosphere will possess an adiabatic lapse rate.

The linear variation of temperature perturbation with altitude does not appear to depend on the special choice of θ; it has been verified also for θ = 1.

The circulation follows a three cell pattern in which a concentrated, relatively high velocity jet at z=h (10 km) carries the heated gases away from the subsolar point. After cooling, these gases return to the subsolar point in a nearly symmetrical pattern giving rise to the first two cells. The third one, extending from approximately 2h to infinity, corresponds to the circulation usually found in atmospheres that are heated along the bottom only. The occurence of the second cell between z=h and z=2h, in which the flow is downward at the subsolar point, can be explained most conveniently by considering the evolution of the above flow pattern. Let us imagine that the boundary conditions, specifying

the temperatures at $z=0$ and $z=h$, are turned on in an atmosphere at rest with the unperturbed lapse rate γ_o. Because of the prescribed vanishing of the vertical velocity at $z=h$, the flow will initially try to follow a two cell pattern with the same sense of rotation in both cells for $\theta > 0$. However, such a configuration generates a concentrated shear layer at $z=h$ which, in time, thickens into a full size cell between $z=h$ and $z=2h$. In that cell the circulation is in the opposite direction so as to minimize the shearing stresses inside the fluid.

The above described general pattern is illustrated in Figure 1 where, in addition, the magnitude of the horizontal velocity component is presented in terms of the temperature perturbation, i.e. the value read off from the graph, when multiplied by $T_s°C$, yields the wind velocity in $\frac{km}{hr}$. The same holds for Figure 2 which represents the variation in the vertical velocity at the anti-solar point.

For a typical value of $T_s = 40°C$ Figure 1 yields about $40 \frac{km}{hr}$ as a representative vertical average of the horizontal wind velocity at the point midway between the subsolar and anti-solar points. Since this velocity decreases in both directions away from that point, 25 or 30 $\frac{km}{hr}$ would be an appropriate average to consider. This value, however, is of the same order of magnitude as the velocity of the subsolar point along the surface of the planet due to rotation relative to the Sun which is 13 $\frac{km}{hr}$.* Therefore, a steady, two-dimensional treatment may not be truly representative of the actual conditions. We have presently completed an analysis of the three-dimensional, time dependent circulation in the atmosphere of Venus, using the approach presented there. These results will be described separately in the near future.

V. Discussion of the Analysis

In the present section we verify the consistency of the approximations, using the explicit form of the solution, and advance some plausibility arguments for the validity of our model. The background investigations which led to the present analysis will also be briefly discussed.

When the neglected convective acceleration terms are computed for the conditions specified in our problem, they are found to be only about 1% of the viscous terms retained in equations (3) and (4). This is much smaller than one would expect

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* This fact has been pointed out to the author by G. A. Briggs.

from the expansion in powers $\frac{T_s}{T_o}$ because such expansion predicts
that the ratio of the neglected terms to those retained should be on
the order of $\frac{T_s}{T_o}$, which, for values used in our computations --

$T_s = 40°$ and $T_o = 600°$ -- is almost 10%. The reason for this for-
tuitous circumstance may be deduced from the observation that the
vertical velocity, w, is only about one thousandth of the hori-
zontal velocity, u, (see Figs. 1 and 2) while the horizontal scale
length, R, is 600 times as large as the vertical one, h. Therefore,
in the convective transport process, the relatively large velocity
u convects only a small derivative in the x-direction, and the
relatively large derivative in the vertical, z, direction is con-
vected only with the small velcity w. This fact, which indicates
that the velocity vector and the gradient operator are nearly per-
pendicular, will be discussed below in more detail. The retained
viscous momentum diffusion terms are always dominated by gradients
in the z direction; this is reflected in the approximation leading
to Eq. (16).

        The largest error incurred from the neglect of convective
transport mechanism occurs in Eq. (2) and amounts to about 10% -
15%, depending on the value of γ. Nevertheless, the fact that the
temperature perturbation, τ, is very nearly a linear function of
z (see Fig. 1) indicates that the solution is controlled mainly
by conduction.

        We now would like to mention some plausibility arguments
which suggest that our analysis, neglecting convective terms, is
not fortuitously self-consistent, but may be a valid description
of the fluid model, to which the analysis is applied. These arguments
indicate, moreover, that the orthogonality of velocity and gradient
vectors does not solely depend on the smallness of the ratio h/R.

        The typical velocities occuring within our model result
in a Reynolds number, $Re = \frac{uh}{\nu}$, of about 200. Therefore, one is
justified in questioning the applicability of Eqs. (3) and (4) to
the present situation since these equations are usually thought of
as a good approximation in cases when $Re \ll 1$. However, there are
other conditions which may render them valid.

        These conditions are illustrated by the following
considerations. The convective acceleration terms neglected
in equations (3) and (4) equal the scalar product of the velocity
with the gradient of the velocity. Now, consider a flow with
rotational symmetry, i.e., where the streamlines are circular.
Under these conditions the velocity and its gradient are always
perpendicular and the convective acceleration vanishes identically

reducing the Navier-Stokes equations to the present linear Stokes approximation. Thus the reason for the validity of this approximation - sometimes known as the Stokes linearization - is purely kinematical and independent of the Reynolds number. A systematic study of all such flows has been carried out by Weinbaum and O'Brien[4].

In our case, of course, the flow pattern is not composed of circular cells, but we may surmise that any configuration, in which the fluid travels along closed, everywhere convex, streamlines, approximates a circular pattern in some sense. This expectation is supported to some extent by the work of Weiss and his collaborators[5], who solved numerically the exact Navier-Stokes equations and the Stokes approximation for a triangular circulation cell and found that the solutions began to differ only for Reynolds numbers greater than about 300 when the flow began to exhibit an inviscid behavior away from the boundaries.

Equations (2), (3), (4) and (5) constitute an adaptation of the classical Boussinesq approximation to the compressible flows derived by Spiegel and Veronis[2]. They were previously employed by Ohring, Tang, and Mariano[3] to describe the atmosphere of Venus, however with simpler boundary conditions prescribing the temperature only along the planet's surface. Therefore, the solution of Ohring and coworkers corresponds qualitatively to the behavior of our model above the altitude $z = 20$ km. Since they did not encounter the presence of shallow circulation cells they had no opportunity to use results similar to those in References 4 and 5 to establish the validity of their model. Instead, they attempted to verify the approximations by solving the nonlinear version of Equations (2), (3) and (4) but failed to complete the task[3]. Our treatment is also more general because it does not rely on the possibility of reducing the system to a single high order differential equation.

Goody and Robinson[6] analyzed the consequences of the absorption of radiative flux at the top of the atmosphere with the simplifying assumption that no heat flux reaches the bottom of the atmosphere. They also linearized the problem but did so about the adiabatic lapse rate and not an arbitrary one, as we have done.

In their work, Goody and Robinson used, instead of the Stokes approximation, the Prandtl boundary layer equations. These equations provide an excellent description of the flow in thin layers along nearly straight walls or slip surfaces but are incapable of describing the turning of such layers in corners where the flow direction changes from horizontal to vertical. The present writer attempted to develop, within the framework of Prandtl's boundary layer theory, a method which would allow to compute

the turning of the flow through an arbitrary angle in a relatively sharp corner. This work[7] has not yet been completed, and the problem remains outstanding. Thus, Goody and Robinson's approach cannot give a complete picture of the flow pattern.

In addition, simplifications leading to the boundary layer equations imply a definite scaling, namely stretching of one coordinate with $\sqrt{Re}$ and the magnification of the corresponding velocity component by the same factor. Therefore, subsequent scalings performed by Goody and Robinson should not lead to any different approximation. This fact is demonstrated, for example, in the case when the authors discard terms of the order $\epsilon^{1/12}$, where ϵ is their small parameter. When, however, $\epsilon = .15$, -- a reasonably small value suggested by the authors --, $\epsilon^{1/12} = .85$ and is no longer negligible in comparison with unity. Thus their approximations are not always justified and their conclusions, therefore, unreliable.

Goody and Robinson's formulation has also been employed by Stone[8] to analyze a Hadley regime of circulation and his results differ from those of the former authors. It should be pointed out, however, that the solution of the governing system of equations is not determined either in Ref. 6 or in Ref. 8; only certain scaling properties of the equations are discussed from which the authors attempt to infer the flow pattern.

VI. Concluding Remarks

The analysis presented here indicates that, under the assumptions introduced, substantial large scale circulation is possible in the presence of an isothermal layer above the surface of Venus. These assumptions imply that the distribution of absorptive properties in the atmosphere permits representation of the radiative energy input by concentrated heat sources along the surface of the planet and at some prescribed altitude level at which, in addition, the vertical convective velocity of the atmospheric gases is required to vanish. If supplementary investigations of the atmospheric heat balance substantiate these assumptions, the present results will provide a consistent picture of certain dynamical processes in the atmosphere of Venus that should be useful in the description of the planet's atmospheric environment.

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Attachments
  Figures 1 & 2

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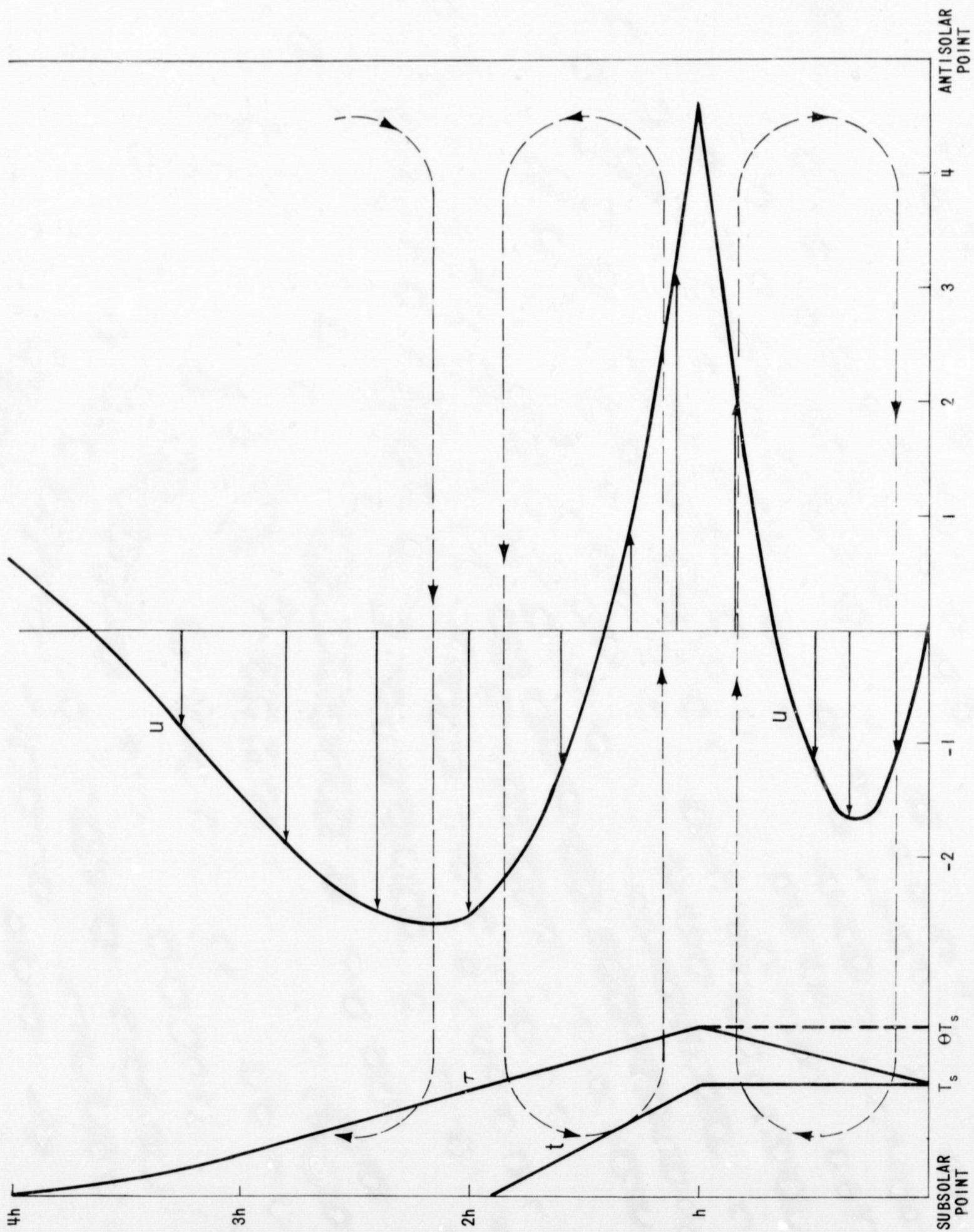
FIGURE I - CIRCULATION PATTERN IN A VERTICAL PLANE. TEMPERATURE (t) AND
TEMPERATURE PERTURBATION (τ) SHOWN AT THE SUBSOLAR POINT;
HORIZONTAL VELOCITY DISTRIBUTION (u) SHOWN MIDWAY BETWEEN
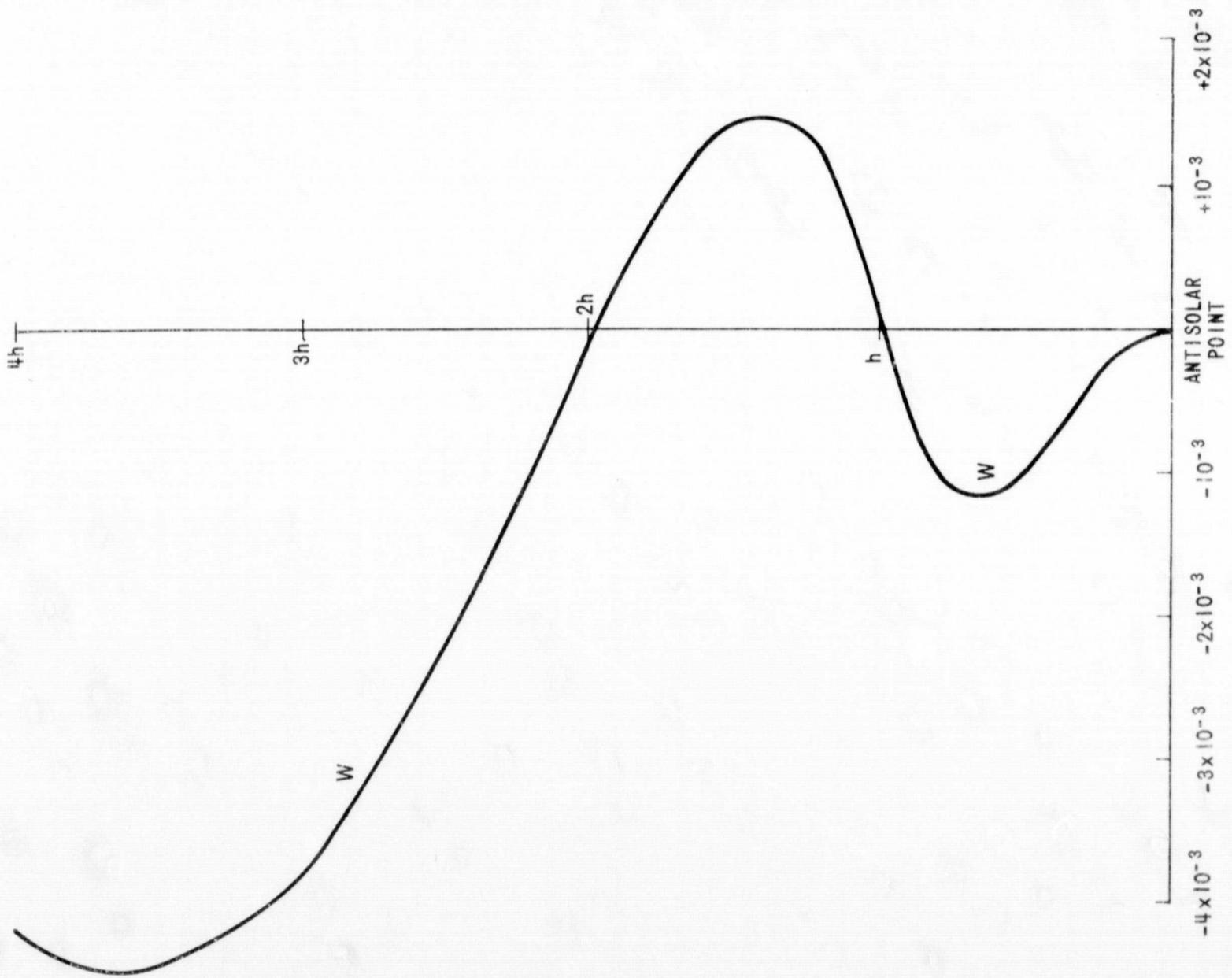SUBSOLAR AND ANTISOLAR POINTS

FIGURE 2 – DISTRIBUTION OF VERTICAL VELOCITY (w) AT THE ANTISOLAR POINT