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COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Determination of Dynamic Loads and Response of a Space Vehicle Using Flight Data

TM-69-2031-4

DATE- August 14, 1969

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AUTHOR(S)- S. N. Hou

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ABSTRACT

An analytical method based on statistical properties of power spectral density functions obtained from actual flight has been developed for determining the maximum intensity level of the residual thrust oscillation and dynamic response of a space vehicle.

The method yields the expected intensity, the standard deviation, and the distribution of the maximum intensity across the ensemble of either excitation or response, which are treated as non-stationary, random processes. It is found that the square of such maximum intensity follows a Gumbel distribution. Thus, a limited number of flight records can provide sufficient information for structural design by using the expected maximum intensity, plus certain standard deviations, or by finding an intensity level which represents a specified probability of occurrence.

The advantages offered by this method are to provide: (1) an analytical base for establishing the residual thrust oscillation specification for a space vehicle, (2) a direct estimate of maximum structural response levels over combined loading effects, and (3) information for structural anomaly studies as well as for structural design.

To verify the theory and demonstrate the technique, two numerical examples are presented. The first example uses four F-1 engine thrust records and their associated power spectral density functions from the Apollo 6 flight. the second example uses 1,000 simulated random samples of structural response; at the 95% confidence level, the Komogorov-Smirnoff goodness of fit test indicates that the assumed distribution is acceptable.

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TECHNICAL MEMORANDUM

INTRODUCTION

An assessment of space vehicle structural capability under launch loads requires knowledge of the nominal force exerted on the vehicle, and the variation of this force as a function of time during the boost phase. This latter quantity, which has been referred to as residual thrust oscillation, is a random phenomenon, and as such should be treated statistically. Structural capability should also be assessed by a direct statistical evaluation of maximum structural response. Conditions can occur such that both excitation and response are a combination of random and deterministic oscillations, (i.e., POGO). Additional considerations, which are not included in this text, should be given for the additional effects of such deterministic excitations to the random process.

As we know, a power spectral density function across the ensemble of the process can be obtained directly by the spectral analysis of the sample vibration records. It would be convenient if we could associate all the statistical properties of the process with the power spectral density function, $S(\omega)$. However, we expect that such a power spectral density function will be non-stationary, with time variation in its overall intensity and the power distribution.

In the following, analytical solutions are presented for solving the mean, variance, and distribution of the maximum oscillation of a random process, based on a fixed $S(\omega)$. Then, a piecewise stationary approach is introduced for extending such solutions to the process with time varying $S(\omega)$. The influence of the means, variances, and correlations of those dominant parameters on the distribution of the process so solved is found in approximate analytical terms.

STATIONARY PROCESS -- TIME-INVARIANT $S(\omega)$

A power spectral density function, $S(\omega)$, can be interpreted as a statistical average of frequency decomposed sample functions of a random process. By its physical meaning, the area under the curve of $S(\omega)$ is equivalent to the mean square value, x^2, of the process:

$$x^2 = \int_{\omega} S(\omega)d\omega, \tag{1}$$

where x may be considered as an averaged intensity of the process. However, for structural design and vibration studies, the extreme peak which may occur in a finite duration of time should receive more attention than just an averaged intensity alone. Since the value of x is directly associated with $S(\omega)$ and can be obtained easily through equation (1), let us choose x for expressing other statistical properties of the process.

Let excitation or structural response be a stationary random process, $Q(t)$, of zero-mean, and let $\dot{Q}(t)$ be the slope of $Q(t)$ at any time. For any arbitrary level of $Q=a$, the expected number of crossings per unit time with positive slope is designated as N_a, an ensemble average. According to S. O. Rice[4]

$$N_a = \int_0^\infty \dot{Q}\, p(a,\dot{Q})d\dot{Q}, \tag{2}$$

where $p(a,\dot{Q})$ is a joint distribution of $Q(t)$ and $\dot{Q}(t)$, with $Q=a$, and is assumed to be a joint Gaussian distribution

$$p(Q,\dot{Q}) = \frac{1}{2\pi\sigma_Q\sigma_{\dot{Q}}} \exp\left[-\frac{1}{2}\left(\frac{Q^2}{\sigma_Q^2} + \frac{\dot{Q}^2}{\sigma_{\dot{Q}}^2}\right)\right], \tag{3}$$

where

$$\sigma_Q^2 = \int_{-\infty}^{\infty} S(\omega)d\omega = x^2 \tag{4}$$

$$\sigma_{\dot{Q}}^2 = \int_{-\infty}^{\infty} \omega^2 S(\omega)d\omega. \tag{5}$$

Let us designate ω_0 and T_0 as

$$\omega_0^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S(\omega)d\omega}{\int_{-\infty}^{\infty} S(\omega)d\omega} = \frac{\int_0^{\infty} \omega^2 S(\omega)d\omega}{\int_0^{\infty} S(\omega)d\omega} = \frac{\sigma_{\dot{Q}}^2}{\sigma_Q^2}, \tag{6}$$

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \frac{\sigma_Q}{\sigma_{\dot{Q}}}, \tag{7}$$

which can be interpreted as the mean frequency and period of Q(t), respectively. Thus, substituting these expressions into equation (2), we have:

$$N_a = \frac{1}{2\pi} \frac{\sigma_{\dot{Q}}}{\sigma_Q} \exp\left(-\frac{a^2}{2\sigma_Q^2}\right) \tag{8}$$

or

$$N_a = \frac{1}{T_0} \exp\left(-\frac{a^2}{2x^2}\right). \tag{9}$$

One approach for finding the expected extreme peak value, m, in a specific duration S is to let the expected number of crossings over the double levels, Q(t) = m and Q(t) = -m, equal to one:

$$N_{|m|}S = (2N_m)S = \frac{2S}{T_0} \exp\left(-\frac{m^2}{2x^2}\right) = 1. \tag{10}$$

Solving for m, the expected extreme peak in a duration S is:

$$m = \left[2 \log_e \left(\frac{2S}{T_0}\right)\right]^{1/2} x. \tag{11}$$

Another approach, based on an additional assumption[3] that the time $T^{(j)}$ for a sample function $q^{(j)}(t)$ to reach the double levels $q(t) = \pm a$ for the first time follows Poisson process across the ensemble, will lead to a similar conclusion as follows.

The probability distribution for n crossings in a duration S is:

$$P(n, N_{|a|}S) = \frac{(N_{|a|}S)^n}{n!} e^{-N_{|a|}S}. \tag{12}$$

Then

$$\begin{aligned} F(a) &= P\begin{bmatrix}\text{Extreme peak} < a \\ \text{in a duration S}\end{bmatrix} \\ &= P\begin{bmatrix}\text{No crossing over} \\ \text{double levels } |a| \\ \text{in a duration S}\end{bmatrix} \\ &= P(0, N_{|a|}S) \\ &= e^{-N_{|a|}S}. \end{aligned} \tag{13}$$

Let $\tau = N_{|a|}S$, we get

$$\begin{aligned} p(a)da &= dF(a) \\ &= de^{-\tau} \\ &= -e^{-\tau}d\tau . \end{aligned} \tag{14}$$

From equation (9), we get

$$\tau = N_{|a|}S = (2N_a)S = \frac{2S}{T_0}\exp\left(-\frac{a^2}{2x^2}\right) \tag{15}$$

$$a = \left(2\log_e \frac{2S}{T_0} - 2\log_e\tau\right)^{1/2}x$$

$$= \left[\left(2\log_e \frac{2S}{T_0}\right)^{1/2} - \frac{\log_e\tau}{\left(2\log_e \frac{2S}{T_0}\right)^{1/2}} - \frac{1}{2}\frac{\log_e^2\tau}{\left(2\log_e \frac{2S}{T_0}\right)^{3/2}} + \cdots\right] x. \tag{16}$$

Hence, the expected extreme peak in a duration S is:

$$m = E[a] = \int_{\infty}^{\infty} ap(a)da = x\int_0^{\infty}\left[\left(2\log_e \frac{2S}{T_0}\right)^{1/2} - \frac{\log_e\tau}{\left(2\log_e \frac{2S}{T_0}\right)^{1/2}} - \frac{1}{2}\frac{\log^2\tau}{\left(2\log_e \frac{2S}{T_0}\right)^{3/2}} + \cdots\right]e^{-\tau}d\tau$$

$$= \left[\left(2\log_e \frac{2S}{T_0}\right)^{1/2} + \frac{C}{\left(2\log_e \frac{2S}{T_0}\right)^{1/2}} - \frac{1}{2}\frac{\frac{\pi^2}{6} + C^2}{\left(2\log_e \frac{2S}{T_0}\right)^{3/2}} + \cdots\right]x, \tag{17}$$

where

$$C = -\int_0^{\infty}\log\tau\exp(-\tau)d\tau = 0.5772$$

= Euler's constant.

Notice that the solution given by Equation (11) is equivalent to the first term of the solution given by equation (17). Generally, the ratio between S and T_0 is large. Thus these two solutions are very close (See Example 1, Appendix).

Similarly, the mean square of "a" can be found as:

$$E[a^2] = \int_{-\infty}^{\infty} a^2 p(a)da = x^2 \int_0^{\infty} \left[2\log_e \frac{2S}{T_0} - 2\log_e \tau\right] e^{-\tau} d\tau$$

$$= \left[2\log_e \frac{2S}{T_0} + 2C\right] x^2 . \qquad (18)$$

Thus, the variance, σ^2, of the extreme peak in a duration S is:

$$\sigma^2 = E[a^2] - m^2$$

$$= x^2 \left[\left(\log_e \frac{2S}{T_0} + 2C\right) - \left(2\log_e \frac{2S}{T_0} + 2C + \frac{C^2}{2\log_e \frac{2S}{T_0}} - \frac{\frac{\pi^2}{6} + C^2}{2\log_e \frac{2S}{T_0}} + \cdots\right)\right]$$

$$= x^2 \left[\frac{\pi^2}{6} \frac{1}{2\log_e \frac{2S}{T_0}} + \cdots\right] , \qquad (19)$$

and the standard deviation of the extreme peak in a duration S is

$$\sigma = \frac{\pi}{\sqrt{6}} \left(2\log_e \frac{2S}{T_0}\right)^{-1/2} x . \qquad (20)$$

After completing the derivation of m and σ in terms of x for the extreme peak in a specific duration S, now let us study the probabilistic distribution for such peak to occur at any arbitrary level "a". Let us designate

$$y = -\log_e N_{|a|}S. \tag{21}$$

From equation (15), we have:

$$\begin{aligned} y &= -\log_e\left[\frac{2S}{T_0}\exp\left(-\frac{a^2}{2x^2}\right)\right] \\ &= \frac{a^2}{2x^2} - \log_e \frac{2S}{T_0} \\ &= \frac{1}{2x^2}\left[a^2 - 2x^2 \log_e \frac{2S}{T_0}\right]. \end{aligned} \tag{22}$$

Substituting into equation (13), we get

$$F(a) = e^{-e^{-\frac{1}{2x^2}\left[a^2 - 2x^2 \log_e \frac{2S}{T_0}\right]}}. \tag{23}$$

Thus, the cumulative probability distribution of extreme peak over any double levels of "$\pm$a" can be expressed in form of a Gumbel distribution[5] as follows;

$$\begin{aligned} F(a) &= e^{-e^{-\alpha(a^2-\mu)}} && \text{for} \quad a^2 \geq 0 \\ &= 0 && \text{for} \quad a^2 < 0, \end{aligned} \tag{24}$$

where $\alpha = \frac{1}{2x^2}$ and $\mu = 2x^2 \log_e \frac{2S}{T_0}$.

NONSTATIONARY PROCESS -- TIME-VARYING $S(\omega)$

The spectral analysis of vibration records from space vehicles indicates that the power spectral density function across the ensemble varies with time. However, such variations are in a gradual transition fashion, which means that the function usually maintains a certain degree of consistency in the overall intensity and the power distribution of its contained frequencies for successive time intervals. Such intervals may have durations ranging from several seconds to several tens of seconds. As we know, when a process has small intensity change in an interval, and such an interval is relatively much longer than the periods of its autocorrelation function, then "local stationarity" can be applied to the process.(1) Thus, equivalent solutions for statistical properties in each interval are obtained by using a nominal (averaged) power spectral density function and a fixed duration. The feasibility and accuracy of such solutions can be judged by the coefficients of variation of the statistical properties so obtained. These coefficients are derived in approximate analytical terms as follows:

According to the solutions of m and σ given in equations (11) and (20), their values are determined by three parameters:

1. duration, S,

2. expected intensity, x, and

3. mean period, T_0, of the process.

Since S is rationally fixed, only x and T_0 remain random variables. With a given time-varying power spectral density function (which can be expressed in the form of a three-dimensional model), the variances of x and T_0, designated as σ_x^2 and $\sigma_{T_0}^2$ respectively, and their covariance can be estimated numerically over duration S. Now, to find the coefficients of variations for m and σ in terms of known quantities σ_x and σ_{T_0}:

$$\frac{\partial m}{\partial x} = \left(2\log_e \frac{2S}{T_0}\right)^{1/2} = \frac{m}{x}, \qquad (25)$$

$$\frac{\partial m}{\partial T_0} = - \frac{x}{T_0}\left(2\log_e \frac{2S}{T_0}\right)^{-1/2}$$

$$= - \frac{x^2}{T_0 m}, \tag{26}$$

$$\frac{\partial \sigma}{\partial x} = \frac{\pi}{\sqrt{6}}\left(2\log_e \frac{2S}{T_0}\right)^{-1/2} = \frac{\sigma}{x}, \tag{27}$$

$$\frac{\partial \sigma}{\partial T_0} = \frac{\pi}{\sqrt{6}} \frac{x}{T_0}\left(2\log_e \frac{2S}{T_0}\right)^{-3/2}$$

$$= \frac{\sigma}{T_0}\left(2\log_e \frac{2S}{T_0}\right)^{-1} = \frac{\sigma}{T_0} \left(\frac{x}{m}\right)^2 . \tag{28}$$

Thus, based on the theoretical explanation given in references (2) and (6), the variances for random functions m, and σ are found in approximate analytical terms as:

$$\mathrm{Var}[m] = \left(\frac{\partial m}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial m}{\partial T_0}\right)^2 \sigma_{T_0}^2 + 2\left(\frac{\partial m}{\partial x}\right)\left(\frac{\partial m}{\partial T_0}\right)\mathrm{Cov}[x\ T_0]$$

$$= \left(\frac{m}{x}\right)^2 \sigma_x^2 + \left(\frac{x^2}{T_0 m}\right)^2 \sigma_{T_0}^2 - \frac{2x}{T_0} \mathrm{Cov}[x\ T_0], \tag{29}$$

$$\mathrm{Var}[\sigma] = \left(\frac{\partial \sigma}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial \sigma}{\partial T_0}\right) \sigma_{T_0}^2 + 2\left(\frac{\partial \sigma}{\partial x}\right)\left(\frac{\partial \sigma}{\partial T_0}\right)\mathrm{Cov}[x\ T_0]$$

$$= \left(\frac{\sigma}{x}\right)^2 \sigma_x^2 + \left(\frac{\sigma x^2}{T_0 m^2}\right)^2 \sigma_{T_0}^2 + \frac{2\sigma^2 x}{T_0 m^2} \mathrm{Cov}[x\ T_0]. \tag{30}$$

The coefficients of variation are found as:

$$\nu_m = \frac{1}{m}\sqrt{\mathrm{Var}[m]}$$

$$= \left\{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{x^2 \sigma_{T_0}}{T_0 m^2}\right)^2 - \frac{2x}{T_0 m^2} \mathrm{Cov}[x\ T_o]\right\}^{1/2}, \tag{31}$$

$$\nu_\sigma = \frac{1}{\sigma}\sqrt{Var[\sigma]}$$

$$= \left\{ \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{x^2 \sigma_{T_0}}{T_0 m^2}\right)^2 + \frac{2x}{T_0 m^2} \; Cov[x \; T_0] \right\}^{1/2}$$

where quantities of x, m, and T_0 are evaluated at the "nominal" condition.

CONCLUSIONS AND APPLICATIONS

The technique has applications to space vehicle structural design and anomaly studies, which depend on the correct prediction and maximum intensity level of excitation or response. Direct measurements of time varying engine thrust, acceleration, or displacement at any location of the vehicle should be used.

Based on the theories so derived, two alternative ways are presented to get the maximum peak distribution:

1. Setting data points on the standard Gumbel distribution plots (see Appendix, Example 2), the whole distribution may easily be defined by drawing a best-fit line.

2. Obtain power spectral density functions and compute the distribution analytically (see Appendix, Example 1).

Two alternative criteria for determing dynamic loads or response for design are presented, which allow for a determination of a probabilistic oscillation intensity level:

1. A maximum intensity of oscillation which is equal to the expected value plus certain standard deviations. Let $a = m + 3\sigma$ and use "a" for design.

2. An assigned probability which indicates the confidence level for the occurrence of the maximum intensity. Let $F(a) = 0.99$, which means that in 99 out of 100 chances, the maximum intensity will fall within the design limit. Then, solve equation (24) for "a" and use "a" for design.

Since any unusual energy concentrations are reflected in the power spectral density functions and eventually effect the result of the maximum intensity distribution, this technique will yield information for structural anomaly studies, as well as for structural design. Because the maximum response is analyzed by records measured directly from the site, distribution of maximum response represents a combined effect of all possible loads to the site. Thus, it saves us from the tedious and complicated work of evaluating individual loading effects. As such, errors caused by unrealistic superpositions are eliminated.

In order to demonstrate the technique, two numerical examples are presented in the Appendix. Owing to a limited number of available flight records, the first example uses only four F-1 engine thrust records from the Apollo 6 flight. Their associated power spectral density functions are attached. Even though the number of samples used is small and some non-random excitation occurred during this time interval (i.e. "POGO"), at the 95% confidence level, the Komogorov-Smirnoff goodness of fit test indicates that the assumed distribution is acceptable. The second example uses 1,000 stationary white noise samples generated by A. G. Brady[9]. Since a large number of sample simulations are not only costly, but also time consuming, we will take advantage of his work to verify our maximum peak distribution theory. Results indicate that it does indeed follow the Gumbel distribution.

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2031-SNH-scs S. N. Hou

Attachments
- Appendix
- Notations
- References

APPENDIX

Numerical Examples

Two numerical examples are presented to illustrate the techniques so derived in actual engineering applications: The first example shows the procedures of estimating the distribution of the maximum intensity of the residual thrust by either analytically using the power spectral density functions, or directly using oscillation records. Results of these two approaches are compared. The second example uses extreme peak values of structural response from 1000 simulated random excitations[9], and verify that the nature of the maximum intensity indeed follow the Gumbel distribution. Both examples are treated as stationary oscillations with finite durations. As explained in the main text, the non-stationary aspects can be dealt with similarly by means of piecewise stationary procedures.

I. Example 1:

Residual thrust oscillations measured from four F-1 engines of Apollo 6 at time slice +110.00 to +120.00 (10 second duration) are used as stationary samples (Figures 1-4). Their corresponding power spectral density functions are shown in Figures 5-8. The following properties are observed:

Sensor No.	Mean Pressure (psi)	Maximum Pressure from mean	Area Under PSD curve
D8-101	1151.17	16.7	19.1
D8-102	1154.66	14.5	27.8
D8-104	1156.00	17.0	20.0
D8-105	1151.33	13.3	21.6

Thus, the mean pressure across the ensemble is:

$$\frac{1}{4}\,(1151.17 + 1154.66 + 1156.00 + 1151.33) = 1153.29 \text{ psi.} \quad (33)$$

A. Analytical solutions using Power Spectral Density Functions:

From equation (1), the area under PSD curve, $S(\omega)$, is equivalent to the mean square intensity of the residual pressure. Thus the mean square intensity across the ensemble is:

$$x^2 = \left(\int_\omega S(\omega)d\omega \right)_{ave.}$$

$$= \frac{1}{4} (19.1 + 27.8 + 20.0 + 21.6)$$

$$= 22.1 \text{ (psi)}^2. \quad (34)$$

From equations (6) and (7), the expected mean frequency and its corresponding period is estimated as:

$$\omega_o = \left(\frac{\int_\omega \omega^2 \, S(\omega)d\omega}{\int_\omega S(\omega)d\omega} \right)^{1/2}_{ave.} \approx 5.5 \text{ cps.} \quad (35)$$

$$T_o = \frac{1}{5.5} = 0.182 \text{ sec.} \quad (36)$$

The duration of records is 10 seconds,

$$S = 10 \text{ sec.} \quad (37)$$

Thus, by evaluating the following two parameters,

$$\alpha = \frac{1}{2x^2} = \frac{1}{2(22.1)} = 0.0226, \quad (38)$$

$$\mu = 2x^2 \log_e \frac{2S}{T_o}$$

$$= 2(22.1) \log_e \frac{(2)(10)}{0.182}$$

$$= (44.2)\ (4.7)$$

$$= 207.5 \ , \quad (39)$$

the expected cumulative distribution of the maximum residual pressure "a" is defined from equation (24) as:

$$F(a) = e^{-e^{-0.026(a^2 - 207.5)}}, \tag{40}$$

which is plotted in Figure 9 as a straight line $a^2 = \mu + \frac{1}{\alpha}$.

The mean and standard deviation of the maximum residual pressure are computed using equations (17) and (20) as:

$$m = E[a]$$

$$= \left[\left(2 \log_e \frac{2S}{T_o}\right)^{1/2} + \frac{0.5772}{(2 \log_e \frac{2S}{T_o})^{1/2}} + \ldots\right] x$$

$$= \left(2x^2 \log_e \frac{2S}{T_o}\right)^{1/2} + \frac{0.5772x^2}{(2x^2 \log_e \frac{2S}{T_o})^{1/2}} + \ldots$$

$$= \mu^{1/2} + \frac{0.5772x^2}{\mu^{1/2}}$$

$$= \sqrt{207.5} + \frac{(0.5772)(22.1)}{\sqrt{207.5}}$$

$$= 14.4 + 0.885 = 15.29 \text{ psi}, \tag{41}$$

$$\sigma = \frac{\pi}{\sqrt{6}} \left(2 \quad \log_e \frac{2S}{T_o}\right)^{-1/2} x$$

$$= \frac{\pi}{\sqrt{6}} \frac{x^2}{\left(2x^2 \log_e \frac{2S}{T_o}\right)^{1/2}}$$

$$= \frac{\pi}{\sqrt{6}} \frac{x^2}{\mu^{1/2}}$$

$$= 1.283 \left(\frac{22.1}{\sqrt{207.5}}\right) = 1.97 \text{ psi.} \tag{42}$$

Notice that the second term on the right in the computation of mean is 0.885, which is much smaller in comparison with the first term, 14.4. Thus, equation (11) is an acceptable approximation of equation (17).

Based on information so computed, two criteria for determining maximum intensity level of engine thrust, which may be used in design are illustrated as follows:

1) Use the mean of maximum pressure plus certain (say 1) standard deviations. Thus,

$$1153.29 + 15.29 + 1 \times 1.97 = \underline{\underline{1170.55}} \text{ psi} \tag{43}$$

is the design pressure. From equation (40) it represents a confidence level with a probability of 0.86:

$$(15.29 + 1.97)^2 = 300$$

$$F\left(\sqrt{300}\right) = \underline{\underline{0.86}}. \tag{44}$$

2) Use assigned confidence level. Let the probability of occurrence be 0.8,

$$F(a) = 0.80. \tag{45}$$

Using equation (40) or curve in Figure 9, we get

$$a^2 = 276$$
$$a = 16.6 \text{ psi}, \qquad (46)$$

Thus the design pressure is

$$1153.29 + 16.6 = \underline{\underline{1169.9}} \text{ psi}, \qquad (47)$$

B. Direct Solutions using Data Plotted from Oscillation Records:

The maximum pressures, one value per record and measured from mean value, are listed in ascending order. Their cumulative probabilities are computed as follows

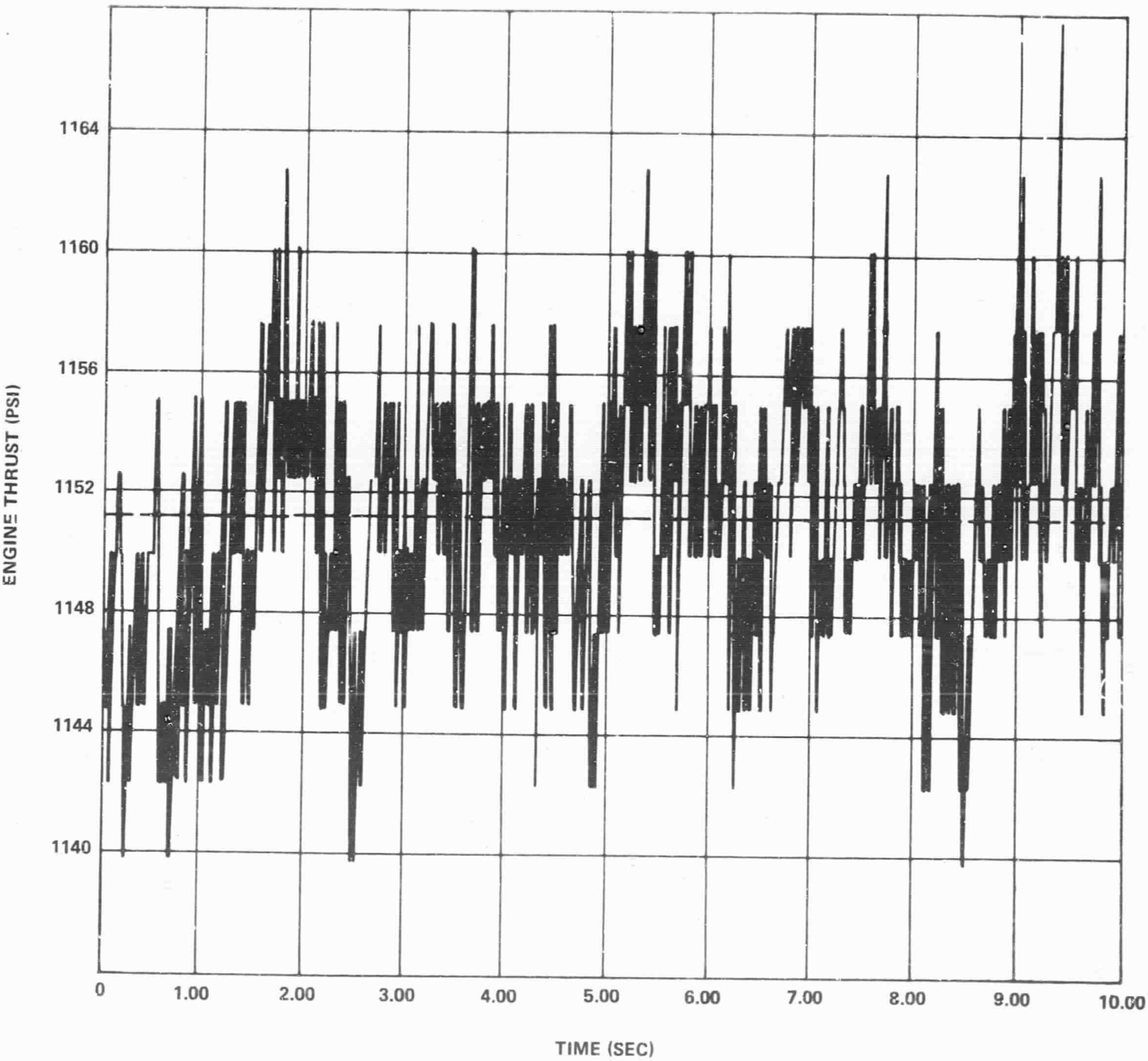
Order	Maximum Pressure		Freq. of Occurrence	Cumulative Freq. Ω	$F(a) = \frac{\Omega}{1+N}$
i	a	a^2			
1	13.3	176	1	1	0.2
2	14.5	210	1	2	0.4
3	16.7	278	1	3	0.6
4	17.0	288	1	4	0.8

where N = total number of samples. Then plot the data points on the special Gumbel paper (see Figure 9) using computed a^2 and F(a). Thus, a distribution of the maximum pressure may be obtained by drawing a best fit line through these data points.

The mean and standard deviation of the maximum pressure can also be computed directly from oscillation records as:

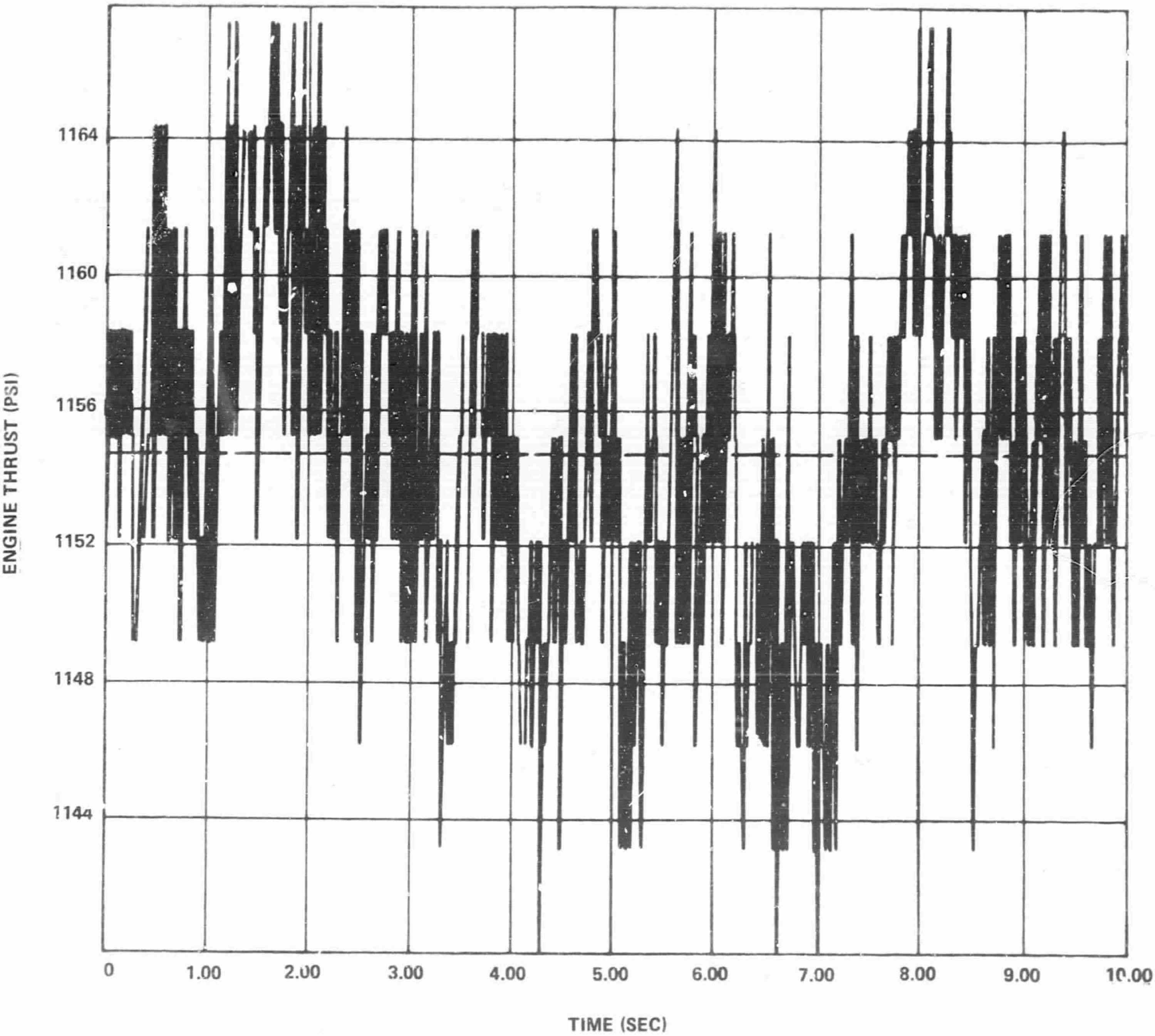
$$m = \frac{1}{4}(13.3 + 14.5 + 16.7 + 17.0) = 15.4 \qquad (48)$$

$$\sigma = \frac{1}{4}\left[\overline{2.1}^2 + \overline{0.9}^2 + \overline{1.3}^2 + \overline{1.6}^2\right]^{1/2} = 1.53. \qquad (49)$$



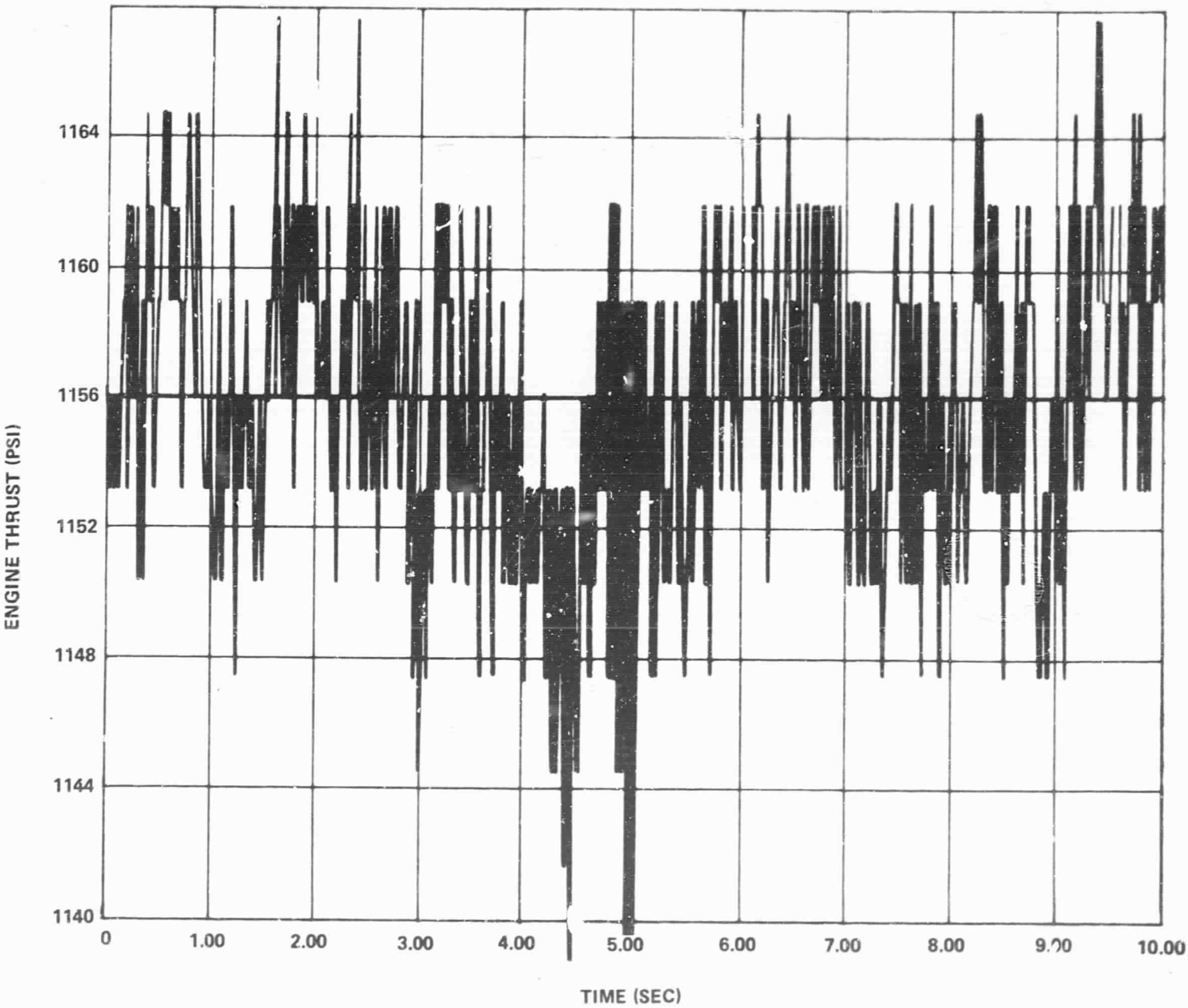
SENSOR NO. D8 - 101

FIGURE 1



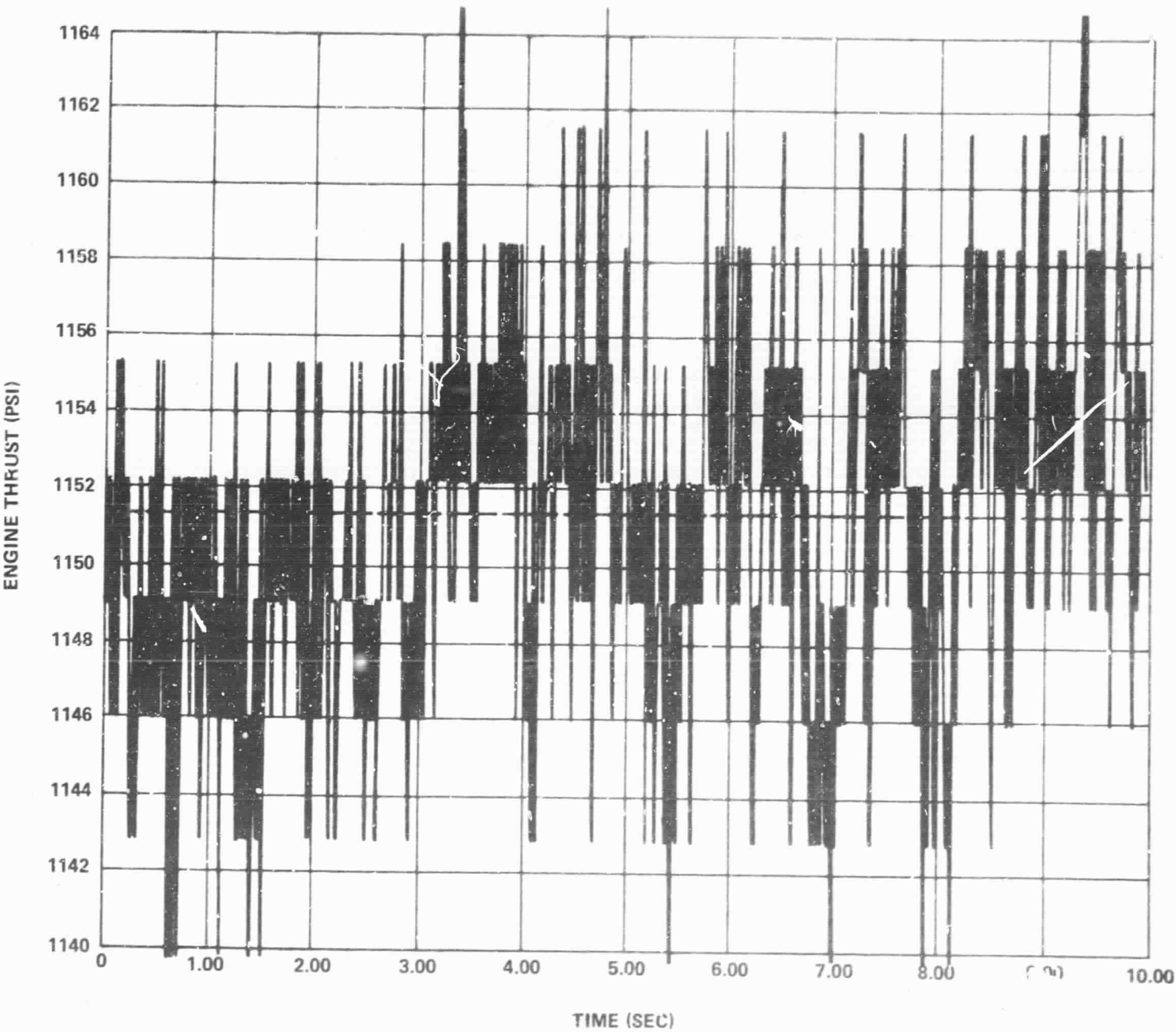
SENSOR NO. D8 - 102

FIGURE 2



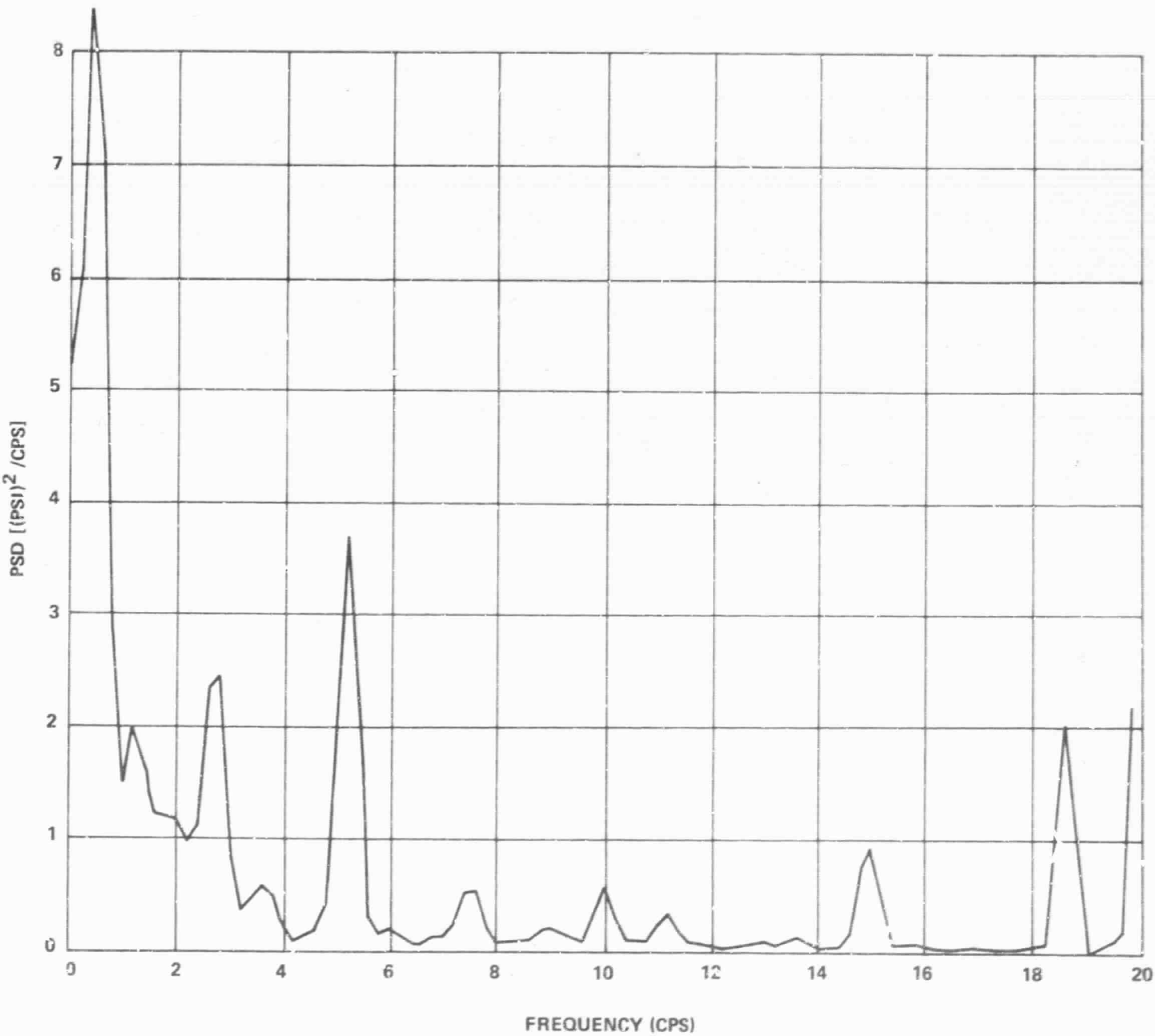
SENSOR NO. D8 -104

FIGURE 3



SENSOR NO. D8 -105

FIGURE 4



SENSOR NO. D8-101

FIGURE 5

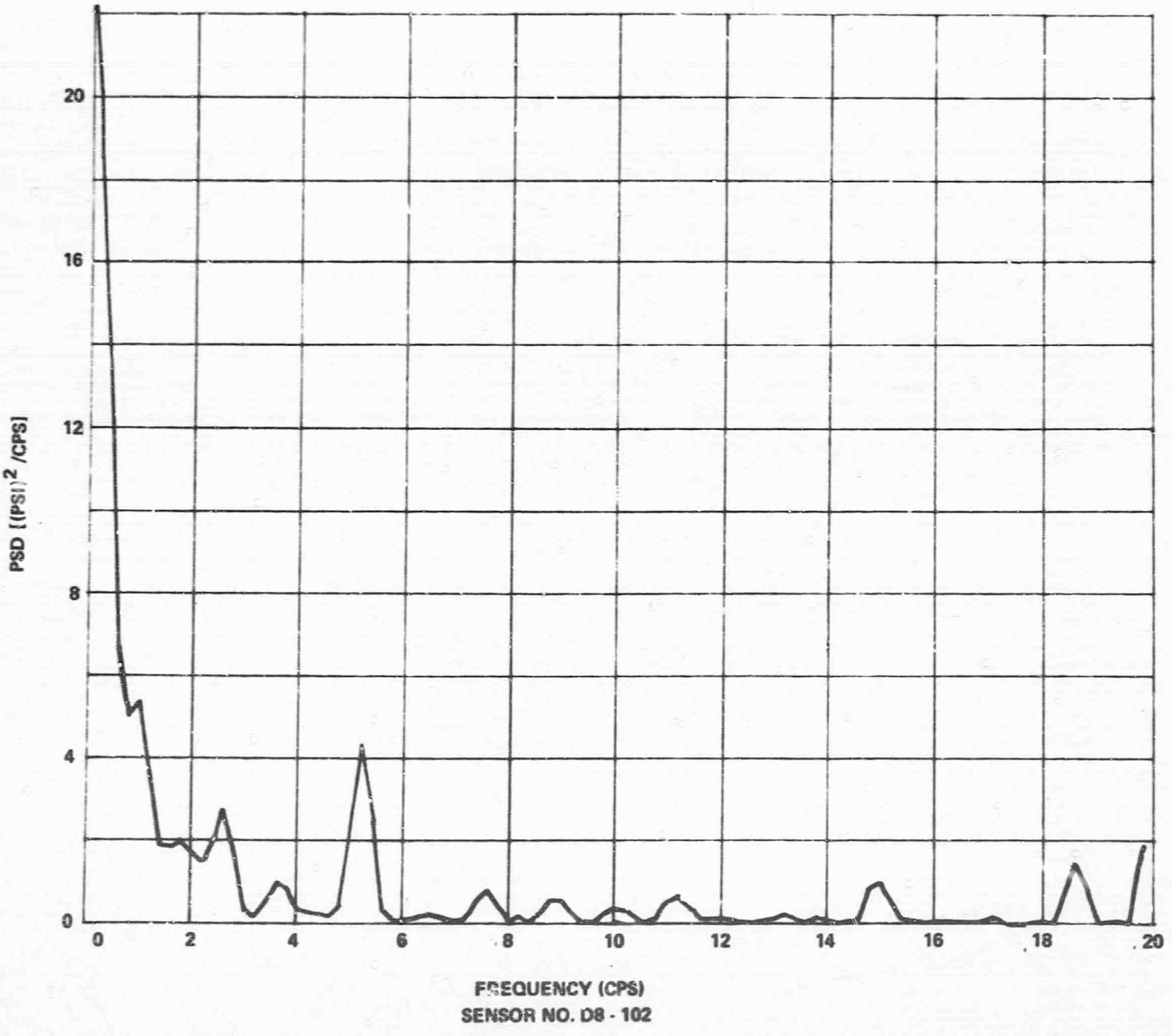
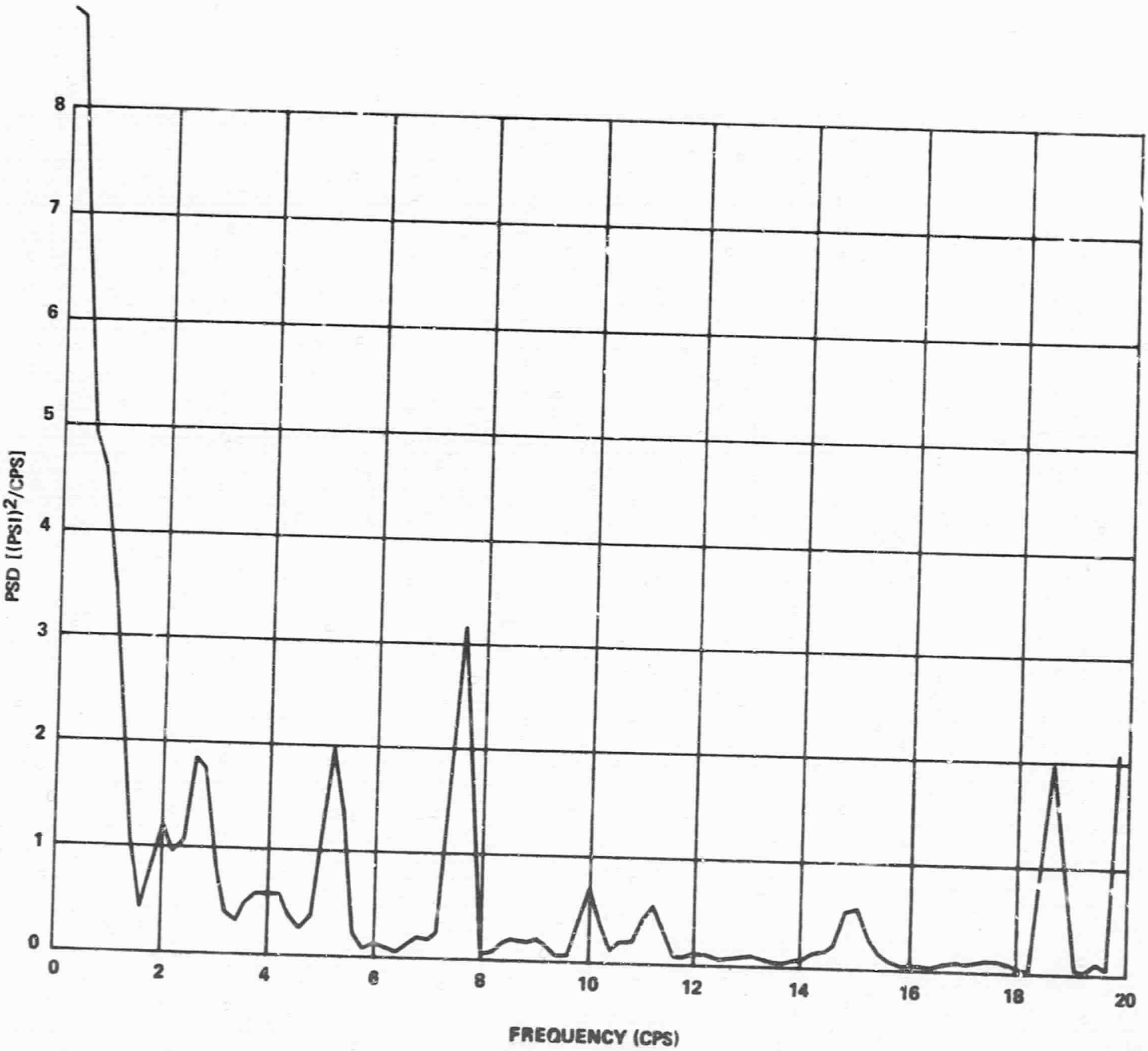
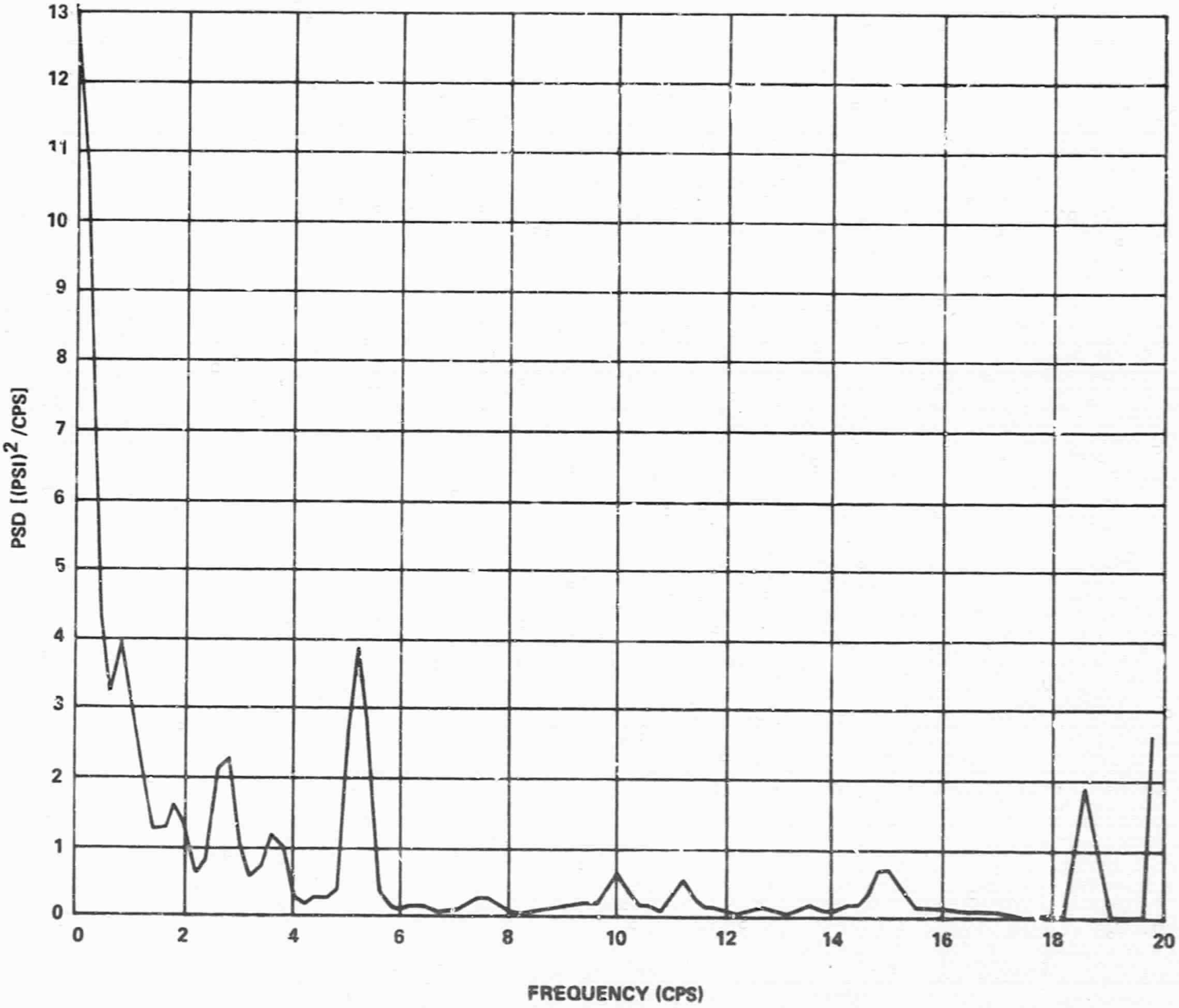

PSD [(PSI)^2 /CPS]
20
16
12
8
4
0
0
2
4
6
8
10
12
14
16
18
20
FREQUENCY (CPS)
SENSOR NO. D8 - 102

FIGURE 6



SENSOR NO. D8 - 104

FIGURE 7



SENSOR NO. D8 - 105

FIGURE 8

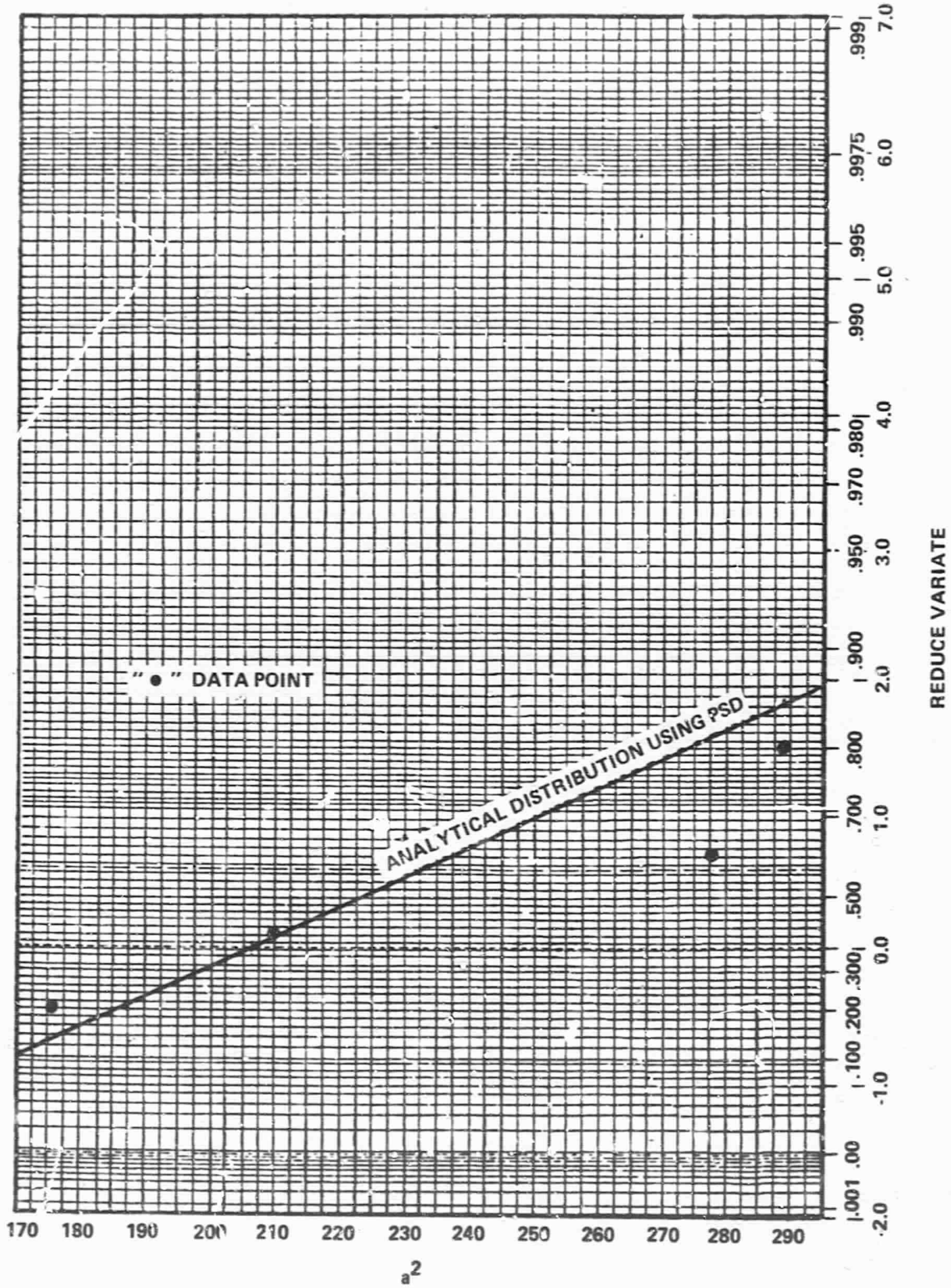

"●" DATA POINT
ANALYTICAL DISTRIBUTION USING PSD
170 180 190 200 210 220 230 240 250 260 270 280 290
a2
.001 .00 .100 .200 .300 .500 .700 .800 .900 .950 .970 .980 .990 .995 .9975 .999
-2.0 -1.0 0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0
REDUCE VARIATE

FIGURE 9

C. Comparison between Analytical Solutions and Direct Solutions:

Even though the number of samples used is extremely small, the results are encouraging:

	Analytical Solution	Direct from Data
m	15.29	15.4
σ	1.97	1.53

The distribution of the maximum pressure obtained analytically is shown by a straight line in Figure 9. Now, comparing this line with the data points obtained directly from records, it is found that at 95% confidence level (or 5% significant level), the Kolmogorov-Smirnoff goodness of fit test indicates that the computed distribution is acceptable.

II. Example 2·

Now, let us study 1000 stationary samples simulated by A. G. Brady[9]. The following table are from Brady's results of the maximum response of a single degree-of-freedom linear structure having a natural period of one second, with 2% of critical damping:

Maximum Response a	a^2	Freq. of Occurrence	Cumulative Freq. Ω	$F(a) = \frac{\Omega}{1+N}$
10	100	3	3	0.0029
11	121	8	11	0.0109
12	144	22	33	0.0329
13	169	49	82	0.0819
14	196	60	142	0.1418
15	225	91	233	0.2327
16	256	104	337	0.3366
17	289	112	449	0.4485
18	324	112	561	0.5604
19	361	104	665	0.6643
20	400	88	753	0.7522
21	441	56	809	0.8081
22	484	47	856	0.8551
23	529	37	893	0.8921
24	576	32	925	0.9240
25	625	25	950	0.9490
26	676	12	962	0.9610
27	729	14	976	0.9750
28	784	10	986	0.9850
29	841	3	989	0.9880
30	900	4	993	0.9920
31	961	4	997	0.9960
32	1024	2	999	0.9980
33	1089	1	1000	0.9990

where N = total number of records = 1000.

By plotting the square value of the maximum response "a^2", and their corresponding cumulative distribution "F(a)" on the Gumbel paper (See Figure 10.), we can draw a best fit line of

$$a^2 = 275 + 12.34\ y \quad ,$$

which gives

$$\mu = 275 \quad ,$$

$$\alpha = \frac{1}{12.34} = 0.0081 \quad .$$

The readings at point A is $a_A^2 = 275$, and at point B is $a_B^2 = 345$. Hence, the mean, standard deviation, and the distribution of the maximum response are:

$$m = \mu^{1/2}$$

$$= \sqrt{275} = 16.6 \quad ,$$

$$\sigma = \sqrt{345 - 275} = \sqrt{70} = 8.37 \quad ,$$

$$F(a) = e^{-e^{-0.0081(a^2-275)}} \quad .$$

At 95% confidence level (or 5% significant level), the Komogorov-Smirnoff goodness of fit test indicates that the assumed distribution is acceptable.

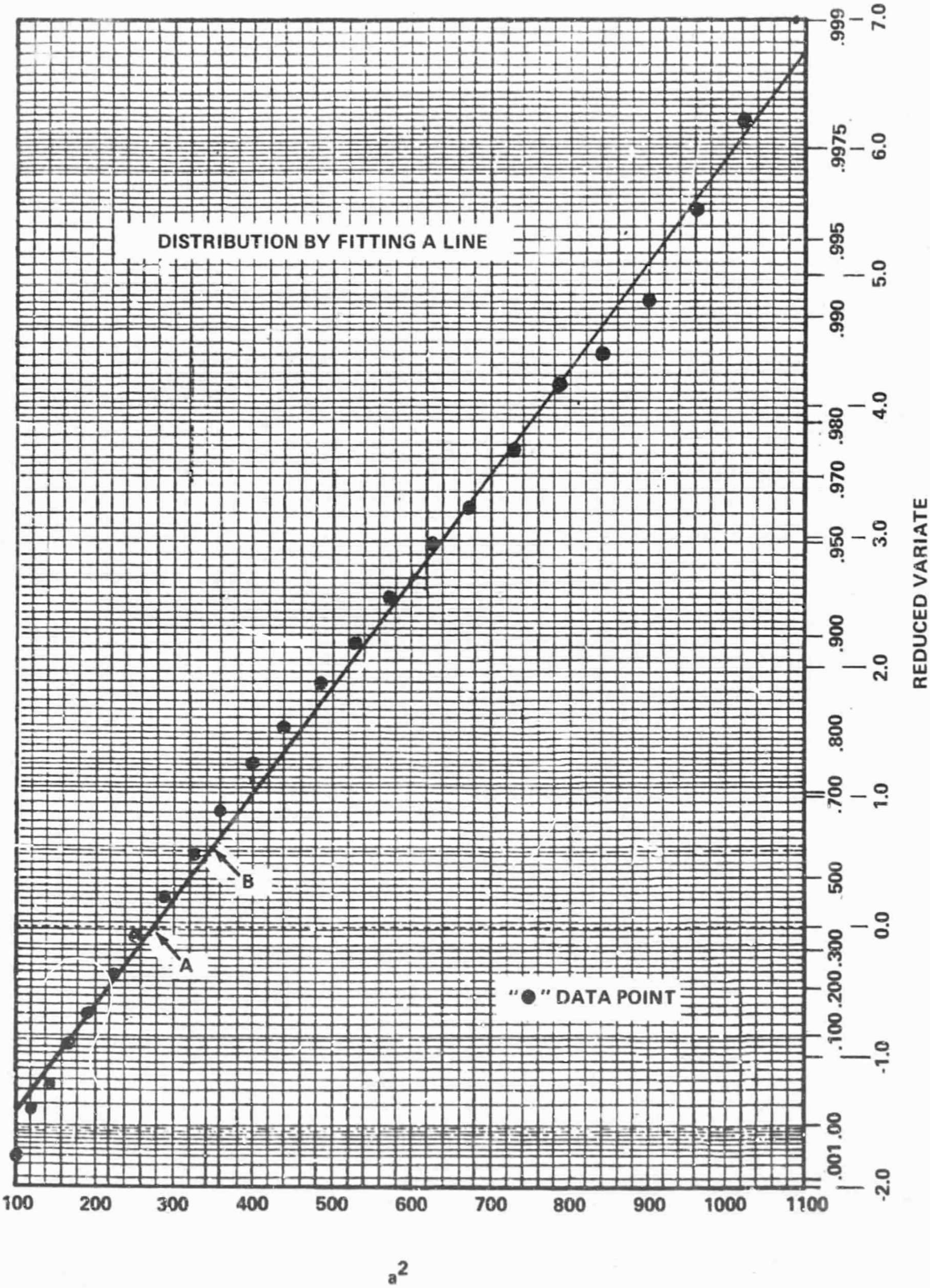

DISTRIBUTION BY FITTING A LINE
"●" DATA POINT
A
B
100
200
300
400
500
600
700
800
900
1000
1100
a²
.999
.9975
.995
.990
.980
.970
.950
.900
.800
.700
.500
.300
.200
.100
.00
.001
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
REDUCED VARIATE

FIGURE 10

NOTATIONS

a	An arbitrary intensity level for the process $Q(t)$.
c	Euler's constant.
$E[a^2]$	Mean square of "a".
$F(a)$	Cumulative distribution of extreme peak up to an intensity level "a" in a duration S of the process $Q(t)$.
m	The expected intensity of extreme peak in a duration S of the process $Q(t)$.
N_a	The expected number of crossings of $Q(t)$ per unit time over an arbitrary intensity level ($Q(t)=a$) with positive slope.
$p(Q,\dot{Q})$	The probability density function of $Q(t)$ and $\dot{Q}(t)$ in joint distribution.
$Q(t)$	A stationary random process with Gaussian distribution and zero mean.
$S(\omega)$	The power spectral density function of the process $Q(t)$.
t	Time.
T_o	The equivalent period of frequency ω_o.
x	Root mean square of the process $Q(t)$, or the averaged intensity of the process.
α	A parameter in Gumbel distribution, which is defined as $\frac{1}{2x^2}$ in the distribution of extreme peak of the process $Q(t)$.
μ	A parameter in Gumbel distribution, which is defined as $2x^2 \log_e(\frac{2S}{T_o})$ in the distribution of extreme peak of the process $Q(t)$.

ν_m	Coefficient of variation for m.
ν_σ	Coefficient of variation for σ.
σ	The standard deviation of the extreme peak in a duration S of the process Q(t).
σ_Q	Standard deviation of the process Q(t).
τ	The expected number of crossing of the process Q(t) with positive slope over double intensity levels (Q(t)=+a and Q(t)=-a) in a duration S.
ω	Vibration frequency.
ω_o	Mean frequency of the process Q(t).

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