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TITLE- Time to Cycle Slip in First and Second Order Phase Lock Loops

TM-69-2034-8

DATE- December 19, 1969

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AUTHOR(S)- L. Schuchman

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ABSTRACT

In this paper, a recursive differential equation for the n^{th} moment of the time-to-cycle slip parameter ($\hat{T}$) is derived for both the first and second order phase lock loop. Explicit results are presented for the first four moments of $\hat{T}$ for both the first and second order phase lock loops. While the moments obtained for $\hat{T}$ in a second order loop are approximately correct, the solutions for the first order loop are exact.

The mean of $\hat{T}$ had been obtained previously by Viterbi and Tauseworthe. The results presented here agree identically with those of these investigators. In addition, the results indicate that the distribution of $\hat{T}$ can be approximated by a Pearson Type III distribution.

The approach taken in deriving results for the first order loop, was to demonstrate that the phase error in the loop is a Markov process so that its transitional frequency function satisfies the Chapman-Kolmogorov equation. Then starting with the Chapman-Kolmogorov equation, a differential equation is derived which the transitional frequency function satisfies. Using this differential equation, one can derive the recursive differential equation satisfied by the moments of $\hat{T}$. This same approach was used by Tikonov and Viterbi in the derivation of the stationary probability density of the phase error. A similar unified approach to the solution of the characterization parameters of a phase lock loop is extended to a second order

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UCLA
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USC
W. C. Lindsey

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*Abstract Only

loop. In addition to the results obtained for the moments of $\hat{T}$, the phase error stationary density distribution is derived. The phase error results agree identically with Viterbi's findings.

The solution for the moment generating function for the first passage time of a process that satisfies the Ornstein-Uhlenbech equation is also given, and explicit expressions for the moments derived.

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

SUBJECT: Time to Cycle Slip in First and Second Order Phase Lock Loops - Case 900

DATE: December 19, 1969

FROM: L. Schuchman

TM-69-2034-8

TECHNICAL MEMORANDUM

I. INTRODUCTION

The estimation of the phase of a sine wave in additive white Gaussian noise is a problem whose solution is of great practical significance. Control people require phase information for tracking purposes while communication types use the phase for coherent reception of carriers so that they may optimally demodulate information transmitted by the carrier. One device used to determine the phase is a phase lock loop. There are two parameters used in measuring the performance of a phase lock loop. The first is based on the assumption that the phase estimate of the device does not differ by more than π radians and is defined as the steady state probability density of the phase error. However, at random times, the noise causes the phase to jump more than π radians. Thus the instantaneous frequency of the sine wave will change by a cycle each time the phase error jumps $\pm 2\pi$. It is common to take the time to skip a cycle ($\hat{T}$), a random variable, as a second parameter whose statistics are a measure of the loops performance.

The phase lock loop is schematically depicted in Figure 1.

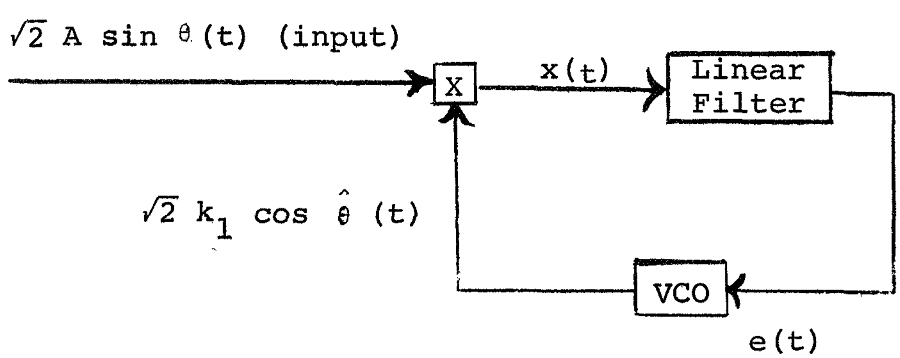

Figure 1. Phase-Lock Loop

The loop consists of three components, a multiplier, a time-invarient linear filter, and a voltage controlled oscillator (VCO). If the loop is operating on a noiseless sine wave then the steady state output of the VCO is a quadrature replica of the input signal which may differ only in amplitude. The multiplier output $x(t)$, under these conditions, will have only a double frequency term which the linear filter and VCO configuration will not pass. The linear filter has a second and equally important function which is to reduce the effects of noise that normally gets into the loop. A phase lock loop is defined to be of n+1 order when there are n finite poles in the Laplace transfer function of its linear filter under the restriction that the number of zeroes of this Laplace transform is no greater than n.

Tikhonov[1,2] was the first to attack the problem of determining the behavior of the phase lock loop in the presence of noise. He determined the stationary phase-error distribution. He did this by showing that the Fokker-Planck differential equation describes the behavior of a first-order phase lock loop. Viterbi[3] extended Tikhonov's work by obtaining an approximate solution for higher order loops and also obtained the mean of $\hat{T}$ for the first order loop. In addition, Viterbi[4] obtained the optimum detector for several binary modulation systems when one has only a noisy estimate of the carrier phase. The significance of this work is the following. In coherent FSK, for example, results had been derived previously based on the assumption of coherent detection. Coherent detection implied one knew the carrier phase perfectly or that the phase estimate given by the phase lock loop was the actual phase of the received signal. Since noise is always present, the assumption is not valid. To obtain a more realistic bit error probability, Viterbi showed that the bit error probability given the phase error between the carrier phase and the loop estimate $P(E/\Theta_{error})$ had to be derived and then averaged with respect to Θ_{error} to obtain $P(E)$.

$$P(E) = \int_{-\Pi}^{\Pi} P(E/\Theta_{error}) P(\Theta_{error}) d\Theta_{error} \tag{1}$$

where

$P(\Theta_{error})$ is the probability density function of the phase error and is taken equal to the stationary phase error distribution.

Viterbi has evaluated equation (1) assuming an optimum partially coherent detection. Lindsey[5] has extended Viterbi's work and has evaluated equation (1) for several real but non-optimum detectors.

Tausworthe[6] extended Viterbi's results on cycle slipping by determining an approximate solution for the mean of $\hat{T}$ for an n^{th} order phase lock loop.

In this memorandum, additional results on cycle slipping are obtained. A technique for obtaining the n^{th} moment of $\hat{T}$ exactly for a first order phase lock loop is given. This result is based on the work of Siegert and Darling.[7] Results are obtained specifically for the first four central moments of $\hat{T}$. An approximate solution for the n^{th} moment of $\hat{T}$ for a second order phase lock loop is derived. This solution is in agreement with Tausworthe's result obtained for the first moment. In addition it is shown that the density function of $\hat{T}$ can best be fitted by a Pearson Type III density function.

In the process of derivation, a technique for characterizing the second order phase lock loop operation in terms of the Kolmogorov differential equations is derived. This approach differs somewhat from the one taken by Viterbi but leads to a simpler derivation of Viterbi's result and in addition other results are obtained.

Finally, it should be pointed out that the Kolmogorov equation to be solved is related to the Ornstein-Uhlenbeck equation used to understand the behavior of certain types of diffusion processes. Thus physicists may find this memorandum of interest since the statistics for the first passage times to a symmetrical boundary for such a process are also given.

II. FIRST ORDER PHASE LOCK LOOP

In this section, the statistics of $\hat{T}$ are obtained. To do this, we briefly sketch the argument which shows that the loop operation can be described as a homogeneous Markov process. This allows one to describe the probabilistic behavior of the loop in terms of differential equations known generally as the Kolmogorov differential equations. The first of these equations is also known as the Fokker-Plank equation and was used by Viterbi[3] to determine the steady state probability density of the first-order loop phase error. The second equation was used by Darling and Siegert[7] to derive results relating to the first passage times of diffusion processes. Our task is then simply to show how Darling's and Siegert's results can be used to obtain the statistics of T.

It is assumed that the loop input is a sine wave plus a stationary narrow band Gaussian process n(t) of zero mean. n(t) may be represented as

$$n(t) = \sqrt{2}\ [n_1(t)\sin\omega_o t + n_2(t)\cos\omega_o t] \qquad (2)^*$$

where $n_1(t)$ and $n_2(t)$ are independent Gaussian processes of zero mean and identical spectral densities. The parameter $\omega_o \frac{rad}{sec}$ is taken as the quiescent frequency output of the VCO. The spectral densities of n(t), $n_1(t)$ and $n_2(t)$ are assumed flat over a sufficiently wide frequency range so that they can be approximated as white with a magnitude equal to No/2.** If we define the input to the phase lock loop as $\sqrt{2}\ A \sin\Theta(t) + n(t)$, where $\Theta(t) = \omega_o t + \Theta_1(t)$, and the VCO output as $\sqrt{2}\ K_1 \cos\hat{\Theta}(t)$ where $\hat{\Theta}(t) = \omega_o t + \Theta_2(t)$ then we may write the multiplier output x(t) as

$$\begin{aligned} x(t) = {} & AK_1\sin[\Theta_1(t)-\Theta_2(t)] - K_1 n_1(t)\sin\Theta_2(t) + K_1 n_2(t)\cos\Theta_2(t) \\ & + AK_1\sin[2\omega_o t+\Theta_1(t)+\Theta_2(t)] + K_1 n_1(t)\sin[2\omega_o t+\Theta_2(t)] \\ & + K_1 n_1(t)\cos[2\omega_o t+\Theta_2(t)] \end{aligned} \qquad (3)$$

Since the VCO will not pass the double frequency terms, they can be ignored. Therefore, when the control signal to the VCO is applied the VCO frequency becomes $\omega_o + K_2 e(t)$ (where K_2 is a proportionallity constant at the VCO whose dimensions are radians per second per volt)and the time derivative of the phase angle of the VCO output phase is given by

$$\frac{d\hat{\Theta}(t)}{dt} = \omega_o + K_2 e(t) \quad \text{or} \quad \frac{d\Theta_2(t)}{dt} = K_2 e(t) \qquad (4)$$

---

*The reader is referred to Section 2.7 of Reference 3 for a more detailed development of the noise model.

**N_o = kT° watts/cps and is the one sided noise spectral density where k = Boltzmann's constant and T° = system noise temperature, °Kelvin.

where

$$e(t) = AK_1\sin[\Theta_1(t)-\Theta_2(t)] - K_1n_1(t)\sin\Theta_2(t) + K_1n_2(t)\cos\Theta_2(t)$$

Defining

$$\phi(t) = \Theta_1(t) - \Theta_2(t)$$

$$K = K_1K_2$$

$$n'(t) = -n_1(t)\sin\Theta_2(t) + n_2(t)\cos\Theta_2(t)$$

equation (4) can be written as

$$\frac{d\phi(t)}{dt} = \frac{d\Theta_1(t)}{dt} - K[A\sin\phi(t)+n'(t)] \tag{5}$$

A process is defined to be Markov if its transition probability distribution (or density function) is a function only of the present value of the process and not of its past values. Letting $\Theta_1(t) = (\omega-\omega_o)t$, (assuming an input signal of constant frequency ω rad/sec) it is seen that the rate of change of the phase error is dependent only upon the present value of the phase error $\phi(t)$ and the noise $n'(t)$. Since Viterbi has shown that we may treat $n'(t)$ as a Gaussian process with a flat spectral density of magnitude No/2 over a sufficiently wide double-sided band of frequencies, we may assume the noise is white. Thus $n'(t)$ is statistically independent of its past and the phase error process is Markov in a first order loop. Viterbi also has shown the following

$$\lim_{\Delta t\to 0} \frac{E[\Delta\phi|\phi]}{\Delta t} = B_1[\Delta\phi|\phi] = (\omega-\omega_o) - AK\sin\phi \tag{6}$$

$$\lim_{\Delta t\to 0} \frac{E[\Delta\phi^2|\phi]}{\Delta t} = B_2[\Delta\phi^2|\phi] = \frac{K^2N_o}{2} \tag{7}$$

$$\lim_{\Delta t \to 0} \frac{E[\Delta\phi^n | \phi]}{\Delta t} = 0 \quad n \geq 3 \tag{8}$$

Equations (6) through (8) are obtained by integrating equation (5) over an infinitesimal interval of time and noting that if ϕ is known $\Delta\phi$ is a Gaussian random process.

We next define $p(\phi_o,\tau;\ \Theta,t)$ as the following conditional probability density function.

$$p(\phi_o,\tau;\ \phi,t) = p(\text{phase error} = \phi \text{ at time } t/\text{phase error} = \phi_o \text{ at time } \tau) \tag{9}$$

Then if we assume the following

$$p(\phi_o,\tau;\ \phi,t),\ \frac{\partial}{\partial\phi}[B_1(\Delta\phi|\phi)p(\phi_o,\tau;\ \phi,t)]$$

and

$$\frac{\partial^2}{\partial\phi 2}[B_2(\Delta\phi^2|\phi)p(\phi_o,\tau;\ \phi,t)]$$

exist, $p(\phi_o,\tau;\ \phi,\ t)$ satisfies the forward Kolmorgorov equation or Fokker-Planck equation which is:

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial\phi}[B_1[\Delta\phi|\phi]p] + \frac{1}{2}\frac{\partial}{\partial\phi}[B_2[\Delta\phi^2|\phi]p] \tag{10}$$

where we have used the short hand notation,

$$p = p(\phi_o\tau;\ \phi,t)$$

Viterbi used equation (10) to derive the long term steady-state (stationary) probability density of the phase error. We can use equations (5) through (8) together with the assumption that $\frac{\partial p}{\partial \phi_o}$ and $\frac{\partial^2 p}{\partial \phi_o^2}$ exist,* to state that p satisfies the backward Kolmogorov equation given by[8]

$$- \frac{\partial p}{\partial \tau} = B_1[\Delta\phi_o|\phi_o] \frac{\partial p}{\partial \phi_o} + \frac{1}{2} B_2[\Delta\phi_o{}^2|\phi_o] \frac{\partial^2 p}{\partial \phi_o^2} \quad (11)$$

Since the Markov process in question is homogeneous $p(t,\tau) = p(t-\tau)$, equation (11) can be written as**

$$\frac{\partial p}{\partial t} = B_1[\Delta\phi_o|\phi_o] \frac{\partial p}{\partial \phi_o} + \frac{1}{2} B_2[\Delta\phi_o^2|\phi_o] \frac{\partial^2 p}{\partial \phi_o^2} \quad (12)$$

If we define $\phi_o(0) = \phi_o$ with $a > \phi > b$, then the first passage time $T_{ab}(\phi_o)$ is the random variable defined as the time to reach either boundary a or b the first time, given that $\phi(t)$ starts at ϕ_0. Mathematically we say

$$T = T_{ab}(\phi_o) = \sup\{t|a > \phi(\gamma) > b \quad 0 \leq \gamma \leq t\} \quad (13)$$

Now we will use the following theorem derived by Darling and Siegert[7].

---

*It is normally the practice of Engineers to assume functions arising from descriptions of parameters in the real world are well behaved when there is no evidence to prove the contrary.

**It is sufficient to show that $B_1(\phi,t) = B_1(\phi)$ and $B_2(\phi,t) = B_2(\phi)$ for homogeneity.

Let $\phi_o(t)$ satisfy the hypothesis of equation (12) and if $T = T_{ab}(\phi_o)$ is a proper random variable whose moments of order $n \leqq n_o$ exist, then $E[T_{ab}{}^n(\phi_o)]$ satisfies

$$\frac{B_2[\Delta\phi^2|\phi]}{2}\frac{d^2E[T^n]}{d^2\phi_o} + B_1[\Delta\phi|\phi]\frac{dE[T^n]}{d\phi_o} = -nE[T^{n-1}] \quad n \leqq n_o$$

$$E[T_{ab}{}^n(a)] = E[T_{ab}{}^n(b)] = 0 \tag{14}$$

$$E[T^o] = 1$$

Thus we can use equation (14) to find the moments of $T_{ab}(\phi_o)$. In particular, we are interested in the time to slip a cycle under the assumption that $\omega = \omega_o$. In this case, we have $a = 2\pi$, $b = 2\pi$ and $\phi_o = 0$ so that $\hat{T} = T_{2\pi\ 2\pi}(0)$. Now if we divide equation (14) by $\frac{B_2[\Delta\phi^2|\phi]}{2}$ and use equations (6) and (7) then the moments of $\hat{T}$ satisfy

$$\frac{d^2E[\hat{T}^n]}{d^2\phi_o} - \rho\sin\phi_o\frac{dE[\hat{T}^n]}{d\phi_o} = \frac{-n\rho}{4B_L}E[\hat{T}^{n-1}] \tag{15}$$

where

$$\rho = \frac{A^2}{N_O B_L}$$

and

$$B_L = \frac{AK}{4}$$

B_L is defined as the one sided effective noise bandwidth computed from a transfer function at baseband.

ρ is defined as the loop signal to noise ratio. However, one should note that A^2 is the incoming signal power at the carrier frequency and N_o is the noise spectral density measured at the carrier frequency. The parameter ρ is a useful measure of performance. For example when ρ is large, the stationary phase error density function tends to a Gaussian shape and the variance of the phase error $\sigma_e^2 = \frac{1}{\rho}$.

There are several ways that we can solve for $E[\hat{T}^n]$. The most direct method is to solve equation (14) analytically and iteratively starting with E[T]. In Appendix A, the solution for E[T] is derived in this manner with the result that

$$E[\hat{T}] = \frac{\rho\pi}{2B_L} I_o^2(\rho) \tag{16}$$

This result is identical to the one obtained by Viterbi.[3]

To proceed further in this manner results in expressions which are in terms of multiple infinite series of Bessel functions which must be programmed if particular solutions are required; the mulitplicity increasing more than linearly with n (the n^{th} ordered moment).

A second method is to solve equation (15) directly by means of an analogue computer as depicted in Figure 2. It can be seen that two constants C_1^n and C_2^n are required in order to solve for $E[\hat{T}^n]$, once $E[\hat{T}^{n-1}]$ is known.

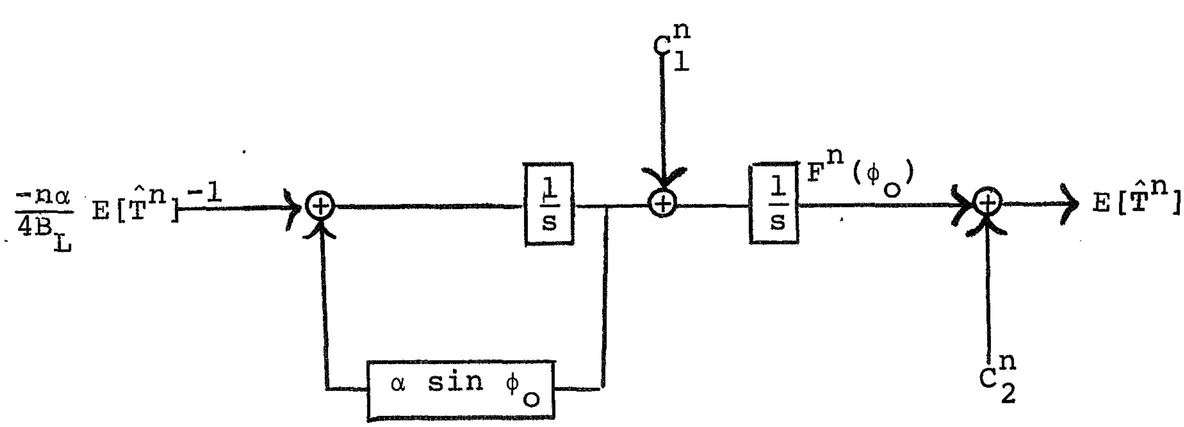

Figure 2. Analogue Model for the Solution of the n^{th} moment of $\hat{T}$

It is shown in Appendix A that

$$C_1^n = 0 \qquad \text{for all } n \qquad C_2^n = -F^n(2\pi) \tag{17}$$

so that

$$E[\hat{T}^n(\phi_o)] = F^n(\phi_o) - F^n(2\pi) \tag{18}$$

The analogue model given in Figure 2 was simulated on an 1108 Univac computer using a Univac program called "Mimic". Although the accuracy of the program in generating the mean could be measured against Viterbi's results, that of the higher moments had to be determined. This was done by programming Mimic to find the moments of $\hat{T}$ (here interpreted as the first passage time) for the Ornstein-Uhlenbech diffusion process. The differential equation for the first passage times of this process is very similar, although not identical, to equation 15. The moment generating function and the moments of the first passage time are derived in Appendix B. In addition the moments generated by Mimic and those from the equations given in Appendix B are compared in Appendix B. The results show that Mimic can generate accurate results.

Returning to the first order loop, the first four normalized moments of the time-to-cycle slip as computed from the simulation of Figure 2 are presented in Table 1. The normalization used in this memorandum (commonly used by statisticians) is the following:

$$\text{mean} = E[\hat{T}]$$

$$\text{variance} = \sigma^2 = E[(\hat{T})^2] - E^2[\hat{T}]$$

$$\text{skewness} = \frac{E[(\hat{T}-E[\hat{T}])^3]}{\sigma^3}$$

$$\text{Excess} = \frac{E[(\hat{T}-E[\hat{T}])^3]}{\sigma^4} - 3$$

We assume the mean and variance is sufficiently understood that they need not be explained except for the following. Both the mean and variance of $\hat{T}$ are functions of the bandwidth B_L. However, the relationship is such that if we define m_N and σ_N^2 to be the mean and variance of $\hat{T}$ when $B_L = 1$, then to obtain the mean and variance for any loop bandwidth we have the relationship

$$m = \frac{m_N}{B_L}$$

$$\sigma^2 = \frac{\sigma_N^2}{B_L^2}$$

Because of normalization, both the skewness and excess are independent of the loop bandwidth.

The skewness is a measure of the lack of symmetry in the density distribution. Thus, the skewness of a normal distribution is 0 since it is perfectly symmetrical while that of the exponential is 2 which is a large skewness. The excess is a measure of peakedness relative to a normal distribution. Therefore, the normal distribution has no excess while that of the uniform distribution is -1.2 showing that it is less peaked than the normal distribution. Table II, taken from reference 9, is a listing of several commonly used distributions together with some of their statistical characteristics.

TABLE I

MOMENTS OF THE TIME TO SKIP A CYCLE $\hat{T}$

FOR A FIRST ORDER PHASE LOCK LOOP

ρ	m_n	σ_n	Skewness	Excess
.03125	.154286	.12599258	1.95966869	5.82889491
.0625	.309024	.25246617	1.95990123	5.82981819
.125	.621676	.508793623	1.96078305	5.83377093
.25	1.27269	1.04875172	1.9640932	5.84319295
.5625	3.24177	2.75806586	1.97719129	5.9047234
1.	7.91004	7.1249047	1.99208036	5.96758202
2.	51.287	50.0598978	1.99972241	5.99881557
3.	352.669	351.194499	1.99999235	5.99990874
4.	2,521.33	2,519.70137	1.9999827	5.99995228
5.	18,308.2	18,306.4418	1.9999959	5.99998094

SOME ONE-DIMENSIONAL CONTINUOUS DISTRIBUTION FUNCTIONS

Name	Domain	Probability Density Function f(x)	Restrictions on Parameters	Mean	Variance	Skewness γ_1	Excess γ_2
Error function	$-\infty<x<\infty$	$\frac{h}{\sqrt{\pi}} e^{-h^2x^2}$	$0<h<\infty$	0	$\frac{1}{2h^2}$	0	0
Normal	$-\infty<x<\infty$	$\frac{1}{\sigma\sqrt{2x}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$	$-\infty<m<\infty$ $0<\sigma<\infty$	m	σ^2	0	0
Cauchy	$-\infty<x<\infty$	$\frac{1}{\pi\beta} \frac{1}{1+\left(\frac{x-a}{\beta}\right)^2}$	$-\infty<a<\infty$ $0<\beta<\infty$	not defined	not defined	not defined	not defined
Exponential	$a\leq x<\infty$	$\frac{1}{\beta} e^{-\left(\frac{x-a}{\beta}\right)}$	$-\infty<a<\infty$ $0<\beta<\infty$	$a+\beta$	β^2	2	0
Laplace, or double exponential	$-\infty<x<\infty$	$\frac{1}{2\beta} e^{-\left\|\frac{x-a}{\beta}\right\|}$	$-\infty<a<\infty$ $0<\beta<\infty$	a	$2\beta^2$	0	3

Table II. The moments of some well known continuous distribution functions (taken from Reference 9)

SOME ONE-DIMENSIONAL CONTINUOUS DISTRIBUTION FUNCTIONS

Name	Domain	Probability Density Function f(x)	Restrictions on Parameters	Mean	Variance	Skewness γ_1	Excess γ_2
Extreme-Value[1] (Fisher-Tippett Type I or doubly exponential)	$-\infty<x<\infty$	$\frac{1}{\beta}\exp(-y-e^{-y})$ with $y=\frac{x-a}{\beta}$	$-\infty<a<\infty$ $0<\beta<\infty$	$a+\gamma\beta$	$\frac{(\pi\beta)^2}{6}$	1.3	2.4
Pearson Type III	$a\le x<\infty$	$\frac{1}{\beta\Gamma(p)}y^{y-1}e^{-y}$ with $y=\frac{x-a}{\beta}$	$-\infty<a<\infty$ $0<\beta<\infty$ $0<p<\infty$	$a+p\beta$	$p\beta^2$	$\frac{2}{\sqrt{p}}$	6/p
Gamma distribution	$0\le x<\infty$	$\frac{1}{\Gamma(p)}x^{p-1}e^{-x}$	$0<p<\infty$	p	p	$\frac{2}{\sqrt{p}}$	6/p
Beta distribution	$0\le x\le 1$	$\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$	$1\le a<\infty$ $1\le b<\infty$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$	$\frac{2(a-b)}{a+b+2}$	See footnote 2.
Rectangular, or uniform	$m-\frac{h}{2}\le x\le m+\frac{h}{2}$	$\frac{1}{h}$	$-\infty<m<\infty$ $0<h<\infty$	m	$\frac{h^2}{12}$	0	-1.2

[1] γ (Euler's constant) = .57721 56649 . . . .

[2] $$\gamma_2=\sqrt{\frac{a+b+1}{ab}}\left\{\frac{3(a+b+1)[2(a+b)^2+ab(a+b-6)]}{ab(a+b+2)(a+b+3)}-3\right\}.$$

TABLE II - Continued

The results indicate that for all values of ρ, the distribution is very skewed and peaked, looking like a Pearson Type III distribution over its first four moments which became identical to those of an exponential distribution for large values of ρ.

III. SECOND ORDER PHASE LOCK LOOP

In this section an outline of the derivation of an approximate solution of a second order phase lock loop is sketched. The details of the derivation are given in Appendix C. The results are then compared, where possible, with those obtained analytically and experimentally.

We assume the transfer function of the linear filter (G(s)) in the phase lock loop (Figure 1) to be

$$G(s) = 1 + \frac{\alpha}{s} \tag{19}$$

where α is a constant and thus the phase lock loop differential equation with $\omega = \omega_o$ is given by

$$\frac{d\phi}{dt} = -k[A\sin\phi(t) + n'(t)] - \alpha f(t) \tag{20}$$

where

$$f(t) = K\int_0^t [A\sin\phi(u) + n'(u)]du$$

Using this equation we can derive the following differential equation which is satisfied by the transition density function

$$\frac{\partial p(\phi_1|\phi_o;t)}{\partial t} = \frac{\partial}{\partial\phi_1}[aK\sin\phi_1(t) + \alpha E[f_t|\phi_1]]p(\phi_1|\phi_o;t) + \frac{K^2N_o}{4}\frac{\partial^2 p(\phi_1|\phi_0,t)}{\partial\phi_1^{\,2}} \tag{21}$$

where $E[f_t|\phi_1]$ is the expectation of f_t given the phase of the received signal at time t.

Tauseworthe[6] has shown that for a second order loop the following linear approximation can be made.

$$E[f_t|\phi_1] \approx - e\phi_1(t) \tag{22}$$

where

$$\alpha e = AK - \frac{\ell^2(0)}{4B_L} \tag{23}$$

$\ell(0)$ is the impluse response of the linearized phase lock loop evaluated at t=0. Therefore we have

$$E[f_t|\phi_1] \approx - \frac{AK}{AK+\alpha}\, \phi(t) \tag{24}$$

where $\xi_\phi(\tau)$ is the normalized covariance function of the stationary process $\phi(t)$.

Thus we can write equation (21) as

$$\frac{\partial p(\phi_1|\phi_o;t)}{\phi\partial\ t} = \frac{\partial}{\partial\phi_1}\ [\rho(\frac{1+r}{r})\sin\ \phi_1 - \frac{\rho}{r}\ \phi_1]\ p(\phi_1|\phi_o;t) + \frac{\partial^2 p(\phi_1|\phi_o,t)}{\partial\phi_1{}^2} \quad (25)$$

where $r = Ak/\alpha$, $\gamma = \frac{K^2N_o}{4}$, $\rho = A^2/N_oB_L$, $B_L = \frac{AK+\alpha}{4}$.

We note that equation (25) describes a $\phi(t)$ which is a homogeneous Markov process whose behavior is determined by the differential equation

$$\frac{d\phi(t)}{dt} = -\ AK\sin\dot{\phi}(t) + \frac{\alpha AK}{AK+\alpha}\ \phi(t) - kn'(t) \quad (26)$$

Examining equation (26) and comparing with our results from the previous section we conclude that for sufficiently large signal to noise ratios, the second order phase lock loop's behavior can be modeled as a homogeneous Markov process. Therefore, a recursive relationship for the time to skip a cycle parameter $\hat{T}$ can be derived in a manner analogous to the one derived for the first order phase lock loop with the result that

$$\frac{d^2E[\hat{T}^n]}{d^2\ \phi_o} - [\rho\frac{(1+r)}{r}\ \sin\ \phi_1 - \frac{\rho}{r}\ \phi_1]\ \frac{dE[\hat{T}^n]}{d\phi_o} = \frac{-n}{\gamma}\ E[\hat{T}^{n-1}] \quad (27)$$

where $E[\hat{T}^o] = 1$

$$E[\hat{T}^n(2\pi)] = E[\hat{T}^n(-2\pi)] = 0$$

$$\gamma = \frac{K^2N_o}{4} = \frac{4B_L}{(1+\frac{1}{r})^2} \quad (28)$$

Equation (26) is identical with Tausworthe's result[6] for the calculation of E[T]* presented in Reference 6. It can be seen in this referenced work that there is close agreement between the theoretical and experimental findings. Equation (27) was programmed for the first four moments of $\hat{T}$. The results are presented in Table III. To interpret the results we compare them with the conclusion reached by Tausworthe[6], based upon experimental work[13], that the probability distribution on the time-to-cycle slip is given by a Pearson Type III density

$$f(\hat{T}) = \frac{\gamma^{\gamma-1} e^{-\gamma}}{z\Gamma(p)} \tag{29}$$

where

$$\gamma = \frac{\hat{T}}{z}$$

$$z = \frac{\sigma^2}{u}$$

$$p = \left(\frac{\mu}{\sigma}\right)^2$$

If this hypothesis is true then we can see from Table II that the skewness and excess must be given by

$$\text{skewness} = \frac{2}{\sqrt{p}} \tag{30}$$

$$\text{excess} = \frac{6}{p} \tag{31}$$

---

*Tausworthe used a filter $G(s) = \frac{1+\tau_2 s}{\tau_1 s}$ for such a filter $r = \frac{AK\tau_2^2}{\tau_1}$. In addition Tausworthe works with the two sided fiducial "bandwidth" (ω_ℓ) here equal to $2B_L$ and noise spectral density $\hat{N}_o = \frac{N_o}{2}$. The net effect is that $\hat{E}[T]$ when normalized with respect to ω_ℓ is twice the value of E[T] when normalized with respect to B_ℓ. Tausworthe shows in a footnote that the moment generating function of $\hat{T}$ satisfies a differential equation. Use of this result leads directly to equation (27).

TABLE III

MOMENTS OF $\hat{T}$ FOR A SECOND ORDER PHASE LOCK LOOP

[1] RHO	[2] r	[3] MEAN=μ	[4] VARIANCE-σ^2	[5] σ	[6] SKEWNESS	[7] EXCESS	[8] $\sqrt{P}$	[9] $2/\sqrt{P}$	[10] $6/P$
.5	2.0	2.7878+00	4.3690+00	2.0902+00	1.9449+00	5.7673+00	1.3338+00	1.4995+00	3.3729+00
1.0	2.0	5.7033+00	2.1827+01	4.6719+00	1.9819+00	5.9255+00	1.2207+00	1.6383+00	4.0262+00
2.0	2.0	1.8544+01	2.9141+02	1.7071+01	1.9976+00	5.9902+00	1.0863+00	1.8411+00	5.0846+00
3.0	2.0	5.7487+01	3.1025+03	5.5700+01	1.9997+00	5.9988+00	1.0321+00	1.9378+00	5.6327+00
4.0	2.0	1.7975+02	3.1587+04	1.7773+02	2.0000+00	5.9998+00	1.0114+00	1.9775+00	5.8658+00
5.0	2.0	5.6889+02	3.2114+05	5.6669+02	2.0000+00	6.0000+00	1.0039+00	1.9923+00	5.9536+00
.5	4.0	2.6366+00	4.2588+00	2.0637+00	1.9531+00	5.8018+00	1.2776+00	1.5654+00	3.6757+00
1.0	4.0	5.9172+00	2.5222+01	5.0222+00	1.9859+00	5.9423+00	1.1782+00	1.6975+00	4.3221+00
2.0	4.0	2.4476+01	5.3796+02	2.3194+01	1.9990+00	5.9959+00	1.0553+00	1.8953+00	5.3880+00
3.0	4.0	1.0103+02	9.9008+03	9.9503+01	1.9999+00	5.9997+00	1.0154+00	1.9697+00	5.8194+00
4.0	4.0	4.2683+02	1.8072+05	4.2512+02	2.0000+00	6.0000+00	1.0040+00	1.9920+00	5.9521+00
5.0	4.0	1.8297+03	3.3410+06	1.8278+03	2.0000+00	6.0000+00	1.0010+00	1.9980+00	5.9879+00
.5	10.0	2.6694+00	4.7366+00	2.1764+00	1.9643+00	5.8496+00	1.2265+00	1.6306+00	3.9883+00
1.0	10.0	6.6420+00	3.3785+01	5.8125+00	1.9889+00	5.9545+00	1.1427+00	1.7502+00	4.5950+00
2.0	10.0	3.4465+01	1.1054+03	3.3248+01	1.9995+00	5.9980+00	1.0366+00	1.9294+00	5.5838+00
3.0	10.0	1.8571+02	3.3956+04	1.8427+02	2.0000+00	5.9999+00	1.0078+00	1.9845+00	5.9071+00

[1] RHO	[2] r	[3] MEAN=μ	[4] VARIANCE-σ^2	[5] σ	[6] SKEWNESS	[7] EXCESS	[8] $\sqrt{P}$	[9] $2/\sqrt{P}$	[10] 6/P
4.0	10.0	1.0326+03	1.0630+06	1.0310+03	2.0000+00	6.0000+00	1.0015+00	1.9969+00	5.9815+00
5.0	10.0	5.8260+03	3.3923+07	5.8243+03	2.0000+00	6.0000+00	1.0003+00	1.9994+00	5.9965+00
.5	1000.0	2.7888+00	5.5406+00	2.3539+00	1.9743+00	5.8924+00	1.1848+00	1.6880+00	4.2742+00
1.0	1000.0	7.8912+00	5.0489+01	7.1056+00	1.9921+00	5.9675+00	1.1106+00	1.8009+00	4.8647+00
2.0	1000.0	5.1023+01	2.4797+03	4.9797+01	1.9997+00	5.9843+00	1.0246+00	1.9519+00	5.7150+00
3.0	1000.0	3.4988+02	1.2139+05	3.4841+02	2.0000+00	6.0000+00	1.0042+00	1.9916+00	5.9495+00
4.0	1000.0	2.4942+03	6.2131+06	2.4926+03	2.0000+00	6.0000+00	1.0007+00	1.9987+00	5.9921+00
5.0	1000.0	1.8059+04	3.2606+08	1.8057+04	2.0000+00	6.0000+00	1.0001+00	1.9998+00	5.9988+00

It is to be noted that this Pearson Type III density function with the parameter a=0 (see Table II) generates the exact first two moments of $\hat{T}$ and approximates all of the others. To see how good this approximation is the values of the skewness and excess calculated from equations (30) and (31) are presented. These results are compared with those calculated from equation (27) (columns 6 and 7). This comparison is also made graphically in Figures 2, 3 and 4. It can be seen that as r and p increase the fitted $3^{\underline{rd}}$ and $4^{\underline{th}}$ moments approach the values of the actual $3^{\underline{rd}}$ and $4^{\underline{th}}$ moments of $\hat{T}$. We can however make use of the Pearson Type III density function to generate the first three moments of T exactly and to approximate the excess with exceptional accuracy. To do this we let $a = \mu k(r,\rho)$ where $k(r,\rho)$ is a constant for a given value of r and ρ. Then we have the following

$$z = \frac{\sigma^2}{\mu - a}$$

$$p = \left(\frac{\mu-a}{\sigma}\right)^2 = \frac{\mu^2}{\sigma^2}\,(1-k(r,\rho))^2$$

Thus

$$p = p(a=0)\,(1-k(r,\alpha))^2$$

Since we want to fit the skewness of $\hat{T}$ we equate

$$\frac{4}{p} = \frac{4}{p(a=0)}\,\frac{1}{(1-k(r,\alpha))^2}$$

or

$$1-k(r,\rho) = \frac{\frac{2}{\sqrt{p(a=0)}}}{\frac{2}{\sqrt{p}}} = \frac{\text{column [9] of Table III}}{\text{column [6] of Table III}} \qquad (32)$$

The values of $k(r,\rho)$ found using equation (32) are graphically presented in Figure 5 while a comparison of the fitted excess with the excess of T is given in Table IV.

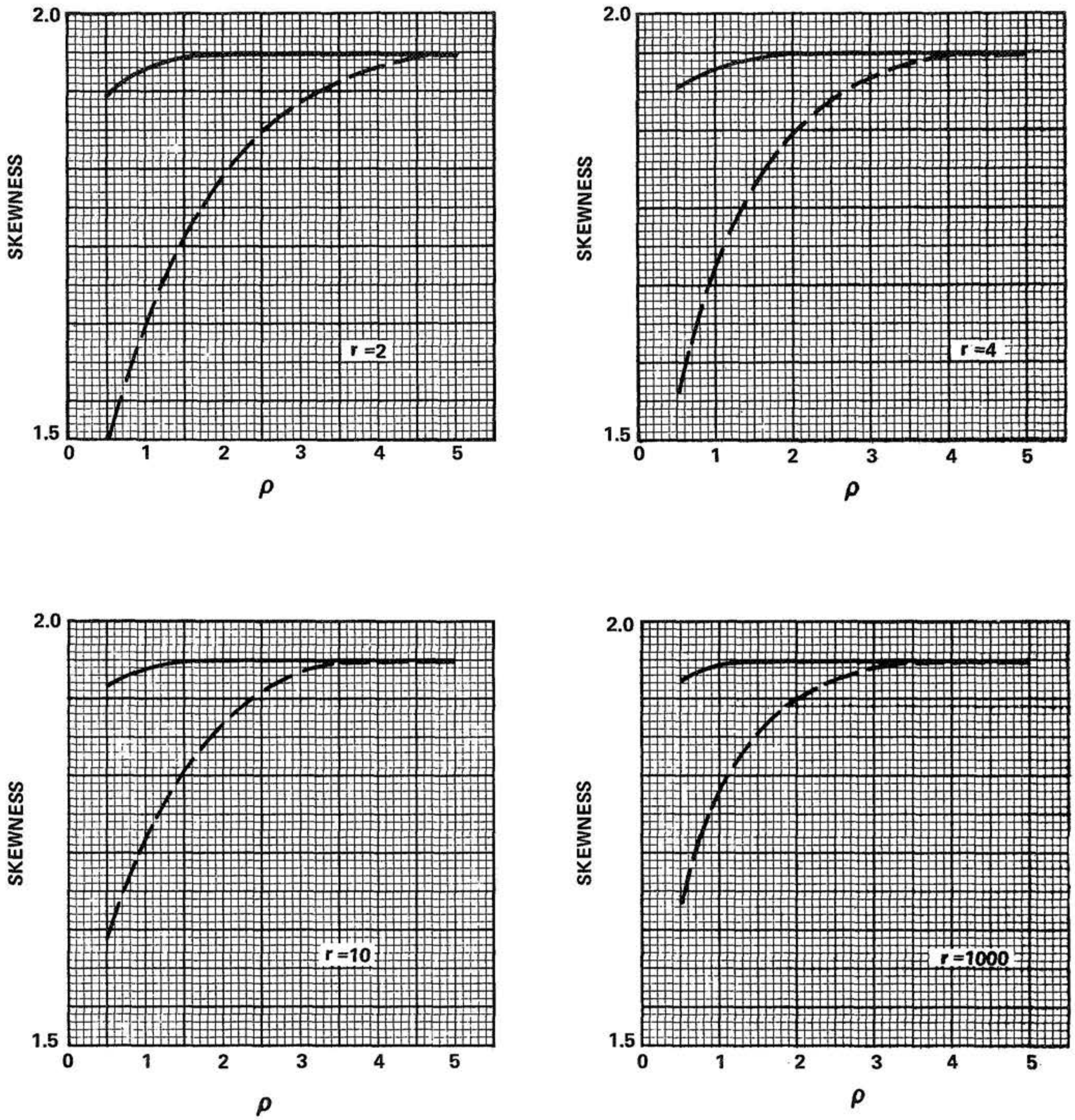

FIGURE 2 - COMPARISON OF SKEWNESS OF $\hat{T}$ WITH THE SKEWNESS OF THE FITTED PEARSON TYPE III DENSITY FUNCTION. (a=0)

[—— $\hat{T}$, – – – PEARSON TYPE III]

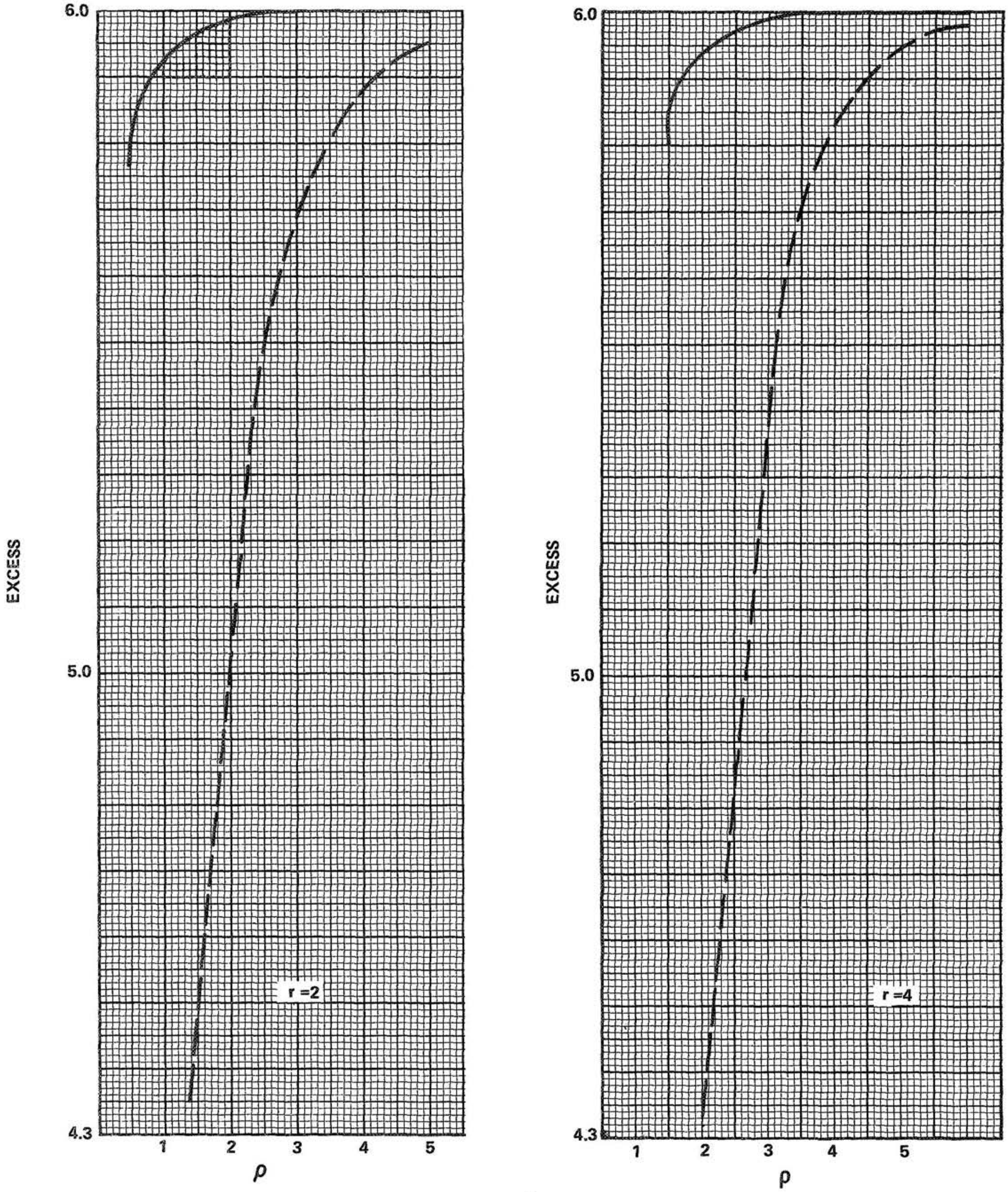

FIGURE 3 - COMPARISON OF EXCESS OF $\hat{T}$ WITH THE EXCESS OF THE FITTED PEARSON TYPE III DENSITY FUNCTION (a=0)

[—— $\hat{T}$, — — — PEARSON TYPE III, r = 2, r = 4]

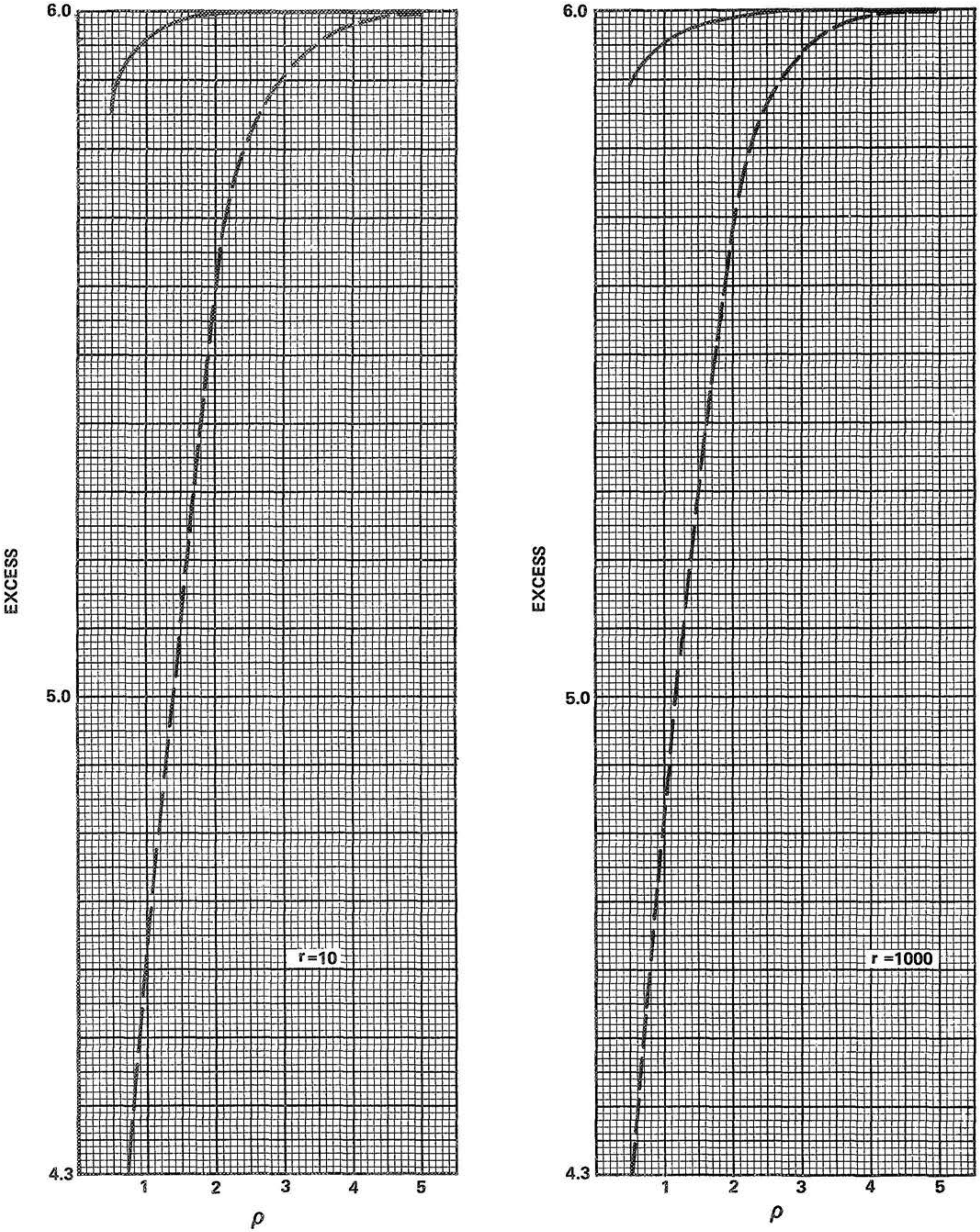

FIGURE 4 - COMPARISON OF EXCESS OF $\hat{T}$ WITH THE EXCESS OF THE FITTED PEARSON TYPE III DENSITY FUNCTION (a=0)

[—— $\hat{T}$, — — — PEARSON TYPE III, r = 10, r = 1000]

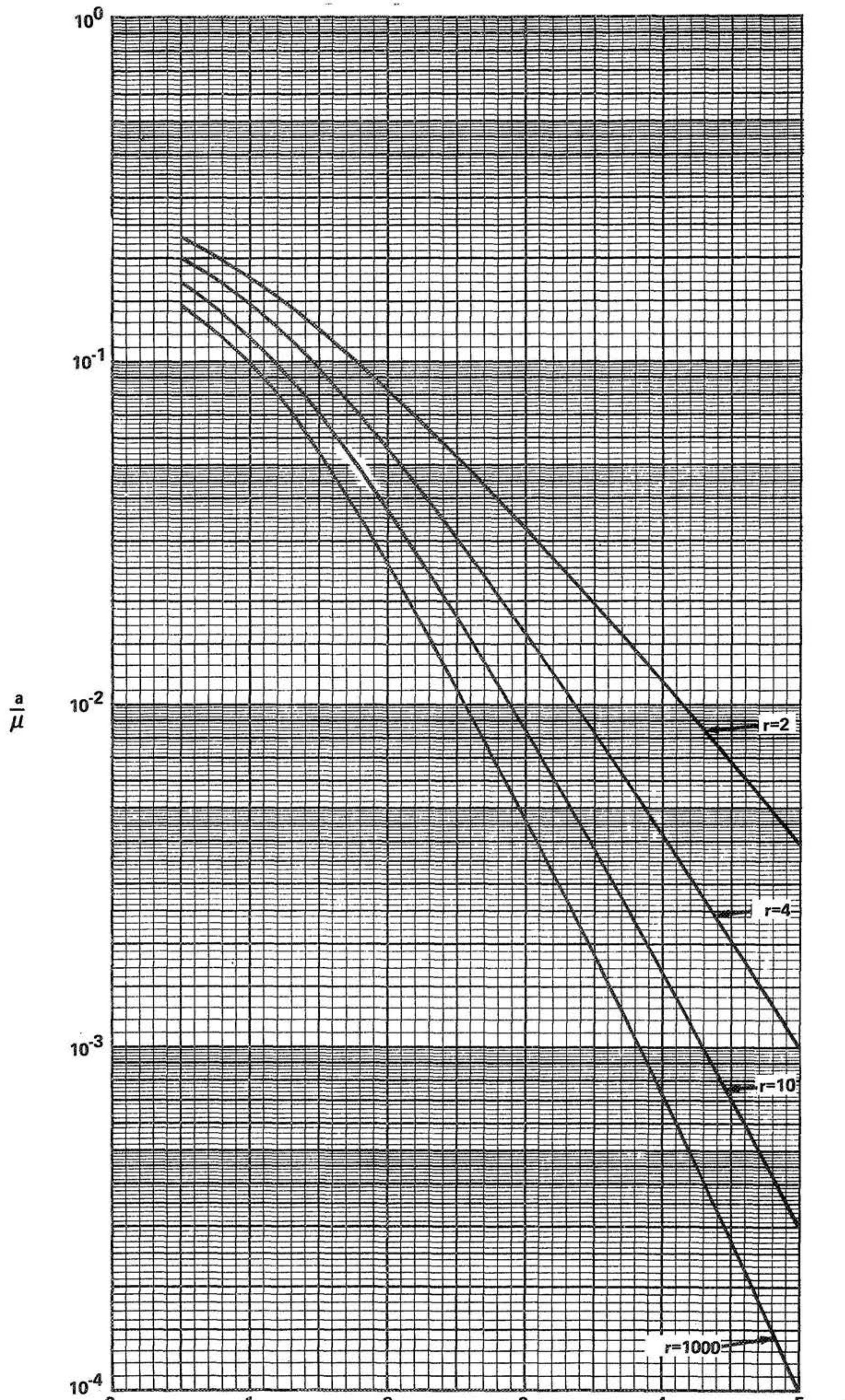

FIGURE 5 - THE VALUE OF a/μ FOR A "BEST FIT" PEARSON TYPE III DENSITY FUNCTION TO THE DENSITY FUNCTION OF $\hat{T}$.

TABLE IV

A comparison of the Excess of $\hat{T}$ to the Excess generated by a Pearson Type III density function fitted to the first three moments of $\hat{T}$.

r	ρ	Excess $\hat{T}$	Excess--fitted Pearson Type III
2	1/2	5.7673	5.6741
	1	5.9255	5.8921
	2	5.9902	5.9856
	3	5.9988	5.9983
	4	5.9998	6.0000
	5	6.0000	6.0000
4	1/2	5.8028	5.7218
	1	5.9423	5.9158
	2	5.9959	5.9938
	3	5.9997	5.9992
	4	6.0000	6.0000
	5	6.0000	5.9999
10	1/2	5.8496	5.7885
	1	5.9545	5.9336
	2	5.9980	5.9976
	3	5.9999	5.9997
	4	6.0000	6.0001
	5	6.0000	6.0000
1000	1/2	5.8924	5.8471
	1	5.9675	5.9532
	2	5.9843	5.9981
	3	6.0000	5.9999
	4	6.0000	5.9999
	5	6.0000	6.0000

Finally, we note that for the steady state distribution of the phase error equation (23) reduces to

$$0 = \frac{\partial}{\partial\phi_1} [\rho\frac{(1+r)}{r} \sin \phi_1(t) - \frac{\rho}{r} \phi_1]p(\phi_1|\phi_o,t) + \frac{\partial^2 p(\phi_1|\phi_o,t)}{\phi} \qquad (32)$$

Now since the phase error ϕ_1 will nearly always be small $\phi_1 \cong \sin \phi_1$. Substituting $\sin\phi_1$ for ϕ_1 in equation (38) results in

$$0 = \frac{\partial}{\partial\phi_1} [\rho\sin \phi_1(t)]p(\phi_1|\phi_o) + \frac{\partial^2 p(\phi_1|\phi_o)}{\partial\phi^2} \qquad (33)$$

Equation (33) is identical to the result derived by Viterbi[3] and verified experimentally by F. S. Charles and W. C. Lindsey[12]. The form of equation (33) is identical to one derived for the first order loop so that the solution is also of identical form. Thus

$$p(\phi|\phi_o=0) = \frac{\exp(\rho \cos \phi)}{2\pi I_o(\rho)} \qquad (34)$$

ACKNOWLEDGMENTS

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2034-LS-jf L. Schuchman

Attachments
References
Appendix A, B and C

REFERENCES

1. V. I. Tikhonov, "The Effects of Noise on Phase Lock Oscillator Operation", Automat. i Telemakh, Vol. 22, No. 9, 1959.

2. V. I. Tikhonov, "Phase Lock Automatic Frequency Control Application in the Presence of Noise", Automat. i Telemakh, Vol. 23, No. 3., 1960.

3. A. J. Viterbi, Principles of Coherent Communications, Chapter 4, McGraw-Hill, 1966.

4. A. J. Viterbi, "Optimum Detection and Signal Selection for Partially Coherent Binary Communications", IEEE Trans. on Information Theory, Vol. IT-11, pp. 239-246, April 1965.

5. W. C. Lindsey, "Phase Shift-Keyed Signal Detection with Noise Referenced Signals", IEEE Trans. on Aerospace and Electronics Systems, Vol. AES-2, pp. 393-401, July 1966.

6. R. Tausworthe, "Cycle Slipping in Phase-Locked Loops", IEEE Transactions on Communication Technology, Vol. Com-15 No. 3, pp. 417-421, June 1967.

7. D. A. Darling and A. J. E. Siegert,"The First Passage Time Problem for a Continuous Markov Process", Ann. Math. Stat. Vol. 24, pp. 624-639, 1953.

8. Prabhu, N. U., "Stochastic Processes", Chapter 3, MacMillan Co., New York, 1965.

9. "Handbook of Mathematical Functions", U. S. Department of Commerce, National Bureau of Standards, Applied Mathematical Series 55, pg. 930, June 1964.

10. R. C. Tausworthe, "Theory and Practical Design of Phase Locked Receivers", Jet Propulsion Lab., Pasadena, California, Report TR 32-819, February 1966.

11. F. S. Charles and W. C. Lindsey, "Some Analytical and Experimental Phase-Locked Loop Results for Low Signal-to-Noise Ratios", Proc.IEEE, Vol.54, pp.1152-1166,Sept.,1966.

12. E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis", 4th Edition, Chapter 16, Cambridge University Press, 1965.

13. D. Sanger and R. C. Tausworthe, "An Experimental Study of the First-Slip Statistics of the Second Order Phase-Lock Loop", Jet Propulsion Lab., Pasadena, California Space Programs Summary, 37-43, Vol. IV, February 1967.

APPENDIX A

EVALUATION OF THE n^{th} MOMENT OF THE TIME TO SKIP A CYCLE ($\hat{T}$)

For convenience we repeat equation (15) which is a second order differential equation in the n^{th} moment of T and whose solution is dependent upon the knowledge of the $n-1^{st}$ moment of $\hat{T}$. Thus we have

$$\frac{d^2E[\hat{T}^n]}{d^2\phi_o} - \alpha \sin \phi_o \frac{E[\hat{T}^n]}{d\phi_o} = - \frac{n\alpha}{4B_L} E[\hat{T}^{n-1}] \tag{A-1}$$

with $E[\hat{T}^o] = 1$.

If we multiply both sides by $e^{\alpha\cos \phi_o}$ and integrate we have

$$\frac{dE[\hat{T}^n]}{d\phi_o} e^{\alpha\cos\phi_o} = - \int_{\wedge}^{\phi_o} \frac{n2}{4B_L} E[\hat{T}(\theta)]^{n-1} e^{+\alpha\cos\theta} d\theta + c_{1n} \tag{A-2}$$

where $\int_{\wedge}^{\phi_o} g(t)\, dt$ is the integral of $g(t)$ evaluated only at its upper limit. c_{1n} is a constant to be determined.

Multiplying both sides by $e^{-\alpha\cos\phi_o}$ and integrating a second time we have

$$E[\hat{T}^n] = - \frac{n\alpha}{4B_L} \int_{\wedge}^{\phi_o} e^{-\alpha\cos\theta_1} d\theta_1 \int_{\wedge}^{\theta_1} e^{\alpha\cos\theta} E[\hat{T}^{n-1}(\theta)] d\theta$$

$$+ \int_{\wedge}^{\phi_o} e^{-\alpha\cos\theta} d\theta \; c_{1n} + c_{2n} \tag{A-3}$$

where c_{2n} is a constant to be determined.

To evaluate c_{1n} and c_{2n} we make use of the boundary conditions which are

$$E[\hat{T}^n(2\pi)] = E[\hat{T}^n(-2\pi)] = 0 \qquad \text{(A-4)}$$

Using equation (A-4) we have that

$$2\int_{\hat{}}^{2\pi} e^{\alpha\cos\theta} d\theta c_{1n} = \int_{\hat{}}^{2\pi} e^{-\alpha\cos\theta_1} \frac{n\alpha}{4B_L} d\theta_1 \int_{\hat{}}^{\theta_1} E[\hat{T}^{n-1}(\theta)] e^{\alpha\cos\theta} d\theta$$

$$- \int_{\hat{}}^{-2\pi} e^{-\alpha\cos\theta_1} \frac{n\alpha}{4B_L} d\theta_1 \int_{\hat{}}^{\theta_1} E[\hat{T}^{n-1}(\theta)] e^{\alpha\cos\theta} d\theta \qquad \text{(A-5)}$$

$$c_{2n} = \int_{\hat{}}^{2\pi} e^{-\alpha\cos\theta_1} \frac{n_\alpha}{8B_L} \int_{\hat{}}^{\theta_1} E[\hat{T}^{n-1}(\theta)] e^{\alpha\cos\theta} d\theta +$$

$$\int_{\hat{}}^{-2\pi} e^{-\alpha\cos\theta_1} \frac{n\alpha}{8B_L} \int_{\hat{}}^{\theta_1} E[\hat{T}^{n-1}(\theta)] e^{\alpha\cos\theta} d\theta \qquad \text{(A-5)}$$

Now if we start with $E[\hat{T}^o] = 1$ then we find that

$$c_{11} = 0 \qquad \text{(A-6)}$$

$$c_{21} = \int_{\hat{}}^{2\pi} e^{-\alpha\cos\theta_1} \frac{n\alpha}{4B_L} \int_{\hat{}}^{\theta_1} e^{\alpha\cos\theta} d\theta \qquad \text{(A-7)}$$

$E[\hat{T}(\phi_o)]$ is an even function of ϕ_o.

By inductive reasoning we find more generally that

$$c_{1n} = 0 \tag{A-8}$$

$$c_{2n} = \int_{\wedge}^{2\pi} e^{-\alpha\cos\theta_1} \frac{n\alpha}{4B_L} \int_{\wedge}^{\theta_1} E[\hat{T}^{n-1}(\theta)]e^{\alpha\cos\theta}d\theta \tag{A-9}$$

and

$$E[\hat{T}^n(\phi_o)] = \int_o^{2\pi} e^{-\alpha\cos\theta_1} \frac{n\alpha}{4B_L} \int_{\wedge}^{\theta_1} E[\hat{T}^{n-1}(\theta)]e^{\alpha\cos\theta}d\theta \tag{A-10}$$

Equation (A-10) can be interpreted in terms of the analogue simulation of Figure 1 by equation (18) of the text.

Evaluation of equation (A-10) for n=1 leads to

$$E[\hat{T}(\phi_o)] = \frac{\alpha}{4B_L}\left[\left(\frac{(2\pi)^2}{2} - \phi_o^2\right) I_o^2(\alpha) + 2I_o(\alpha)\sum_{k=1}^{\infty} I_k(\alpha)[\cos k\phi_o - 1]\right.$$

$$- 2I_o(\alpha)\sum_{k=1}^{\infty}(-1)^k I_k(\alpha)\phi_o\frac{\sin k\phi_o}{k} + 2I_o(\alpha)\sum_{k=1}^{\infty}(-1)^k I_k(\alpha)\frac{[\cos k\phi_o - 1]}{k^2}$$

$$+ 2\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^m}{2k}\frac{[\cos(m+k)\phi_o - 1]}{m+k} +$$

$$\left. + 2\sum_{\substack{m=1 \\ m\neq k}}^{\infty}\sum_{k=1}^{\infty}(-1)^m[\cos(m+k)\phi_o - 1]\right] \qquad \text{(A-11)}$$

so that

$$E[\hat{T}(o)] = \frac{\alpha\pi^2}{2B_L} I_o^{\,2}(\alpha) \qquad \text{(A-12)}$$

APPENDIX B

DERIVATION OF MOMENTS OF $\hat{T}$ FOR THE ORENSTEIN UHLINBECK PROCESS

In this appendix we simplify equation (5) of the text by linearizing it so that instead of equation (6) we have $B_1[\Delta\phi|\phi] = -AK\phi$ and the background Kolmogorov equation becomes

$$\frac{1}{4B_L}\frac{\partial f}{\partial t} = -\phi_0 \frac{\partial f}{\partial \phi_o} + \frac{1}{\rho}\frac{\partial^2 f}{\partial \phi_o^2} \tag{B-1}$$

Equation (B-1) is well known as the backward equation of the Ornstein-Uhlenbeck process, one of the diffusion processes studied by both physicists and statisticians. If we take the Laplace transform with respect to t we have

$$\frac{\theta\omega}{4B_L} = -\phi_o \frac{\partial \omega}{\partial \phi_o} + \frac{1}{\rho}\frac{\partial^2 \omega}{\partial \phi_o^2} \tag{B-2}$$

where

$$\omega = \int_o^\infty f e^{-\theta t}\, dt$$

Following a technique of Whittaker[12] we assume a solution to (B-2) and if it works it is the solution. We assume therefore that

$$\omega(\phi_o) = \int_o^\infty e^{-\phi_o t - t^2/2\rho}\,(-t)^{\frac{\theta}{4B_L} - 1}\, dt \tag{B-3}$$

then we have that

$$\left[-\frac{\theta}{4B_L} - \phi_o \frac{\partial}{\partial\phi_o} + \frac{1}{\rho}\frac{\partial^2}{\partial\phi_o^2}\right]\omega = \int_0^\infty \left[-\frac{\theta}{4B_L} + \phi_o t + \frac{t^2}{\rho}\right] e^{-\phi_o t - t^2/2\rho}(-t)^{\frac{\theta}{4B_L}-1} dt$$

$$= \int_0^\infty \frac{d}{dt}\left[\left(e^{-\phi_o t - t^2/2\rho}\right)\right](-t)^{\frac{\theta}{4B_L}} dt$$

$$= 0 \text{ (for } \theta \text{ positive)}$$

and $\omega(\phi_o)$ as given by equation (B-3) is a solution to equation (B-2).

In a similar manner one finds $\omega(-\phi_o)$ to be a solution for $\theta > 0$. Thus if we use a theorem given by Darlington and Siegert for symmetrical boundary conditions ϕ_L and $-\phi_L$ the generating function for T, $g(\phi_o, 1/4BL, \rho, \phi_L)$ is given by

$$g(\phi_o, \frac{1}{4B_L}, \rho, \phi_L) = \frac{\omega(\phi_o) + \omega(-\phi_o)}{\omega(\phi_L) + \omega(-\phi_L)} \qquad \text{(B-3)}$$

Darlington and Siegert have derived equation (B-3) for $\hat{T}$ in terms of the ratio of two integrals called Weber functions. The following is a simplification of their result which allows one to derive the moments of $\hat{T}$.

We write equation (B-3) in the following form

$$g(\phi_o, \frac{1}{4B_L}, \rho, \phi_L) = \frac{\int_0^\infty (e^{\phi_o t} + e^{-\phi_o t})\, e^{-t^2/2\rho} (-t)^{\theta/4B_L^{-1}} dt}{\int_0^\infty (e^{\phi_L t} + e^{-\phi_L t})\, e^{-t^2/2\rho} (-t)^{\theta/4B_L^{-1}} dt} \qquad \text{(B-4)}$$

If we let $\frac{t^2}{2\rho^{1/2}} = u$ and if we expand the bracketed pair of exponentials in both the denominator and numerator of equation (B-4) we have

$$g(\phi_o, \frac{1}{4B_L}, \rho, \phi_L) = \frac{\int_0^\infty \sum_{n=0}^\infty \frac{\phi_o^{2n} (\rho . 2)^n}{(2n)!} e^{-u} \cdot u^{\theta/8B_L + n-1} du}{\int_0^\infty \sum_{n=0}^\infty \frac{\phi_L^{2n} (\rho . 2)^n}{(2n)!} e^{-u} \cdot u^{\theta/8B_L + n-1} du}$$

which can be written as

$$g(\phi_o, \frac{1}{4B_L}, \rho, \phi_L) = \frac{[\Gamma(\theta/8B_L)]}{[\Gamma(\theta/8B_L)]} \cdot \frac{\left[1 + \sum_{n=1}^\infty \frac{\phi_o^{2n} (\rho 2)^n}{(2n)!} \left(\frac{\theta}{8B_L} + n-1\right)\left(\frac{\theta}{8B_L} + n-2\right)\cdots\left(\frac{\theta}{8B_L}\right)\right]}{\left[1 + \sum_{n=1}^\infty \frac{\phi_L^{2n} (\rho 2)^n}{(2n)!} \left(\frac{\theta}{8B_L} + n-1\right)\left(\frac{\theta}{8B_L} + n-2\right)\cdots\frac{\theta}{8B_L}\right]}$$

and reduces to

$$g(\phi_o;\frac{1}{4B_L},\rho,\phi_L) = \frac{1+\sum_{n=1}^{\infty}\frac{(\phi_o^2\ \rho 2)^n}{2n!}\left(\frac{\theta}{8B_L}+n-1\right)\left(\frac{\theta}{8B_L}+n-2\right)\cdots\frac{\theta}{8B_L}}{1+\sum_{n=1}^{\infty}\frac{(\phi_L^2\ \rho 2)^n}{2n!}\left(\frac{\theta}{8B_L}+n-1\right)\left(\frac{\theta}{8B_L}+n-2\right)\cdots\frac{\theta}{8B_L}} \quad \text{(B-5)}$$

It is easy to show that equation (B-5) satisfies equation (B-2) and the boundary value conditions so that it is indeed a solution.

Our main interest is determining the statistics of $\hat{T}$ when $\phi_o = 0$. Thus for this case the generating function reduces to

$$g(\phi_o,\frac{1}{4B_L},\gamma) = \frac{1}{1+\sum_{n=1}^{\infty}\frac{(\gamma^2)^n}{2n!}\left(\frac{\theta}{8B_L}+n-1\right)\left(\frac{\theta}{8B_L}+n-2\right)\ \cdots\ \frac{\theta}{8B_L}} \quad \text{(B-6)}$$

where $\gamma^2 = \delta^2 2$

$\delta^2 = \phi_L^{\ 2}\rho$

It is interesting to note the relationship between ϕ_o^2 and ρ. That is for a given value of δ^2 there is an inverse relationship between the square of ϕ_L and ρ.

To determine the mounts of $\hat{T}$ we define

$$D(\theta) = \sum_{n=1}^{\infty}\frac{\gamma^n}{2n!}\left(\frac{\theta}{8B_L}+n-1\right)\ \cdots\ \frac{\theta}{8B_L} = \sum_{n=1}^{\infty}V(\gamma,\theta,n) \quad \text{(B-7)}$$

and differentiate $\omega(\theta)$. Thus we have that

$$\dot{\omega}(\theta) = \frac{-\dot{D}(\theta)}{[1+D/\theta)]^2} \tag{B-8}$$

$$\ddot{\omega}(\theta) = \frac{-\ddot{D}(\theta)}{[1+D(\theta)]^2} + \frac{2[\dot{D}[\theta]]^2}{[1+D(\theta)]^3} \tag{B-9}$$

$$\dddot{\omega}(\theta) = \frac{\dddot{D}(\theta)}{[1+D(\theta)]^2} + \frac{6\ddot{D}(\theta)\dot{D}(\theta)}{[1+D(\theta)]^3} - \frac{6[\dot{D}(\theta)]^3}{[1+D(\theta)]^4} \tag{B-10}$$

$$\ddddot{\omega}(\theta) = \frac{-36\ddot{D}(\theta)\dot{D}^2(\theta)}{[1+D(\theta)]^2} + \frac{8\dddot{D}(\theta)\dot{D}(\theta)+6\ddot{D}^2(\theta)}{[1+D(\theta)]^3} \frac{+24[\dot{D}(\theta)]^4}{[1+D(\theta)]^5} - \frac{\ddddot{D}(\theta)}{[1+D(\theta)]^2} \tag{B-11}$$

where

$$\dot{D}(\theta) = \frac{1}{\gamma B_L} \sum_{n=1}^{\infty} V(\gamma,\theta,n) \sum_{j=0}^{n-1} \frac{1}{(\frac{\theta}{8B_L} + j)} \tag{B-12}$$

$$\ddot{D}(\theta) = \frac{1}{(8B_L)^2} \sum_{n=2}^{\infty} V(\gamma,\theta,n) \sum_{j=1}^{n} \sum_{\substack{h=1 \\ j \neq h}}^{n} \frac{1}{(\frac{\theta}{\gamma B_L} +j-1)(\frac{\theta}{\gamma B_L} +h-1)} \tag{B-13}$$

$$\dddot{D}(\theta) = \frac{1}{(\gamma B_L)^3} \sum_{n=3}^{\infty} V(\gamma,\theta,n) \sum_{j=1}^{n} \sum_{h=1}^{n} \sum_{\substack{\ell=1 \\ j \neq h \neq \ell}}^{n} \frac{1}{(\frac{\theta}{\gamma B_L} +j-1)(\frac{\theta}{\gamma B_L} +h-1)(\frac{\theta}{\gamma B_L} +\ell-1)} \tag{B-14}$$

$$\dddot{D}(\theta) = \frac{1}{(\gamma B_L)^4} \sum_{n=4}^{\infty} V(\gamma,\theta,n) \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} \quad . \qquad \text{(B-15)}$$

$$j \neq k \neq \ell \neq m$$

$$\frac{1}{\left(\frac{\theta}{\gamma B_L}+j-1\right)\left(\frac{\theta}{\gamma B_L}+k-1\right)\left(\frac{\theta}{\gamma B_L}+\ell-1\right)\left(\frac{\theta}{\gamma B_L}+m-1\right)}$$

The normalized central moments are then given by

$$E[\hat{T}] = \dot{D}(0) \qquad \text{(B-16)}$$

$$E[\hat{T}^2] = 2\dot{D}^2(0) - \ddot{D}(0) \qquad \text{(B-17)}$$

Thus the variance is

$$\sigma^2(T) = E^2[T] - \ddot{D}(0) \qquad \text{(B-18)}$$

$$E[\hat{T}^3] = 6E[\hat{T}]\sigma^2(\hat{T}) + \dddot{D}(0) \qquad \text{(B-19)}$$

while the skewness is

$$\frac{E[\hat{T}-E[\hat{T}]]^3}{\sigma^3} = \frac{3E[\hat{T}]}{\sigma(\hat{T})} - \frac{E^3[\hat{T}]}{\sigma^3(T)} + \frac{\dddot{D}(0)}{\sigma^3} \tag{B-20}$$

Since

$$E[\hat{T}^4] = -36\dddot{D}(0)\dot{D}^2(0) + 8\dddot{D}(0)\dot{D}(0) + 6\ddot{D}^2(0) + 24\dot{D}^4(0) - \ddddot{D}(0)$$

which reduces to

$$= 24E^2(\hat{T})\sigma^2(T) - 6E^4[\hat{T}^4] + 6\sigma^4(T) + 8E(\hat{T})\dddot{D}(0) - \ddddot{D}(0) \tag{B-21}$$

Therefore the excess is

$$\text{Excess} = 3 + \frac{6E^2(\hat{T})}{\sigma^2(\hat{T})} - \frac{3E^4(\hat{T})}{\sigma^4(\hat{T})} - \frac{4E[\hat{T}]D(0)}{\sigma^4(\hat{T})} - \frac{\ddddot{D}(0)}{\sigma^4(\hat{T})} \tag{B-22}$$

where

$$\dot{D}(0) = \frac{1}{8B_L}\sum_{n=1}^{\infty}\frac{(2\delta^2)^n(n-1)!}{(2n)!} \tag{B-23}$$

$$\ddot{D}(0) = \frac{2}{(8B_L)^2}\sum_{n=3}^{\infty}\frac{(2\delta^2)^n(n-1)}{(2n)!}\sum_{j=1}^{n-1}\frac{1}{j} \tag{B-24}$$

$$\dddot{D}(0) = \frac{3}{(8B_L)^2}\sum_{n=3}^{\infty}\frac{(2\delta^2)^n(n-1}{(2n)!}\sum_{j=1}^{n-1}\sum_{\substack{k=1 \\ j \neq k}}^{n-1}\frac{1}{j}\frac{1}{k} \tag{B-25}$$

$$\dddot{D}(0) = \frac{4}{(8B_L)^2} \sum_{n=4}^{\infty} \frac{(2\delta^2)^n (n-1)}{(2n)!} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \sum_{\ell=1}^{n-1} \frac{1}{j}\frac{1}{k}\frac{1}{\ell} \qquad j \neq k \neq \ell \tag{B-26}$$

We note that if the moments are to be evaluated by a digital computer then a useful identity is the following

$$\frac{1}{2^n n z(n)} = \frac{2^n (n-1)!}{(2n)!} \tag{B-27}$$

where

$z(1) = 1/2$

$z(2) = 3/2 \cdot 1/2$

$\vdots$

$z(n) = (n-1/2)\ z(n-1)$

with $z(0)$ defined as equal to 1.

The 1108 was programmend to compute the first four moments of $\hat{T}$. These results were compared with those computed by programming Mimic to solve the moment differential equation

$$\frac{d^2 E[\hat{T}^n]}{d^2\ \phi_o} - \rho\phi_o \frac{dE[\hat{T}^n]}{d\phi_o} = -\frac{n\rho}{4B_L} E[\hat{T}^{n-1}] \tag{B-28}$$

The results of these several programs are shown in Table B-1. It can be seen that the two methods generate results which are in close agreement over the four statistics and that this agreement improves as ρ increases.

TABLE B-1

MOMENTS OF $\hat{T}_{ab}$ DERIVED FROM THE ORENSTEIN-UHLENBECK BACKWARD EQUATION (a=2π and b=2π)

a) Results Computed from Moment Density Function

α	m_n	σ_n	Skewness	Excess
.03125	1.9190984×10^{-2}	1.7806721×10^{-2}	2.0240190	6.0921676
.0625	4.9097526×10^{-2}	4.6267151×10^{-2}	2.0144344	6.0559489
.125	1.7539018	1.6962369	2.0042966	6.0158691
.25	1.5627382×10^{1}	1.5518436×10^{1}	2.0001864	6.0007334
.5625	4.6491181×10^{3}	4.6489418×10^{1}	2.0000000	6.0000000
1.	1.9122959×10^{7}	1.9122959×10^{7}	2.0000000	6.0000000

b) Results Generated by "MIMIC"

α	m_n	σ_n	Skewness	Excess
.03125	1.91907×10^{-1}	$1.63051229 \times 10^{-1}$	1.97450445	5.89304482
.0625	4.90968×10^{-1}	$4.32512917 \times 10^{-1}$	1.98498144	5.9376813
.125	1.75387	1.63650849	1.99577757	5.98276668
.25	1.56269×10^{1}	1.54082444×10^{1}	1.99983898	5.99934946
.5625	4.64883×10^{3}	4.64847067×10^{3}	2.00000684	5.99999
1.	1.91517×10^{7}	1.91517×10^{7}	2.00001042	5.99999

APPENDIX C

APPROXIMATE SOLUTIONS FOR THE SECOND ORDER PHASE LOCK LOOP

In this appendix we assume the second order loop transfer function to be

$$F(s) = 1 + \frac{\alpha}{s} \tag{C-1}$$

and thus the phase lock loop differential equation is given by

$$\frac{d\phi}{dt} = (\omega-\omega_o) - k[A\sin\phi(t) + n'(t)] - \alpha k \int_0^t [A\sin\phi(u) + n'(u)]du \tag{C-2}$$

We next define

$$f_t = K \int_0^t [A\sin\phi(u) + n'(u)]du$$

and proceed to derive a differential equation which describes the long term statistical behavior of the phase error. Our approach is similar to the one taken by Viterbi[3] in deriving the Fokker-Planck equation and its relation to the first order phase lock loop.

We next define four variables descriptively in Figure C-1. Thus we may write the joint density distribution of these variables in the following manner.

$$P(\phi, \phi_1, f_t, \phi_o) = P(\phi_1, \phi_o, f_t) \cdot P(\phi | \phi_1, \phi_o, f_t) \qquad \text{(C-2)}$$

But from equation C-2 we see that ϕ_o adds no knowledge to determining the statistical behavior of ϕ, when we know ϕ_1. Thus if we integrate both sides with respect to ϕ_1 we have

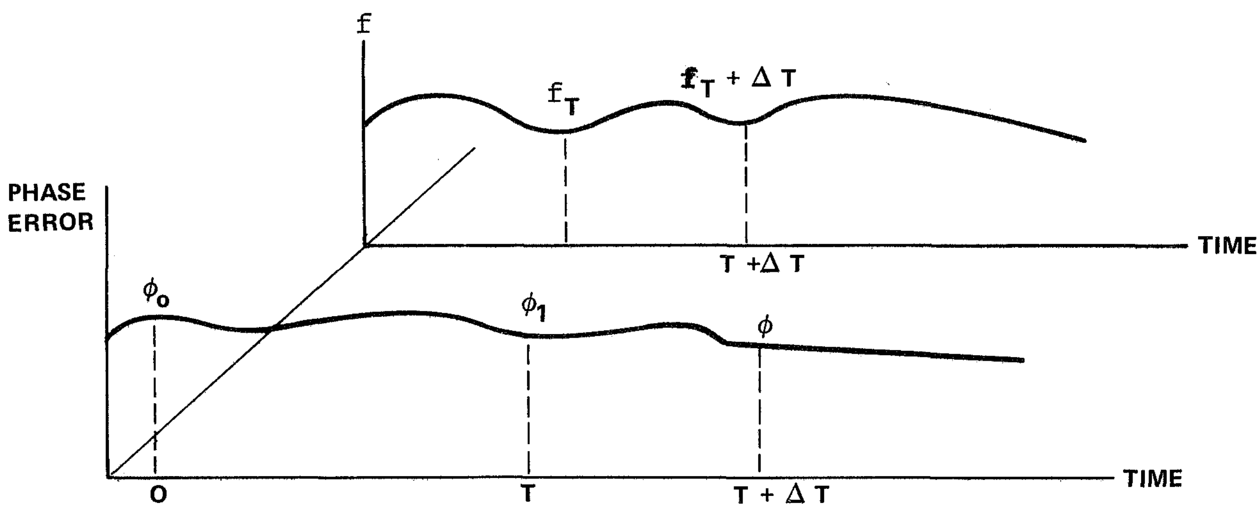

Figure C-1 PHASE ERROR AND f AS A FUNCTION OF TIME

$$P(\phi, f_t, \phi_o) = \int_{-\infty}^{+\infty} P(\phi_1, \phi_o, f_t)\, P(\phi|\phi_1, f_t) d\phi_1 \qquad \text{(C-3)}$$

We can factor out $P(\phi_o, f_t)$ from both sides of equation (C-3) with the result that

$$P(\phi|\phi_o, f_t) = \int_{-\infty}^{+\infty} P(\phi_1|\phi_o, f_t)\, P(\phi|\phi_1, f_t) d\phi_1 \qquad \text{(C-4)}$$

Thus we see that the conditional transitional probabilities satisfy the Chapman Kolmogorov equation given by equation (C-4)).

We shall next derive the corresponding forward and backward conditional differential equations which describe the behavior of the second order loop.

We begin with the integral

$$I = \int_{-\infty}^{+\infty} R(\phi) \frac{\partial P(\phi|\phi_o, f_t)}{\partial t} d\phi \qquad \text{(C-5)}$$

where $R(\phi)$ is an arbitrary analytic function with properties soon to be stated.

Equation (C-5) can be written in the limiting form as

$$I = \lim_{\Delta t \to 0} \int_{-\infty}^{+\infty} R(\phi)d\phi \frac{[P(\phi|\phi_o;\ t+\Delta t,\ f_t) - P(\phi|\phi_o;\ t,\ f_{t-\Delta t})]}{\Delta t} \tag{C-6}$$

Substituting equation (C-4) into equation (C-6) leads to

$$I = \lim_{\Delta t \to 0} \int_{-\infty}^{+\infty} R(\phi)\ \frac{1}{\Delta t}\ [\int_{-\infty}^{+\infty} P(\phi_1|\phi_o;\ t,f_t)P(\phi|\phi_1;\ \Delta t,f_t)d\phi_1$$

$$- P(\phi|\phi_o;\ t,\ f_{t-\Delta t})]d\phi \tag{C-7}$$

Interchanging the order of integration and expanding the analytic function $R(\phi)$ in a Taylor series about ϕ_1 leads to

$$I = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{+\infty} P(\phi_1|\phi_o;\ t,\ f_t)\left[\sum_{n=1}^{\infty} \frac{R^{(n)}(\phi_1)}{n!}\right.$$

$$\left.\int_{-\infty}^{+\infty} (\phi-\phi_1)^n P(\phi|\phi_1;\ \Delta t,f_t)d\phi\right] d\phi_1 + I_1 \tag{C-8}$$

where

$$R^n(\phi) = \frac{dR^n(\phi)}{d\phi^n} \tag{C-9}$$

and

$$I_1 = \lim_{\Delta t \to 0} \int_{-\infty}^{+\infty} R(\phi_1) d\phi_1 \frac{1}{\Delta t} \left[\int_{-\infty}^{+\infty} P(\phi | \phi_1; \Delta t, f_t) d\phi \cdot P(\phi_1 | \phi_o; t, f_t) - P(\phi_1 | \phi_o; t, f_{t-\Delta t}) \right] \tag{C-10}$$

which reduces to

$$I_1 = \lim_{\Delta t \to 0} \int_{-\infty}^{+\infty} R(\phi_1) \frac{1}{\Delta t} \left[(P(\phi_1 | \phi_o; t, f_t) - P(\phi_1 | \phi_o; t, f_{t-\Delta t}) \right] d\phi_1 \tag{C-11}$$

Equation (C-11) can be written as

$$I_1 = \int_{-\infty}^{+\infty} R(\phi_1) \frac{\partial P}{\partial f_t} (\phi_1 | \phi_o; t, f_t) \frac{df_t}{dt} d\phi_1 \tag{C-12}$$

Next we define the limit of the normalized n^{th} conditional moment of the increment $(\phi - \phi_1)$ during the time Δt to be

$$A_n(\phi_1, f_t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{+\infty} (\phi - \phi_1)^n P(\phi | \phi_1; t, f_t) d\phi \quad n \geq 1 \tag{C-13}$$

Substitution of equation (C-13) into equation (C-8) leads to

$$I = \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} R(\phi_1)^{(n)} A_n(\phi_1, f_t) P(\phi_1 | \phi_o;\ t,\ f_t) d\phi_1$$

$$+ \int_{-\infty}^{+\infty} R(\phi_1) \frac{\partial P}{\partial f_t} (\phi_1 | \phi_o;\ t, f_t) \frac{df_t}{dt} d\phi_1 \qquad \text{(C-14)}$$

Under the assumption that $R(\phi_1)$ and its derivatives decrease sufficiently rapidly as ϕ_1 tends to $\pm\infty$ we have

$$R^{n-1}(\phi_1)\ A_n(\phi_1, f_t)\ P(\phi_1 | \phi_o;\ t,\ f_t) \Big|_{-\infty}^{+\infty} = 0$$

$$R^{n-2}(\phi_1)\ A_n(\phi_1, f_t) p(\phi_1 | \phi_o;\ t,\ f_t) \Big|_{-\infty}^{+\infty} = 0$$

$$\vdots$$

$$R(\phi_1)\ \frac{\partial^{n-1}}{\partial \phi} [A_n(\phi, f_t) P(\phi_1 | \phi_o;\ t,\ f_t) \Big|_{-\infty}^{+\infty} = 0$$

Integrating the $n^{\underline{th}}$ term of the sum n times; substituting this result in equation (C-9) and then subtracting this from equation (C-5) leads to

$$\int_{-\infty}^{+\infty} R(\phi_1)d\phi_1 \left\{ \left[\frac{\partial P(\phi_1|\phi_o;\ t,\ f_t)}{\partial t} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \phi_1^n} \right. \right.$$

$$\left. [A_n(\phi_1,\ f_t)P(\phi_1|\phi_o;\ t,\ f_t)] - \frac{\partial P}{\partial f_t}(\phi_1|\phi_o;\ t,\ f_t)\frac{df_t}{dt} \right\} = 0 \qquad \text{(C-15)}$$

Since $R(\phi_1)$ was an arbitrary analytic function except for the above conditions on its derivatives, in order for the integral to vanish the quantity in brackets must be equal to zero. Thus

$$\frac{\partial P(\phi_1|\phi_o;\ t,\ f_t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \phi_1^n} [A_n(\phi_1,\ f_t)P(\phi_1|\phi_o;\ t,\ f_t)]$$

$$+ \frac{df_t}{dt} \frac{\partial P}{\partial f_t} (\phi_1|\phi_o;\ t_1\ f_t) \qquad \text{(C-16)}$$

We next evaluate $A_n(\phi_1,\ f_t)$ using equation (C-2). Integrating equation (C-2) over a small time we have that

$$\Delta\phi = \phi(t+\Delta t) - \phi(t) = \phi - \phi_1 = -\ AK\sin\phi(t)\Delta t$$

$$- \int_0^t AK\sin\phi_1(u)du\ \Delta t - aK \int_t^{t+\Delta t} \int_0^t n'(u)dt - K \int_0^{t+\Delta t} n'(u)du \qquad \text{(C-17)}$$

where we have set $\omega = \omega_o$. Therefore

$$\lim_{\Delta t \to 0} \frac{E[\Delta\phi | \phi_1, f_t]}{\Delta t} = -[AK\sin\phi(t) + \alpha f_t \qquad \text{(C-18)}$$

and

$$\lim_{\Delta t \to 0} E[\Delta\phi^2 | \phi_1, t_f] = \frac{K^2 N_o}{2}. \qquad \text{(C-19)}$$

Since all other $A^n(\phi_1 | f_t)$ terms go to zero as Δt goes to zero equation (C-16) reduces to

$$\frac{\partial P(\phi_1 | \phi_o;\ t,\ f_t)}{\partial t} = \frac{\partial}{\partial \phi_1}\left[\left[AK\sin\phi_1(t) + \alpha f_t\right] P(\phi_1 | \phi_o;\ t\ f_t)\right]$$

$$+ \left(\frac{K^2 N_o}{4}\right) \frac{\partial^2 P(\phi_1 | \phi_o;\ t,\ f_t)}{\partial \phi_1{}^2}$$

$$+\frac{df_t}{dt} \frac{P}{\partial f_t} \quad (\phi_1 | \phi_o;\ t_1\ f_t) \qquad \text{(C-20)}$$

We next expand the left side of equation (C-20) by the chain rule for differentation so that

$$\frac{\partial P(\phi_1 | \phi_o;\ t, f_t)}{\partial t} = \frac{\partial P(\phi_1 | \phi_o;\ t, f_t)}{\partial \phi_1} \frac{d\phi_1}{dt} + \frac{\partial P(\phi_1 | \phi_o;\ t, f_t)}{\partial f_t} \frac{df_t}{dt} \qquad \text{(C-21)}$$

Using equation (C-21), equation (C-20) reduces to

$$\frac{\partial P(\phi_1 | \phi_o;\ t, f_t)}{\partial \phi_1} \frac{d\phi_1}{dt} = \frac{\partial}{\partial \phi_1}\left[\left[AK\sin\phi_1(t) + \alpha f_t\right] P(\phi_1 | \phi_o;\ t,\ f_t)\right]$$

$$+ \frac{K^2 N_o}{4} \frac{\partial^2 P(\phi_1 | \phi_o;\ t,\ f_t)}{\partial \phi_1^2} \qquad \text{(C-22)}$$

We next multiply both sides of equation (C-22) by $P(f_t)$ and then integrate over all values of f_t with the result that

$$\frac{\partial P(\phi_1|\phi_o;\ t)}{\partial t} = \frac{\partial}{\partial \phi_1}\left[AK\sin\phi_1 + \alpha E[f_t|\phi_1]\right]P(\phi_1|\phi_o;\ t) + \frac{K^2N_o}{4}\frac{\partial^2 P(\phi_1|\phi_o;\ t)}{\partial \phi_1^2} \qquad \text{(C-23)}$$

which is equation (21) of the text.