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COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Problems in Attitude Control of Artificial "G" Space Stations with Mass Unbalance

TM- 69-1022-12

DATE- December 23, 1969

FILING CASE NO(S)- 730

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FILING SUBJECT(S) (ASSIGNED BY AUTHOR(S))- Space Stations, Attitude Control

ABSTRACT

An investigation is made into the problem of attitude control of a mass-unbalanced, axially-symmetrical space station consisting of a rotating artificial gravity section and a despun section. It is shown that there exist coning motions of the satellite with amplitudes that can be minimized by several design considerations: Design should be such that the axial moment of inertia of the spinning section is as much as possible greater than the transverse moment of inertia of the space station. This implies that the despun section is much smaller than the spinning section and that the space station presents a low, flat silhouette when viewed from a direction normal to the spin axis. Also, crew compartments and compartments to and from which equipment is to be shifted should be situated as close as possible to the space station mass center.

In addition, for very large satellites, it is found impractical to use control moment gyros (CMGs), even with implementation of these design features, to reduce the coning motion amplitudes to within allowable limits for many experiments which will be carried in future space stations. To alleviate the mass unbalance problem, a means of facilitating an active mass balance system which has been suggested by NASA is discussed and a lighter, simpler, passive system is proposed for further study.

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TECHNICAL MEMORANDUM

1.0 INTRODUCTION

NASA is now considering for the late 1970's or early 1980's a large manned space base which will provide artificial gravity and zero gravity environments simultaneously in separate sections (Ref.[1]). Artificial gravity modes of operation are also being considered for earlier manned orbital programs leading up to Space Base.

For Space Base and the earlier programs, the method being considered for providing artificial gravity is the spinning of the artificial "G" compartments so that objects within experience a radial acceleration towards the spin axis. However, these accelerations are imparted by forces of the satellite on the objects. The torques about the satellite mass center of the equal and opposite reaction forces of the objects on the satellite, if not balanced, are very much greater than gravity gradient, aerodynamic, and other environmental torques and might be expected to have a correspondingly greater effect on attitude motions. Because future space stations will provide a significant degree of flexibility of movement for crew members and equipment, these torques inevitably will not be balanced.

Following is an analysis of the effect of mass center departures of the spinning section of a satellite from its spin axis upon the attitude motion and CMG requirements. Also, alternative methods for reducing the amplitudes of these motions are discussed.

2.0 SYSTEM DESCRIPTION

The space station dynamical model to be considered is shown in Fig. 1 and consists of two sections attached on a common axis. The satellite is stabilized by control moment gyros providing three axis control on one of the sections. The other section is driven by a motor at a constant rate relative to the first about the common spin axis. Displacement of the mass center of the spinning section from the spin axis is represented by a mass particle attached to the spinning section.

System details and terminology are given as follows: The spinning section is termed body B and the other is termed body A. The spin axis intersects the mass centers of both bodies. The composite mass center is denoted by S* and lies along principal axes of inertia of A and B, the corresponding principal moments of inertia being A_3 and B_3. For each body A and B, the principal moments of inertia about any line normal to the spin axis are equal and are denoted by A_1 and B_1, respectively. The rotation rate of B relative to A is ω, a constant. The mass particle is termed P with mass m and its position is designated by a distance ℓ from S* along the satellite spin axis and a distance r from the spin axis. The three CMG torque output axes are fixed with respect to A and lie along the spin axis (designated the 3 axis) and mutually perpendicular lines (designated the 1 and 2 axes) normal to the spin axis. If the orientation of this mutually right-handed orthogonal set of axes is given with respect to an inertially fixed set of axes by a 1,2,3 sequence of three axis Euler rotations ϕ_1, ϕ_2 and ϕ_3, then the CMG control torques are given by

$$T_3^c=-K_{03}\phi_3-K_{13}\dot{\phi}_3 \qquad T_i^c=-K_0\phi_i-K_1\dot{\phi}_i \qquad i=1,2 \tag{1}$$

3.0 EQUATIONS OF MOTION

Details on the formulation of the equations of motion are given in Appendix A. These are obtained by writing the equations of motion of the composite system made up of bodies A and B, and considering the mass particle P to exert on the composite system an external torque given by the torque of the reaction force of P on B. Gravity gradient and other environmental torques are neglected in the determination of the equations of motion because, for the rotation rate ω much greater than orbital rate, their magnitudes are very much smaller than the magnitude of torque associated with the mass particle P.

The equations of motion set forth in Appendix A are as follows:

$$\ddot{\phi}_1[J_1+m(\ell^2+r^2s^2\omega t)]-\ddot{\phi}_2mr^2s\omega tc\omega t-\ddot{\phi}_3mr\ell c\omega t+\dot{\phi}_1[K_1+2\omega mr^2s\omega tc\omega t]$$
$$+\dot{\phi}_2\omega[B_3+2mr^2s^2\omega t]+\dot{\phi}_3 2\omega mr\ell s\omega t+\phi_1K_0=-\omega^2mr\ell s\omega t+F_1(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \tag{2}$$

$$-\ddot{\phi}_1mr^2s\omega tc\omega t+\ddot{\phi}_2[J_1+m(\ell^2+r^2c^2\omega t)]-\ddot{\phi}_3mr\ell s\omega t-\dot{\phi}_1\omega[B_3+2mr^2c^2\omega t]$$
$$+\dot{\phi}_2[K_1-2\omega mr^2s\omega tc\omega t]-\dot{\phi}_3 2\omega mr\ell c\omega t+\phi_2K_0=\omega^2mr\ell c\omega t+F_2(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \tag{3}$$

$$-\ddot{\phi}_1 mr\ell c\omega t-\ddot{\phi}_2 mr\ell s\omega t+\ddot{\phi}_3(A_3+B_3+mr^2)+\dot{\phi}_3K_{13}+\phi_3K_{03}=F_3(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \quad (4)$$

where $s\omega t = \sin\omega t$, $c\omega t = \cos\omega t$ and $F_j(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t)$, $i,j = 1,2,3$, represent nonlinear functions of ϕ_i, $\dot{\phi}_i$, $\ddot{\phi}_i$ which are periodic in t of period $2\pi/\omega$ and which when expanded in Fourier series have all terms of the second power and higher in ϕ_i, $\dot{\phi}_i$, $\ddot{\phi}_i$. J_1 is the principal moment of inertia of the composite A-B body at the composite mass center S* in any direction normal to the spin axis.

Defining a new independent variable

$$\tau = \frac{\omega}{2\pi}\, t = \bar{\omega} t \quad (5)$$

so that

$$\frac{d}{dt}(\ ) = \bar{\omega}\frac{d}{d\tau}(\ ) = \bar{\omega}(\ )' \quad (6)$$

Equations (2) - (4) may be written

$$\phi_1''[1+\varepsilon(d^2+\tfrac{1}{2}(1-c2p\tau))]-\phi_2''\tfrac{\varepsilon}{2}s2p\tau-\phi_3''\varepsilon dcp\tau+\phi_1'(k_1+\varepsilon s2p\tau)+\phi_2'[b+\varepsilon(1-c2p\tau)]$$
$$+\phi_3' 2\varepsilon dsp\tau+\phi_1k_0=-\varepsilon dp^2sp\tau+F_1(\phi_i,\phi_i',\phi_i'',p\tau)/J_1\bar{\omega}^2 \quad (7)$$

$$-\phi_1''\tfrac{\varepsilon}{2}s2p\tau+\phi_2''[1+\varepsilon(d^2+\tfrac{1}{2}(1+c2p\tau))]-\phi_3''\varepsilon dsp\tau-\phi_1'[b+\varepsilon(1+c2p\tau)]+\phi_2'(k_1-\varepsilon s2p\tau)$$
$$-\phi_3' 2\varepsilon dcp\tau+\phi_2k_0=\varepsilon dp^2cp\tau+F_2(\phi_i,\phi_i',\phi_i'',p\tau)/J_1\bar{\omega}^2 \quad (8)$$

$$-\phi_1''\varepsilon decp\tau-\phi_2''\varepsilon desp\tau+\phi_3''(1+\varepsilon e)+\phi_3'k_{13}+\phi_3k_{03}=F_3(\phi_i,\phi_i',\phi_i'',p\tau)/J_1\bar{\omega}^2 \quad (9)$$

where

$$\varepsilon = mr^2/J_1 \qquad k_0 = K_0/\bar{\omega}^2 J_1 \qquad k_1 = K_1/\bar{\omega} J_1$$

$$b = B_3/J_1 \qquad k_{03} = K_{03}/\bar{\omega}^2(A_3+B_3) \qquad k_{13} = K_{13}/\bar{\omega}(A_3+B_3)$$

$$d = \ell/r \qquad p = 2\pi$$

$$e = J_1/(A_3+B_3) \tag{10}$$

Solution of Equations (7) - (9) for ϕ_1'', ϕ_2'', ϕ_3'' and setting

$$X_i = \phi_i \; , \quad i=1,2,3 \qquad X_i = X_{i-3}' \; , \quad i=4,5,6 \tag{11}$$

gives equations of motion in a standard matrix form

$$X' = AX + f(X,\tau) + \varepsilon g(\tau) \tag{12}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_0 & 0 & 0 & -k_1 & -b & 0 \\ 0 & -k_0 & 0 & b & -k_1 & 0 \\ 0 & 0 & -k_{03} & 0 & 0 & -k_{13} \end{bmatrix} \tag{13}$$

and $f(X,\tau)$ and $g(\tau)$ are periodic of period 1, with the average of $g(\tau)$ over a period equal to zero. It can also be shown that $f(X,t)$ consists of terms proportional to at least the first power in ε or nonlinear terms that approach zero as X^2 for X approaching zero, and there is a constant c, related to system parameters, such that*

$$|f(X,t) - f(Y,t)| \leq \varepsilon c|X-Y| \tag{14}$$

for $|X| = \sum_{i=1}^{6} |X_i|$, $|Y| = \sum_{i=1}^{6} |Y_i|$ sufficiently small. In addition Routhian analysis (see Appendix B) can be used to show that the characteristic roots of matrix A have negative real parts for $k_0, k_1, k_{03}, k_{13} > 0$. Consequently, the results of Appendix C establish that for ε sufficiently small, the steady state solution to the equations of motion (Equations (12) or Equations (7) - (9)) is periodic of period 1 in τ. Also, for sufficiently small ε, this solution is accurately represented by the first approximation of the method of successive approximations set forth in Appendix C.

The mass m is assumed to be very small in comparison to the total mass of bodies A and B. Then ε is a very small quantity and applying the method of successive approximations of Appendix C and neglecting terms of higher than the first power in ε yields

$$\phi_1 = \delta[-(1+R_0)\sin p\tau + R_1\cos p\tau]/[(1+R_0)^2 + R_1^2] \tag{15}$$

$$\phi_2 = \delta[R_1\sin p\tau + (1+R_0)\cos p\tau]/[(1+R_0)^2 + R_1^2] \tag{16}$$

$$\phi_3 = 0 \tag{17}$$

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*The significance of Equation (14) is developed in Appendix C.

where

$$\delta = mr\ell/(B_3-J_1)$$

$$R_0 = K_0/\omega^2(B_3-J_1)$$

$$R_1 = K_1/\omega^2(B_3-J_1) \qquad (18)$$

4.0 DISCUSSION OF SOLUTION

For small amplitude motions, the angle ψ which the satellite spin axis makes with its nominal direction (that for which $\phi_1 = \phi_2 = 0$) is accurately given by

$$\psi = \sqrt{\phi_1^2 + \phi_1^2} = \pm\, \delta/\sqrt{(1+R_0)^2 + R_1^2} \qquad (19)$$

Since ψ is constant with respect to τ, the satellite spin axis once each period $T = 2\pi/\omega$ sec traces a right circular cone with half angle equal to ψ.

For the space station motion given by ϕ_1, ϕ_2, ϕ_3 of Equations (15)-(17), the minimum magnitude $\overline{H}$ of the total CMG spin angular momentum necessary to provide the control torques of Equations (1) is determined in Appendix D as

$$\overline{H} = H_0\sqrt{\frac{R_0^2 + R_1^2}{(1+R_0)^2 + R_1^2}} \qquad (20)$$

where

$$H_o = mr\ell\omega \qquad (21)$$

Also, the magnitude $\overline{T}$ of the CMG control torque can be shown to be

$$\overline{T} = \sqrt{T_1^2 + T_2^2 + T_3^2} = \omega\overline{H} \qquad (22)$$

Now the values of ψ/δ and $\overline{H}/H_0$ can be plotted against R_0 for varying R_1. However, the nature of these curves depends upon whether B_3-J_1 is greater than or less than zero. Defining

$$\delta^{\pm} = \pm\delta \qquad R_0^{\pm} = \pm R_0 \qquad R_1^{\pm} = \pm R_1 \tag{23}$$

where the positive sign is taken for $B_3-J_1 > 0$, and the negative sign is taken for $B_3-J_1 < 0$, Figures 2 and 3 show the variation of ψ/δ and $\overline{H}/H_0$ with respect of R_0^+ and R_1^+ and Figures 4 and 5 show the variation of ψ/δ and $\overline{H}/H_0$ with respect to R_0^- and R_1^-. These plots reveal the following:

1) The ψ/δ and $\overline{H}/H_0$ curves for $B_3-J_1 > 0$ always lie below the corresponding curves for $B_3-J_1 < 0$. Consequently, for minimizing the satellite coning angle and required CMG spin angular momentum, it is preferable for the space station design to be such that the moment of inertia of the spinning section about the spin axis is as much as possible greater than the moment of inertia of the space station about a direction normal to the spin axis.

2) For a space station with $B_3-J_1 < 0$, values of K_0 in the region of $K_0 = \omega^2(J_1-B_3)$ should be avoided.

3) For decreasing R_0^+, R_1^+ or R_0^-, R_1^- corresponding to decreasing CMG spin angular momentum, the space station coning angle approaches

$$\psi = \delta = \left|\frac{mr\ell}{B_3-J_1}\right| \tag{24}$$

This limiting value of $\psi = \delta$ shall hereafter be termed the light control coning angle.

For increasing R_0^+, R_1^+ or R_0^-, R_1^-, the control moment gyros cannot have significant effect in decreasing the space station coning angle until the magnitude of their composite spin angular momentum is of the order of the limiting value

$$\bar{H} = H_0 = mr\ell\omega \tag{25}$$

5.0 APPLICATION TO TWO SPACE STATION CONFIGURATIONS

Amplitudes of motion and control system requirements will now be determined for two space station configurations with spinning artificial gravity sections.

The first configuration (Space Base), which has been suggested for study by NASA (Reference [1(a)]), is capable of supporting a crew of fifty and is made up of a hub with a non-rotating section and a rotating section to which three arms are attached in a Y fashion and which is spinning so as to provide artificial gravity compartments at the extremities of one of the arms.

This configuration has the worst unbalance characteristics for a supply spacecraft docked to the non-rotating section. For this case, the space base might be supposed to have the same (to within NASA's margin of error in their moment of inertia estimates) moment of inertia of $J_1 = 7 \times 10^8$ slug ft^2 about any line normal to the spin axis and a spinning section axial moment of inertia of $B_3 = 9.5 \times 10^8$ slug ft^2. Considering an unbalance mass of 300 slugs, equivalent to the mass of the Space Base elevator fully loaded and located at the artificial gravity compartment, then $r = 170$ ft, $\ell = 20$ ft and the light control coning angle is

$$\psi = \delta = 0.23 \text{ degrees}$$

A coning angle of .23 degrees might well be unacceptable for several reasons, among which are

a) This amplitude of motion is outside the pointing accuracies required for some space station experiments, especially those involving astronomical telescopes.

b) For this amplitude of motion and ω = 4rpm, locations within the despun section and at four feet from the space station mass center experience instantaneous accelerations of the order of 10^{-4}G. However, some experiments to be conducted within the despun section, such as those involving crystal growth, require extremely low levels of accelerations, perhaps as low as 10^{-5}G.

c) Although experiments could be located on gimballed platforms to isolate them from space station attitude motion, the weight and complexity of gimballed systems required to maintain a specified pointing accuracy likely increase substantially with amplitude of attitude motion. Also, extremely delicate gimballed systems located at substantial distances from the space station mass center might experience significantly detrimental translational accelerations.

Now, if for these or other reasons the coning angle is unacceptable, then, as declared in statement 3) above, for a significant reduction, the CMGs must have a magnitude of total spin angular momentum of at least the order of H_0,

$$\bar{H} = H_0 = 425{,}000 \text{ ft-lb-sec}$$

a value very much beyond the capability of existing CMGs, optimistically 6000 ft-lb-sec for three CMGs.

The second space station configuration consists of a zero gravity section and a centrifuge section which is utilized for short intervals of time either by crew members periodically to counteract physiological effects of prolonged weightlessness or to reacclimate crew members to accelerations prior to their return to Earth and exposure to high reentry decelerations.

For such a space station having total mass M, made up of two cylindrical sections of depth 1/3R for the centrifuge section and 5/3R for the despun section, both with the same mass distribution and radius R = 15 ft, and with a displaced object of mass m attached as shown in Figure 6, the light control coning angle is

$$\psi = \delta = .11 \text{ degrees}$$

for object P having 0.1% of the total mass of the space station.

Now for P representing a man of mass 6 slugs experiencing a radial acceleration of one G, the magnitude of the total CMG spin angular momentum must be at least about

$$\bar{H} = H_0 = 1970 \text{ ft-lb-sec}$$

(a value within the capability of existing CMGs) if it is desired to significantly reduce the coning angle. By Equation (22) the CMGs must be capable of providing a torque of about 2900 ft-lbs, which, being an order of magnitude higher than maximum output for existing CMGs, may present implementation difficulties.

6.0 REDUCTION OF AMPLITUDE OF MOTIONS

6.1 Mass Balancing System

It has been shown that, for very large artificial "G" space stations, CMGs offer little hope in reducing the amplitude of the angular oscillations induced by mass unbalance. One means of reducing these which has been suggested (Reference [1(b)]) involves providing counterbalance mass shifts by pumping of fluids or by movement of massive bodies.

The influencing factor in producing satellite coning results from the torque of the radial acceleration force of the mass unbalance particle. Components of this torque with respect to a $\beta_1, \beta_2, \beta_3$ axis system with origin at the satellite mass center, β_3 in the direction of the spin axis, and the axis system rotating about β_3 at rate ω are

$$M_1 = mr\omega^2 \ell \sin\theta \tag{26}$$

$$M_2 = mr\omega^2 \ell \cos\theta \tag{27}$$

where θ is the angle between the β_1 axis and the radial line intersecting the unbalance mass.

Now if N movable counterbalance masses m_i can be shifted Δr_i, $\Delta \ell_i$, $\Delta\theta_i$ from their positions, represented by cylindrical coordinates r_i, ℓ_i, θ_i, for which the mass center of the spinning section lies on the spin axis, then for the above unbalance torques M_1 and M_2, these shifts can be used to result in the two components of net torque about the satellite mass center being equal to zero.

$$\omega^2[mr\ell\sin\theta+\sum_{i=1}^{N}(m_i+\Delta m_i)(r_i+\Delta r_i)(\ell_i+\Delta\ell_i)\sin(\theta_i+\Delta\theta_i)-\sum_{i=1}^{N}m_ir_i\ell_i\sin\theta_i]=0 \tag{28}$$

$$\omega^2[mr\ell\cos\theta+\sum_{i=1}^{N}(m_i+\Delta m_i)(r_i+\Delta r_i)(\ell_i+\Delta\ell_i)\cos(\theta_i+\Delta\theta_i)-\sum_{i=1}^{N}m_ir_i\ell_i\cos\theta_i]=0 \tag{29}$$

Since the quantities m_i, r_i, ℓ_i, and θ_i are known quantities (indicating masses and locations of the counterbalance particles), then choosing at convenience all but two of the 4N quantities Δm_i (representing transfer of a fluid mass to or from a location), Δr_i, $\Delta\ell_i$, and $\Delta\theta_i$ (representing mass shifts), Equations (28) and (29) can be solved for the remaining two unknowns, provided the quantities m, r, ℓ, and θ of the unbalance mass can first be found.

These are determined as follows: From Equations (15) and (16), the maximum amplitude L of ϕ_1 and ϕ_2 is given by

$$L = \left| mr\ell \Big/ \left[(B_3-J_1)\sqrt{(1+R_0)^2+R_1^2}\right]\right| \tag{30}$$

where R_0 and R_1 are given by Equations (18). Then, $mr\ell$ can be determined

$$mr\ell = \left|L(B_3-J_1)\sqrt{(1+R_0)^2+R_1^2}\right| \tag{31}$$

Also, values of $\tau=0,1,2,...$ indicate the instants at which the radial line from the spin axis to the unbalance mass is parallel to a unit vector $\underline{a}_1$ (see Appendix A) in the direction of the first CMG output axis. Then, by Equations (16) and (30), at $\tau=0,1,2,...$, the ratio Q of the amplitude of ϕ_2 to its maximum amplitude is given by

$$Q = (1+R_0)/\sqrt{(1+R_0)^2+R_1^2} \qquad (32)$$

Because the satellite oscillations ϕ_1 and ϕ_2 can be obtained with attitude sensors attached to the despun section of the satellite, then, since B_3, J_1, R_0, and R_1 are known quantities and the maximum amplitudes of the ϕ_1 and ϕ_2 oscillations can be measured from the sensor outputs, the quantity $mr\ell$ can be determined from Equation (31). Also, at the instant the measured ϕ_2 oscillation reaches the ratio Q of its maximum value, the radial line intersecting the unbalance mass is parallel to a known line and the angle θ of Equations (26) and (27) can be determined. With this information, Equations (28) and (29) can be solved to give the mass shifts necessary to eliminate the satellite coning.

This method of mass balancing assumes that the spinning body B is axially symmetrical and that the spin axis lies along a principal axis of inertia of B. Consequently, some of the 4N-2 quantities Δm_i, Δr_i, $\Delta \ell_i$, and $\Delta\theta_i$ which are prescribed must be chosen so as to insure body B retains axial symmetry and the principal axis of B remains along the spin axis. Also, steady state conditions have been assumed so that if the usual time intervals between mass shifts of crew and equipment are less than system decay time, then transient effects must be considered.

6.2 Passive System

Since space station experiments requiring very small attitude motions would not likely be housed in the spinning artificial gravity section, it may be possible to bypass the coning problem by allowing the spinning section to experience the coning motion but providing an interconnection to the despun section that does not transmit significant torques in directions normal to the satellite spin axis. Then the amplitude of the motion performed by the despun section could be much smaller than that of the spinning section. One possible connection, shown in Figure 7, might be a bellows-like tube which may or may not enclose a pressurized volume.

This means of alleviating satellite coning effects induced by mass unbalance of a spinning artificial gravity section offers the advantages over the active mass balance system of involving a passive system with greater simplicity and lower weight and, having heretofore not been considered, should be investigated in the future.

7.0 SUMMARY

It has been shown that mass unbalance of the spinning artificial compartments of a space station produces a coning motion of amplitude which can be detrimental to experiments which will be carried on future space stations.

Overall space station design influences the amplitude of motion. It is preferable for the despun section to be much smaller than the spinning section, for the maximum dimensions of the space station in the direction along the spin axis to be much smaller than the maximum radial dimensions of the spinning section, and for crew compartments and compartments to and from which equipment is shifted to be located as close as possible to the satellite mass center.

For small space stations for which only small mass unbalances of the spinning section are expected, CMGs might be suitable for reducing the amplitude of satellite coning motion, although there may be difficulties in implementing the necessary torque levels. However, CMGs are ineffective for very large space stations with significant mass unbalances and special systems may well have to be employed. One such system involves active balancing by shifting of massive bodies or pumping of fluids. The facilitation of such a system is discussed and a lighter and less complex passive system involving a flexible connection between the artificial gravity and despun sections is proposed for further investigation.

1022-RAW-jf/cf

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Attachments
References
Figures 1-7
Appendix A,B,C,D

REFERENCES

1. NASA Request for Proposal, "Space Station, Phase B Study," 1969, (a) p. A-14, (b) p. A-11.

2. Farnell, A. B., Langenhop, C. E., and Levinson, N., "Forced Periodic Solutions of a Stable Nonlinear System of Differential Equations," Journal of Math. Phys., Vol. 29, 1951, pp. 300-302.

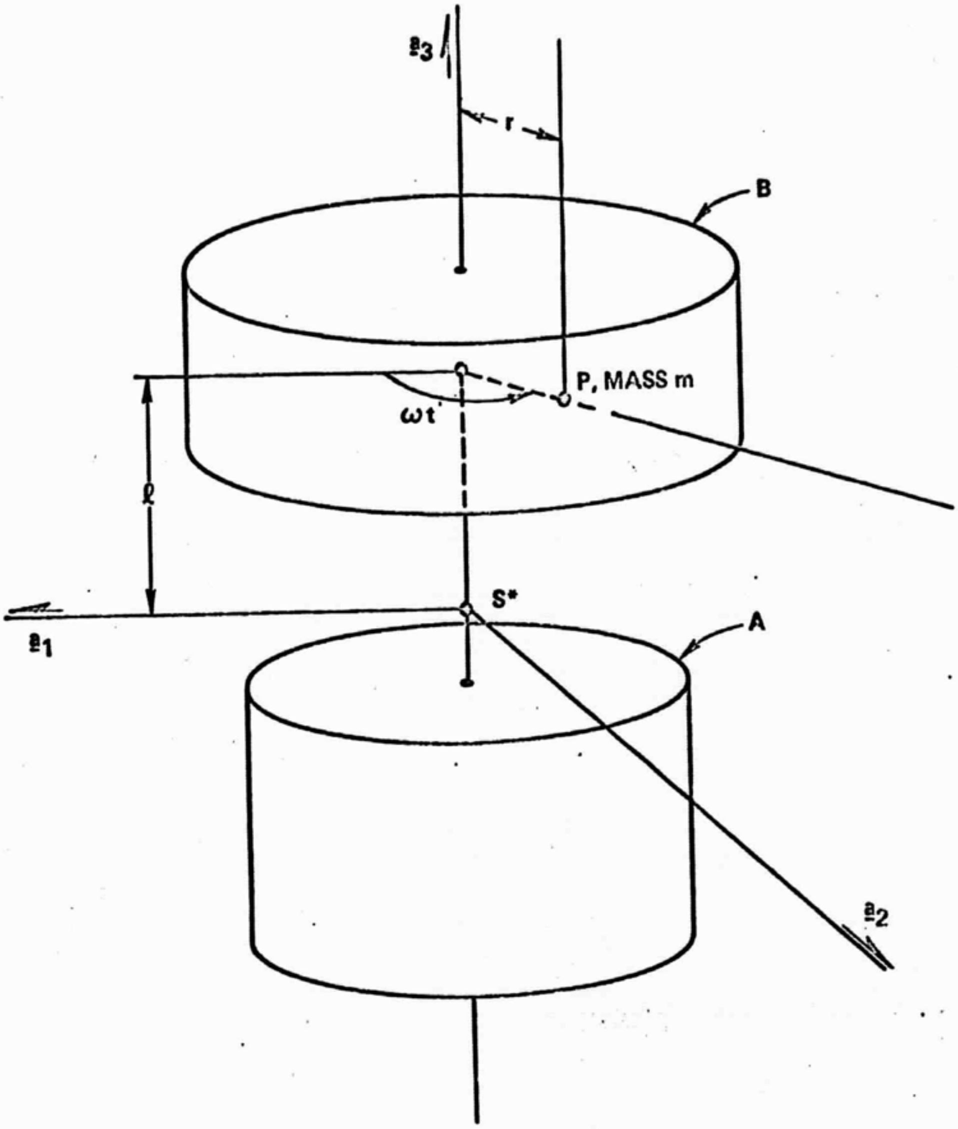

FIGURE 1.

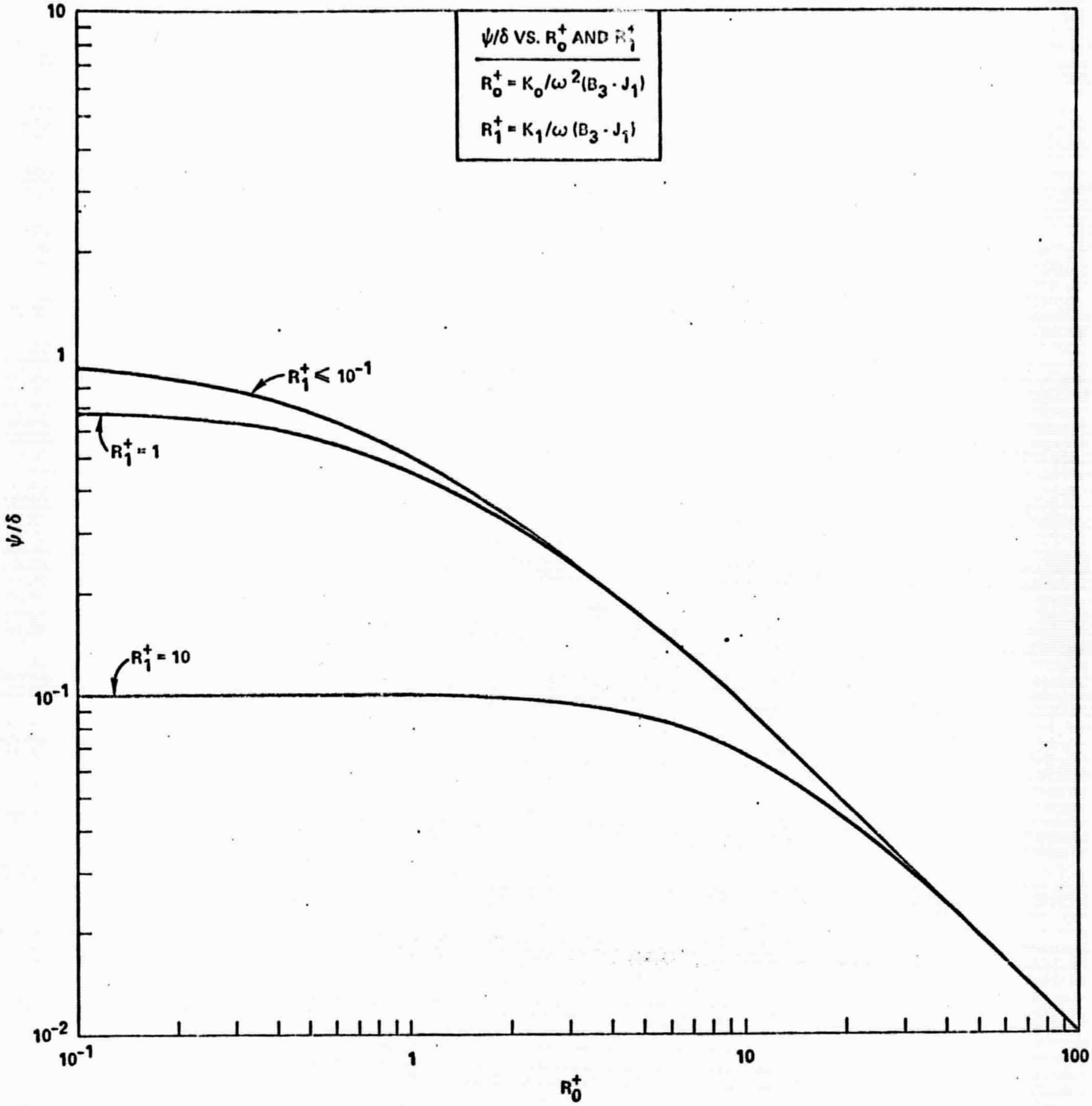

FIGURE 2. SPACE STATION CONING ANGLE FOR $B_3 > J_1$

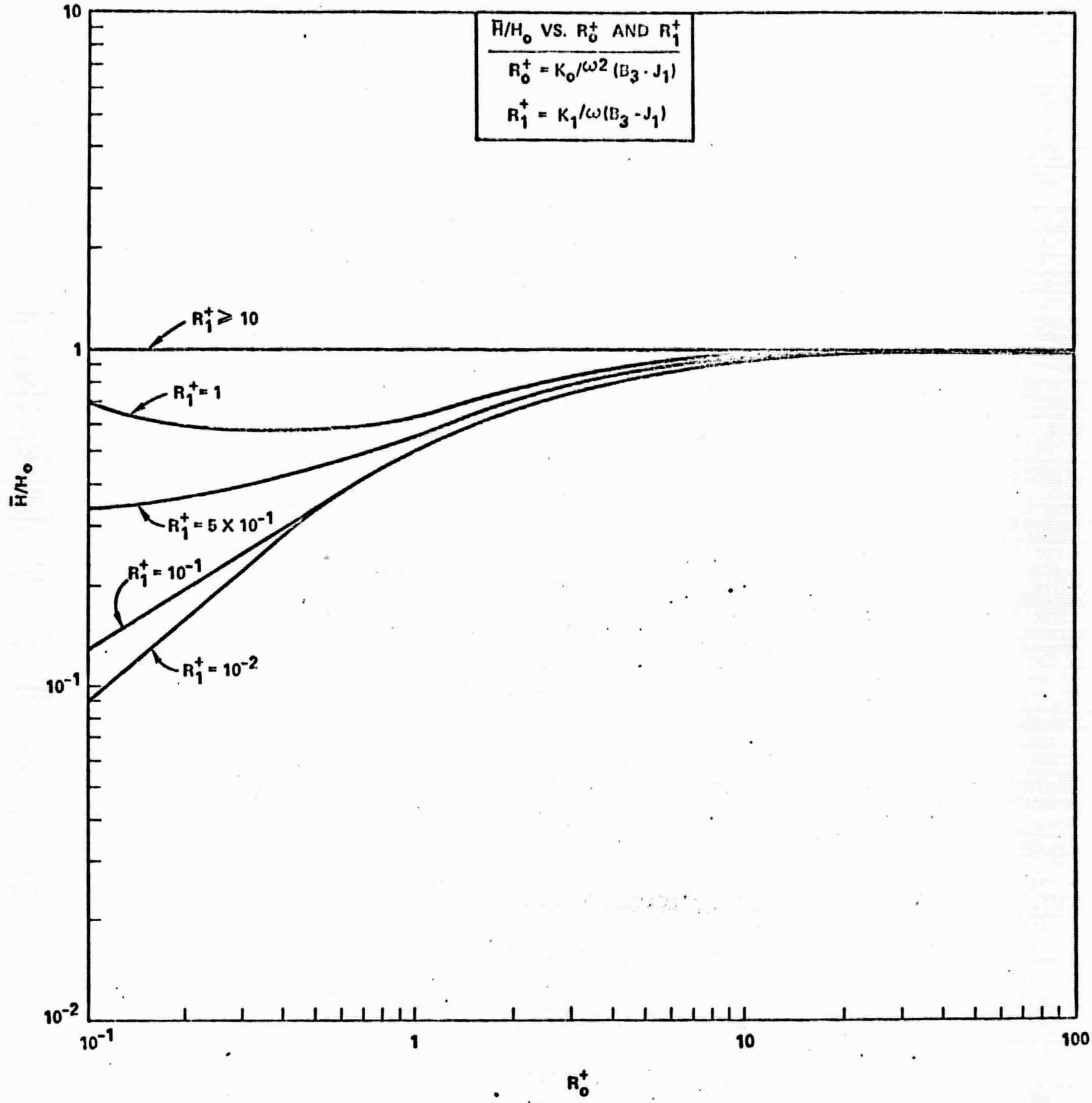

FIGURE 3. MINIMUM CMG ANGULAR MOMENTUM FOR $B_3 > J_1$

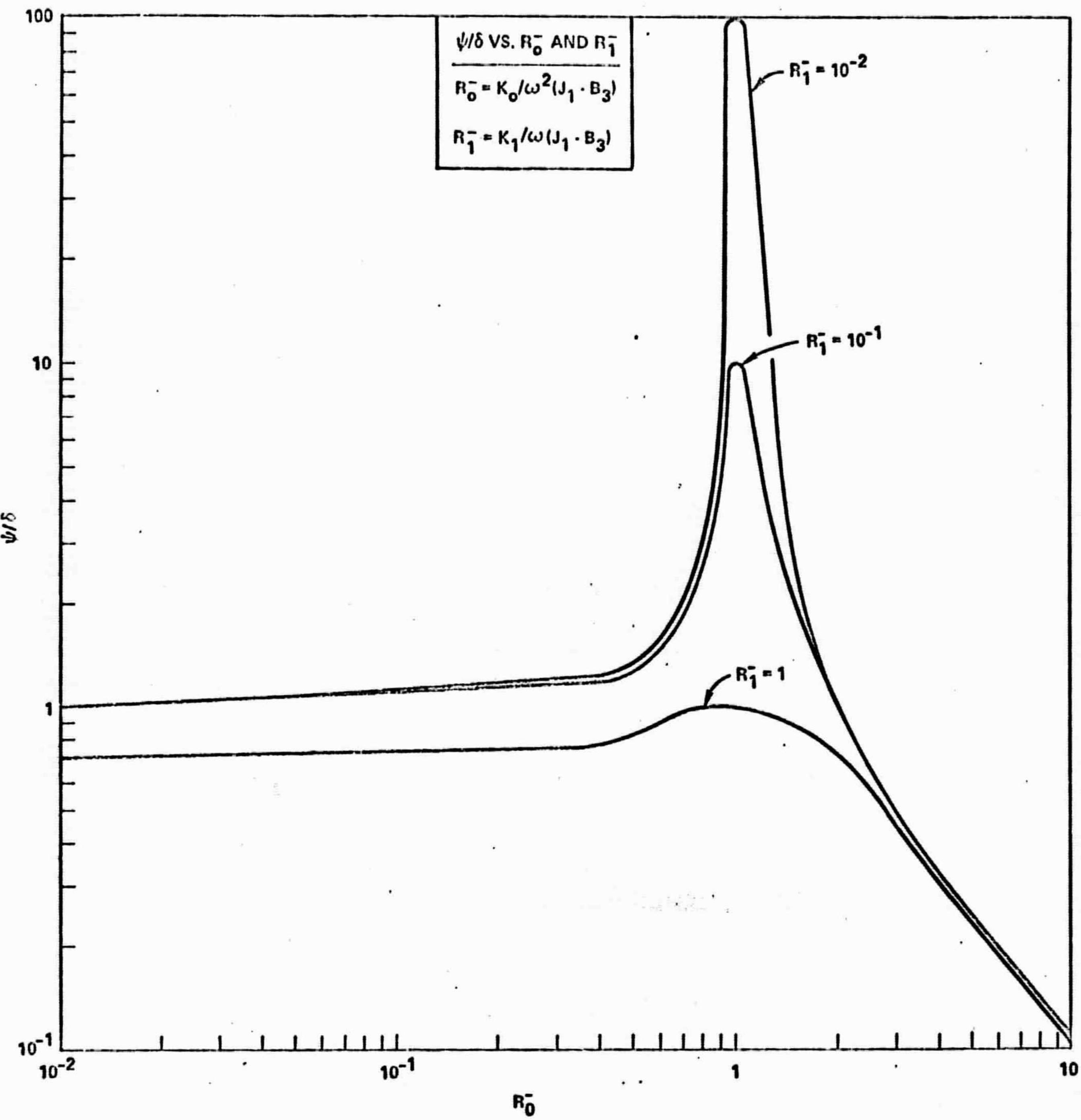

FIGURE 4. SPACE STATION CONING ANGLE FOR $B_3 < J_1$

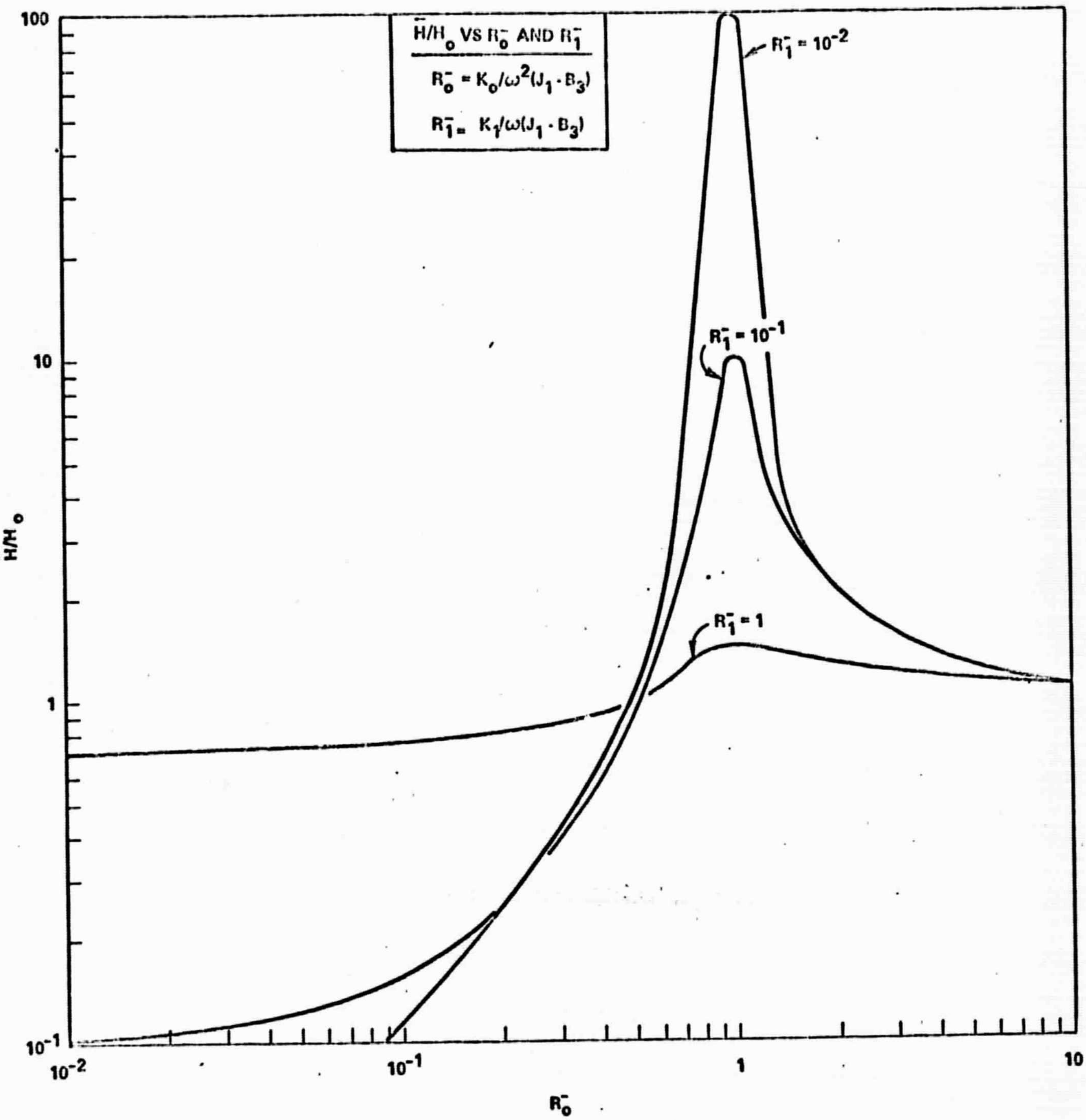

FIGURE 5. MINIMUM CMG ANGULAR MOMENTUM FOR $B_3 < J_1$

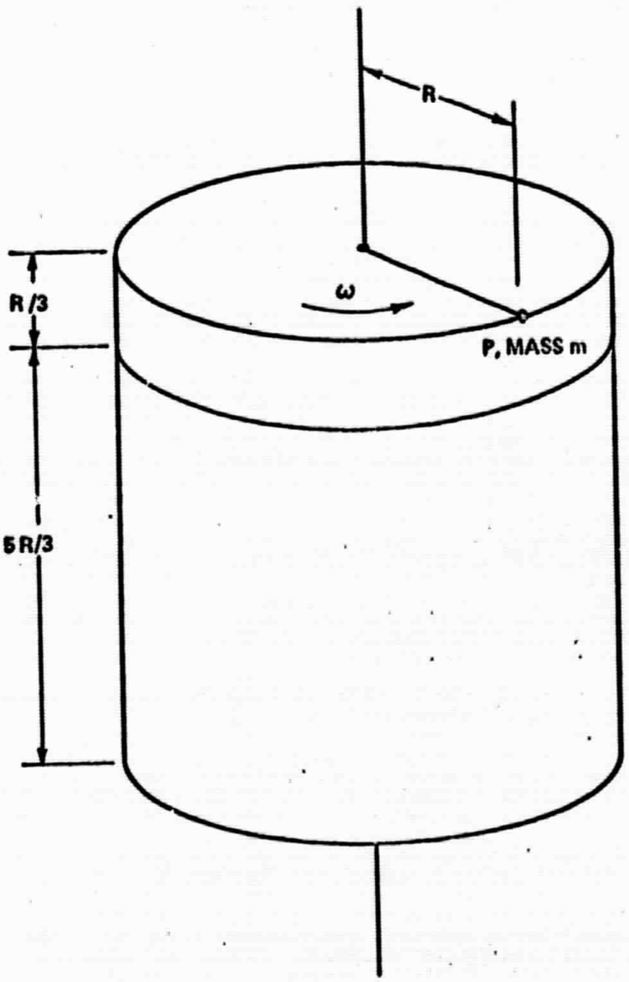

FIGURE 6. STACKED SPACE STATION CONFIGURATION

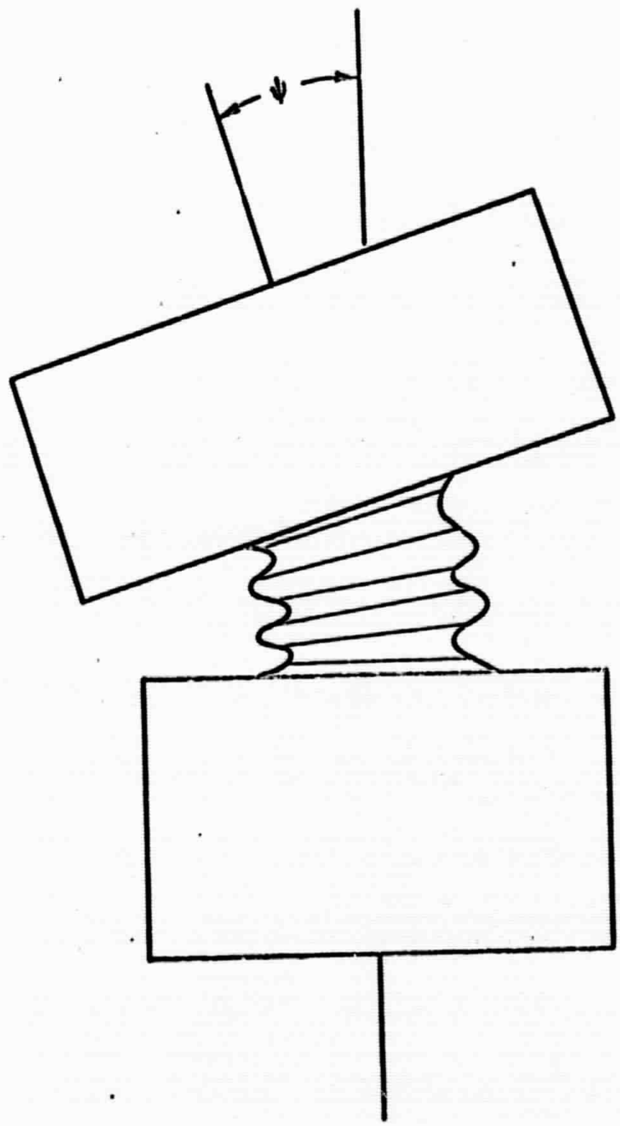

FIGURE 7. FLEXIBLE CONNECTION BETWEEN SPACE STATION SECTIONS (ψ EXAGGERATED)

APPENDIX A

Formulation of Satellite Equations of Motion

The equations of motion for the composite A-B body and the particle P are to be written separately. The equations for each contain the reaction force between A-B and P and elimination of that quantity results in the equations of motion for the whole system made up of A, B, and P. Prior to the formulation of these equations, several additional definitions need to be set forth.

Consider α_1, α_2, α_3 to be a right-handed mutually orthogonal set of axes fixed with respect to A and parallel to the directions of the three control moment gyro output axes. Also, α_3 is parallel to the satellite spin axis. The origin of α_1, α_2, α_3 is at the composite A-B body mass center S*. The 1,2,3 sequence of three axis Euler rotations ϕ_1, ϕ_2, ϕ_3 characterize the orientation of α_1, α_2, α_3 with respect to their initial directions at time t=0.

For unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ parallel to α_1, α_2, α_3 respectively, and the angular velocity $\underline{\omega}^A$ of body A expressed as

$$\underline{\omega}^A = u_1\underline{a}_1 + u_2\underline{a}_2 + u_3\underline{a}_3 \tag{A-1}$$

then the angular velocity $\underline{\omega}^B$ of body B may be written

$$\underline{\omega}^B = u_1\underline{a}_1 + u_2\underline{a}_2 + (u_3+\omega)\underline{a}_3 \tag{A-2}$$

where

$$u_1 = \dot{\phi}_1 \cos\phi_2 \cos\phi_3 + \dot{\phi}_2 \sin\phi_3 \qquad \text{(A-3)}$$

$$u_2 = -\dot{\phi}_1 \cos\phi_2 \sin\phi_3 + \dot{\phi}_2 \cos\phi_3 \qquad \text{(A-4)}$$

$$u_3 = \dot{\phi}_1 \sin\phi_2 + \dot{\phi}_3 \qquad \text{(A-5)}$$

Equations of Motion for A-B

The angular momentum $\underline{H}$ of the composite body A-B is

$$\underline{H} = [A_1+B_1+\ell_A^2 M_A+\ell_B^2 M_B]u_1\underline{a}_1 + [A_1+B_1+\ell_A^2 M_A+\ell_B^2 M_B]u_2\underline{a}_2 + [u_3(A_3+B_3)+B_3\omega]\underline{a}_3 \qquad \text{(A-6)}$$

where M_A and M_B are the masses of bodies A and B respectively and ℓ_A and ℓ_B are the distances between S* and the mass center of A and S* and the mass center of B respectively. Setting the time rate of change of the angular momentum equal to the external torques gives

$$J_1\dot{u}_1 + [(J_3-J_1)u_3+B_3\omega]u_2 = {}^C T_1 + {}^P T_1 \qquad \text{(A-7)}$$

$$J_1\dot{u}_2 + [(J_1-J_3)u_3-B_3\omega]u_1 = {}^C T_2 + {}^P T_2 \qquad \text{(A-8)}$$

$$J_3\dot{u}_3 = {}^C T_3 + {}^P T_3 \qquad \text{(A-9)}$$

where

$$J_1 = A_1 + B_1 + \ell_A^2 M_A + \ell_B^2 M_B$$

$$J_3 = A_3 + B_3 \qquad \text{(A-10)}$$

and ${}^{C}T_i$ are components of control torque given by Equation (1) and ${}^{P}T_i$ are components of the torque of the reaction force of particle P on A-B.

Reaction Torque of P on A-B

The equations of motion of particle P may be written in vector form

$$ {}^{G}\underline{F}^{P} + {}^{S}\underline{R}^{P} = m\underline{a}^{P} \tag{A-11}$$

where ${}^{G}\underline{F}^{P}$ is the force of gravitational attraction on P of the body being orbited, ${}^{S}\underline{R}^{P}$ is the reaction force of the satellite on P, and $\underline{a}^{P}$ is the inertial acceleration of P.

For a circular orbit, i.e., the mass center of A-B tracing a circular path,

$$ \underline{a}^{P} = R_A \omega_0^{2} \underline{C} + \dot{\underline{V}}^{P} \tag{A-12}$$

where R_A is the magnitude and unit vector $\underline{C}$ is in the direction of the vector from S* to the mass center of the attracting body, $\dot{\underline{V}}^{P}$ is the time rate of change of the velocity of P with respect to S*, and ω_0 is the orbital rate.

The gravitational attraction force is given by

$$ {}^{G}\underline{F}^{P} = \frac{mK}{R_A'^{2}} \underline{C}' \tag{A-13}$$

where K is a gravitational constant, and R_A' is the magnitude and $\underline{C}'$ is in the direction of the vector from P to the mass center of the attracting body. Now for dimensions of the satellite very small in comparison to the orbital radius R_A,

$$\underline{C}' \approx \underline{C} \qquad\qquad R_A' \approx R_A \tag{A-14}$$

and

$$^{G}\underline{F}^{P} \approx m \frac{K}{R_A^{\ 2}} \underline{C} \tag{A-15}$$

For a circular orbit

$$K/R_A^{\ 2} = R_A \omega_0^{\ 2} \tag{A-16}$$

Substitution of Equations (A-16), (A-15), and (A-12) into Equation (A-11) yields

$$^{S}\underline{R}^{P} = m\underline{\dot{V}}^{P} \tag{A-17}$$

and the torque $^{P}\underline{T}$ about the space station mass center of the equal and opposite reaction force of P on the space station is

$$^{P}\underline{T} = -m\underline{r}_{S*P} \times \underline{\dot{V}}^{P} \tag{A-18}$$

where $\underline{r}_{S*P}$ is the vector from S* to P which second derivative with respect to t is $\underline{\dot{V}}^{P}$.

If time t=0 is chosen at an instant the radial line from the satellite spin axis to P is parallel to unit vector $\underline{a}_1$, then $\underline{r}_{S*P}$ may be written

$$\underline{r}_{S*P} = r\cos\omega t\underline{a}_1 + r\sin\omega t\underline{a}_2 + \ell\underline{a}_3 \qquad \text{(A-19)}$$

Differentiation twice with respect to t and substitution of the result into Equation (A-18) gives

$$P_{\underline{T}} = P_{T_1}\underline{a}_1 + P_{T_2}\underline{a}_2 + P_{T_3}\underline{a}_3 \qquad \text{(A-20)}$$

where for sωt = sinωt, cωt = cosωt,

$$P_{T_1} = -m\{\dot{u}_1(\ell^2+r^2s^2\omega t)-\dot{u}_2r^2s\omega tc\omega t-\dot{u}_3r\ell c\omega t+u_1[(u_3+2\omega)r^2s\omega tc\omega t-u_2r\ell c\omega t]$$

$$+u_2[(u_3+2\omega)r^2s^2\omega t-u_2r\ell s\omega t-u_3\ell^2]+u_3(u_3+2\omega)r\ell s\omega t+\omega^2r\ell s\omega t\} \qquad \text{(A-21)}$$

$$P_{T_2} = -m\{-\dot{u}_1r^2s\omega tc\omega t+\dot{u}_2(\ell^2+r^2c\omega t)-\dot{u}_3r\ell s\omega t+u_1[-(u_3+2\omega)r^2c^2\omega t+u_1r\ell c\omega t$$

$$+u_2r\ell s\omega t+u_3\ell^2] - u_2(u_3+2\omega)r^2s\omega tc\omega t-u_3(u_3+2\omega)r\ell c\omega t-\omega^2r\ell c\omega t\} \qquad \text{(A-22)}$$

$$P_{T_3} = -mr\{-\dot{u}_1\ell c\omega t-\dot{u}_2\ell s\omega t+\dot{u}_3r+u_1u_2r(c^2\omega t-s^2\omega t) + (u_2^2-u_1^2)rs\omega tc\omega t$$

$$+ u_3\ell(u_2c\omega t-u_1s\omega t)\} \qquad \text{(A-23)}$$

Equations of Motion for Entire System

Substitution of torque components ${}^{C}T_i$ from Equation (1) and ${}^{P}T_i$ from Equations (A-21)-(A-23) into Equations (A-7)-(A-9) and use of Equations (A-3)-(A-5) to eliminate u_i, i=1,2,3 results in the equations of motion for the entire system.

$$\ddot{\phi}_1[J_1+m(\ell^2+r^2s^2\omega t)]-\ddot{\phi}_2mr^2s\omega tc\omega t-\ddot{\phi}_3mr\ell c\omega t+\dot{\phi}_1[K_1+2\omega mr^2s\omega tc\omega t]$$

$$+\dot{\phi}_2\omega[B_3+2mr^2s^2\omega t]+\dot{\phi}_32\omega mr\ell s\omega t+\phi_1K_0 = -\omega^2mr\ell s\omega t+F_1(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \quad \text{(A-24)}$$

$$-\ddot{\phi}_1mr^2s\omega tc\omega t+\ddot{\phi}_2[J_1+m(\ell^2+r^2c^2\omega t)]-\ddot{\phi}_3mr\ell s\omega t-\dot{\phi}_1\omega[B_3+2mr^2c^2\omega t]$$

$$+\dot{\phi}_2[K_1-2\omega mr^2s\omega tc\omega t]-\dot{\phi}_32\omega mr\ell c\omega t+\phi_2K_0 = \omega^2mr\ell c\omega t+F_2(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \quad \text{(A-25)}$$

$$-\ddot{\phi}_1mr\ell c\omega t-\ddot{\phi}_2mr\ell s\omega t+\ddot{\phi}_3(A_3+B_3+mr^2)+\dot{\phi}_3K_{13}+\phi_3K_{03} = F_3(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \quad \text{(A-26)}$$

where $F_j(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t)$, i,j=1,2,3 represent nonlinear functions of $\phi_i,\dot{\phi}_i,\ddot{\phi}_i$ which are periodic in t of period $2\pi/\omega$ and which when expanded in Fourier series have all terms of the second and higher powers in ϕ_i, $\dot{\phi}_i$, $\ddot{\phi}_i$.

APPENDIX B

The characteristic roots λ_j, $j=1,\ldots,6$, of matrix A of Equation (13) are given by

$$\text{determinant } [A-\lambda I] = 0 \qquad \text{(B-1)}$$

where I is the identity matrix. Then

$$[\lambda^2+\lambda k_{13}+k_{03}][\lambda^4+2\lambda^3 k_1+\lambda^2(b^2+k_1{}^2+2k_0) + \lambda 2k_0 k_1 + k_0{}^2] = 0 \qquad \text{(B-2)}$$

Two characteristic roots are given by

$$\lambda^2 + \lambda k_{13} + k_{03} = 0 \qquad \text{(B-3)}$$

or solving for λ

$$\lambda_{1,2} = \frac{-k_{13} \pm \sqrt{k_{13}^2 - 4k_{03}}}{2} \qquad \text{(B-4)}$$

which have negative real parts for k_{13}, $k_{03} > 0$.

The remaining characteristic roots are given by

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \qquad \text{(B-5)}$$

where a_1, ..., a_4 are given by comparison with Equation (B-2).

Routhian analysis states that if Δ_i, i=1,2,3,4, given by

$$\Delta_1 = a_1 \qquad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \qquad \Delta_4 = \begin{vmatrix} a_1 & a_3 & 0 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & 1 & a_2 & a_4 \end{vmatrix} \tag{B-6}$$

are all greater than zero, then all roots of Equation (B-5) have negative real parts. By Equations (B-6), (B-5), and (B-2)

$$\Delta_1 = 2k_1 \tag{B-7}$$

$$\Delta_2 = 2k_1(b^2+k_1^2+k_0) \tag{B-8}$$

$$\Delta_3 = 4k_0k_1^2(b^2+k_1^2) \tag{B-9}$$

$$\Delta_4 = 4k_0^3k_1^2(b^2+k_1^2) \tag{B-10}$$

and for k_0, $k_1 > 0$, Δ_i, i=1,...,4, are all greater than zero and all roots of Equation (B-5) have negative real parts.

APPENDIX C

Periodic Solutions of a Set of Differential Equations

A method of successive approximations has been set forth by Farnell, Langenhop, and Levinson (Reference [2]) which provides arbitrarily good approximations to stable periodic solutions of systems of differential equations of the form

$$x' = Ax + f(x,\omega t) + \varepsilon g(\omega t) \tag{C-1}$$

where x is a vector with m components x_i, A is a constant m x m matrix with characteristic roots having negative real parts, the vector functions f(x,t) and g(t) are continuous and periodic in t of period 1 with the average of g(t) over a period equal to zero, ε and ω are constants, and

$$|f(x,t) - f(y,t)| \le \varepsilon c|x-y| \tag{C-2}$$

for small enough $|x| = \sum_{i=1}^{m} |x_i|$, $|y| = \sum_{i=1}^{m} |y_i|$ uniformly in t and some constant c.

Then if ε is sufficiently small or ω is sufficiently large there exists a periodic solution $x^* = p(t)$ of period $1/\omega$. This is stable for $|x(t_0)|$ sufficiently small and the steady state solution to Equation (C-1) as $t\to\infty$ is $x(t) = p(t)$ provided $\varepsilon/(1+\omega)$ is sufficiently small.

The statement (C-2) is somewhat different from that stated in Reference [2] and the following is a correspondingly modified outline of the proof of the approximation method given there. If

$$x(t) = \varepsilon \int_{-\infty}^{t} e^{A(t-\tau)} g(\omega\tau)d\tau + \int_{-\infty}^{t} e^{A(t-\tau)} f(x(\tau),\omega\tau)d\tau \tag{C-3}$$

has a solution, then Equation (C-1) is satisfied by that solution. If a method of successive approximations is defined

$$x^{(0)}(t) = 0 \tag{C-4}$$

$$x^{(n+1)}(t) = \varepsilon\int_{-\infty}^{t} e^{A(t-\tau)}g(\omega\tau)d\tau + \int_{-\infty}^{t} e^{A(t-\tau)}f(x^{(n)}(\tau),\omega\tau)d\tau \tag{C-5}$$

then

$$x^{(1)}(t) = \varepsilon \int_{-\infty}^{t} e^{A(t-\tau)}g(\omega\tau)d\tau \tag{C-6}$$

and for $|\int g(\omega\tau)d\tau|$ uniformly bounded, it can be shown that there is an N such that

$$|x^{(1)}(t)| \le \max|x^{(1)}(t)| \le \varepsilon N/(1+\omega) \tag{C-7}$$

Now, since matrix A has characteristic roots with negative real parts, there exist k and $\sigma > 0$ such that

$$|e^{At}| \le ke^{-\sigma t} \tag{C-8}$$

and

$$|x^{(n+1)}(t)-x^{(n)}(t)| \le k \int_{-\infty}^{t} e^{-\sigma(t-\tau)}|f(x^{(n)}(\tau),\omega\tau)-f(x^{(n-1)}(\tau),\omega\tau)|d\tau \tag{C-9}$$

For $|x^{(n+1)}(\tau)|$ and $|x^{(n)}(\tau)|$ less than some δ, it has been given that

$$|f(x^{(n)}(\tau),\omega\tau) - f(x^{(n-1)}(\tau),\omega\tau)| \leq \epsilon c|x^{(n)}-x^{(n-1)}| \qquad \text{(C-10)}$$

so that if

$$M_n = \max|x^{(n)}(t)-x^{(n-1)}(t)| \qquad \text{(C-11)}$$

then, using Eqs. (C-9) and (C-10)

$$M_{n+1} \leq (\epsilon ck/\sigma)M_n \qquad \text{(C-12)}$$

and

$$|x^{(n)}| = |x^{(n)}-x^{(n-1)}+x^{(n-1)}-x^{(n-2)}+\ldots+x^{(1)}-x^{(0)}| \leq \sum_{i=1}^{n} M_n$$

$$\leq M_1\left[1+\sum_{j=1}^{n-1}(\epsilon ck/\sigma)^j\right] \qquad \text{(C-13)}$$

For ϵ sufficiently small that $\epsilon ck/\sigma < 1$

$$|x^{(n)}| \leq M_1\left[1+\frac{\epsilon ck/\sigma}{1-(\epsilon ck/\sigma)}\right] \leq \max|x^{(1)}|\left[1+\frac{\epsilon ck/\sigma}{1-(\epsilon ck/\sigma)}\right] \qquad \text{(C-14)}$$

Hence, if $\epsilon/(1+\omega)$ is sufficiently small that

$$\frac{\epsilon N}{1+\omega}\left[1 + \frac{\epsilon ck/\sigma}{1-\epsilon ck/\sigma}\right] < \delta \qquad \text{(C-15)}$$

the above formulas are valid and for ϵ sufficiently small that $\epsilon ck/\sigma$ is a very small quantity, $x^{(1)}(t)$ is a good approximation to the steady state solution of Equation (C-1).

APPENDIX D

Angular Momentum Requirements for CMGs

The equations of motion for the CMG configuration may be written in terms of the configuration composite spin anglar momentum ${}^{C}\underline{H}$ and output torque ${}^{C}\underline{T}$.

$$ {}^{C}\dot{\underline{H}} + \underline{\omega}^{A} \times {}^{C}\underline{H} = -{}^{C}\underline{T} \qquad \text{(D-1)} $$

Or, written in terms of components associated with unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ fixed in A,

$$ \dot{H}_1 + u_2H_3 - u_3H_2 = K_1\dot{\phi}_1 + K_0\phi_1 \qquad \text{(D-2)} $$

$$ \dot{H}_2 + u_3H_1 - u_1H_3 = K_1\dot{\phi}_2 + K_0\phi_2 \qquad \text{(D-3)} $$

$$ \dot{H}_3 + u_1H_2 - u_2H_1 = K_{13}\dot{\phi}_3 + K_{03}\phi_3 \qquad \text{(D-4)} $$

Now, considering steady state conditions, the motion of body A is given to the first power in δ by Equations (15)-(17),

$$ \phi_1 = \delta(-p_1\sin\omega t + p_2\cos\omega t) \qquad \text{(D-5)} $$

$$ \phi_2 = \delta(p_2\sin\omega t + p_1\cos\omega t) \qquad \text{(D-6)} $$

$$ \phi_3 = 0 \qquad \text{(D-7)} $$

where p_1 and p_2 are defined

$$ p_1 = (1+R_0)/[(1+R_0)^2 + R_1^2] \qquad p_2 = R_1/[(1+R_0)^2 + R_1^2] \qquad \text{(D-8)} $$

Substitutions of Equations (D-5)-(D-7) into Equations (A-3)-(A-5) and placement of the resulting values of u_1, u_2, u_3 (carrying only first powers of δ) into Equations (D-2)-(D-4) yields

$$\dot{H}_1 + \dot{\phi}_2 H_3 = K_1\dot{\phi}_1 + K_0\phi_1 \quad \text{(D-9)}$$

$$\dot{H}_2 - \dot{\phi}_1 H_3 = K_1\dot{\phi}_2 + K_0\phi_2 \quad \text{(D-10)}$$

$$\dot{H}_3 + \dot{\phi}_1 H_2 - \dot{\phi}_2 H_1 = 0 \quad \text{(D-11)}$$

Multiplying Equation (D-9) by $\dot{\phi}_1$, Equation (D-10) by $\dot{\phi}_2$, and adding the result, it can be shown that

$$\dot{H}_1\dot{\phi}_1 + \dot{H}_2\dot{\phi}_2 = \delta^2 K_1 (p_1{}^2 + p_2{}^2) \quad \text{(D-12)}$$

which implies that H_1 and H_2 are proportional to δ and by Equation (D-11) H_3 is proportional to δ^2. Then, to the first power in δ,

$$\dot{H}_1 = K_1\dot{\phi}_1 + K_0\phi_1 \quad \text{(D-13)}$$

$$\dot{H}_2 = K_1\dot{\phi}_2 + K_0\phi_2 \quad \text{(D-14)}$$

$$\dot{H}_3 = 0 \quad \text{(D-15)}$$

Integrating and considering the mean value of H_1, H_2, and H_3 to be zero in order to minimize $|{}^C\underline{H}|$, then the amplitude $\overline{H}$ of the ${}^C\underline{H}$ vector is given to the first power in δ by

$$\bar{H} = \sqrt{H_1^2 + H_2^2 + H_3^2} = H_0 \sqrt{\frac{R_0^2 + R_1^2}{(1+R_0)^2 + R_1^2}} \qquad \text{(D-16)}$$

where R_0, R_1 are defined by Equations (18) and

$$H_0 = mr\ell\omega \qquad \text{(D-17)}$$