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TECHNICAL MEMORANDUM

A MECHANISTIC VISUALIZATION OF THE SUN-EARTH-EARTH ORBITAL PLANE SYSTEM

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ABSTRACT

A simplified mechanistic model of the sun-earth-earth orbital plane system is presented. In this model the earth orbital plane simultaneously revolves about the lateral surface and rotates about the vertex of an inertially oriented right circular cone. The cone also revolves about a fixed point with uniform circular velocity with its vertex in a plane. Equations are developed for the angles between the orbital and ecliptic planes and between the orbital plane and solar vector. General sun angle envelopes for both date and time of launch are described. By consideration of appropriate angles in the cone base, the times from orbital insertion to angular extrema between the orbital plane and solar vector are determined. Use of the model facilitates visualization of the many significant parameters of earth-orbital space flight mechanics without use of coordinate transformations.

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TECHNICAL MEMORANDUM

I. INTRODUCTION

For the non-specialist in orbital mechanics kinematic consideration of a long mission time, earth orbiting spacecraft involves a formidable array of intersecting planes, time dependent angles and constant angles. A simplified mechanistic visualization of the sun-earth-earth orbital plane system and the associated time dependent, relative motions is presented. Visualization of the many significant parameters of earth-orbital space flight mechanics is facilitated by use of the model. Determination of the angles between orbital elements and the times between sun angle extrema are accomplished by use of the model. Vector geometry is used in the analysis without use of coordinate transformations.

Two recent papers(1)(2) indicate the importance of the angle between the orbital plane and solar vector in mission planning. Envelopes are developed for this angle considering both date and time of orbital insertion.

II. MODEL DESCRIPTION

The model consists of an inertially oriented right circular cone representing the earth which revolves about a point representing the sun's center at constant angular velocity (ω_E). The cone orbit is circular with the vertex in the ecliptic plane. A geocentric, orthogonal, right hand, inertial coordinate system is positioned at the cone vertex and the positive x and y directions are the winter solstice and vernal equinox directions (Figure 1). Initial positioning of the cone is established with the axis coincident with the earth's rotational axis at winter solstice. The angle (e) between the cone axis and the z-axis is also the angle between the equatorial and ecliptic planes. This positioning fixes the cone base parallel to the equatorial plane. For northern hemisphere launch sites the cone is above (positive z) the ecliptic plane. Angular position of the cone in the ecliptic plane is measured from winter solstice by the angle γ. At the insertion date t_I, $\gamma_I = \omega_E t_I$ where ω_E = 360/365.24 deg/mean solar day.

Cone shape is established by the base angle which is equal to the angle (i) between the orbital and equatorial planes. Cone size is unimportant in the analysis. The orbital inclination is constant during the mission, and for a due east/west launch azimuth is equal to the latitude of the launch site (λ). With a due north/south launch azimuth, i = 90° and the cone degenerates to its axis. For other launch azimuths (α) measured from north, $i=\cos^{-1}(\cos\lambda \sin\alpha)$.

The earth orbital plane is tangent to a cone generator. The normal component of the perturbing force induced by the oblate earth causes the orbital plane to rotate about the conical lateral surface at a uniform angular velocity (ω_θ) relative to the cone axis. This rotation is westward (regressive) for posigrade ($i<90°$) orbits and eastward for retrograde orbits. The perturbing force component in the orbital plane causes the orbit to rotate in its plane at a uniform angular velocity (ω_ν) about the cone vertex. Each of these angular velocities is constant relative to the cone for a particular orbit geometry. These two motions are represented in the model by simultaneously rolling the orbital plane about the cone's lateral surface and rotating the orbit in its plane about the cone's vertex at different uniform angular velocities. Initial positioning of the orbital plane on the cone's lateral surface is uniquely determined for any launch site by the insertion time/date, launch azimuth, and longitude difference between the launch site and winter solstice midnight meridians. Figure 1 shows the cone model and coordinate system. Perturbations resulting from drag forces and attractive forces from solar system bodies other than the earth are not considered.

III. ORIENTATION AND MOTION OF THE ORBITAL PLANE

In the winter solstice position, an inertial point P is fixed in the cone base perimeter. Point P is located by the plane which contains the cone axis and is perpendicular to the ecliptic plane. The cone base center and P define the cone base reference line from which the angle θ is measured. The cone generator to which the orbital plane is tangent at any instant of time is located by θ whose constituent angles are measured in the cone base. The terminal side of θ is thus perpendicular to the line of nodes used in conventional representation. Positive θ for northern hemisphere launch sites is in the direction of a right-hand (to the east) rotation about the cone axis. Determination of θ incorporates the launch site location, launch azimuth, orbital insertion date, mission interest date and appropriate angular velocities to establish the orbital plane tangent position on the cone's lateral surface.

The angle γ_I determines the mean sun direction in the ecliptic plane at orbital insertion t_I days after winter solstice. The angle $\gamma_{\theta I}$ locates the local mean noon direction in the cone base and corresponds to γ_I in the ecliptic plane. The angles $\gamma_{\theta I}$ and γ_I are not equal except for the four integer multiples of 90° per year when each season begins. This angular inequality is a consequence of the cone axis inclination (e) relative to the ecliptic plane normal. The angular relationship is

$$\gamma_{\theta I} = \tan^{-1} (\tan \gamma_I / \cos e), \qquad (1)$$

At winter solstice the phase angle between the launch site and local midnight meridian planes depends on the respective longitudes involved. The time difference between these two longitudes corresponds to the phase angle and is used to establish the time origin at winter solstice. The fraction of a day ($\Delta\rho$) corresponding to the time difference between these longitudes can be determined with a current ephemeris.

Both the insertion (t_I) and mission interest (t_M) dates are measured from winter solstice and can include fractional days. If d is an integer number of mean solar days measured at the launch site from winter solstice to the launch day and Δd is a fraction of a day, $t_I = d + \Delta d \pm \Delta\rho$. The sign of $\Delta\rho$ is determined from the launch site location relative to the midnight meridian at winter solstice. If the launch site position is east/west of the midnight winter solstice meridian the appropriate sign of $\Delta\rho$ is +/-. Figure 2 shows these times. The launch site insertion time is thus 24 ($\Delta\rho + \Delta d$) hours and the corresponding cone base angle measured from midnight is 360 ($\Delta\rho + \Delta d$) = B°.

Ascent time to orbital altitude is ignored since it is small compared with mission time. Orbital insertion is thus assumed to occur in the launch meridian plane for due east/west launch azimuths.

For launch azimuths not due east/west, an additional angle must be included in the cone base to locate the orbital plane. For a given launch azimuth α at latitude λ, the orbital plane tangent position is east or west of that corresponding to due east/west launches. This angle is

$$\xi = \cot^{-1} (\tan \alpha \sin \lambda)^*$$

In conventional representation $90 - \xi$ is east or west of the launch meridian and locates the line of nodes for non-due east/west launch azimuths.

The cone base angle which locates the orbital plane tangent position at insertion is thus $\theta_I = \gamma_{\theta I} + 180 + B + \xi$. Figure 2 shows the cone base in the winter solstice and orbital insertion time positions. For clarity this view normal to the ecliptic plane does not show the cone axis inclination relative to the ecliptic plane normal.

The orbital plane's angular velocity (ω_θ) relative to the cone base is induced by the earth's oblateness[(3)(4)]

$$\omega_\theta \simeq 10(R_E/a)^{3.5} \cos i/(1-\varepsilon^2)^2 \text{, deg/day}$$

Since the mission duration of interest is $(t_M - t_I) = \Delta t$ days, the total angle of regression (westward rotation) of the orbital plane for posigrade orbits is $\omega_\theta \Delta t_M$. This cone base angle is measured in a negative direction from the time of orbit insertion. For posigrade orbits the angle θ_M after a mission of Δt_M days is

$$\theta_M = \gamma_{\theta I} + 180 + B + \xi - \omega_\theta \Delta t_M = \theta_I - \omega_\theta \Delta t_M, \qquad (2)$$

The sign of term $\omega_\theta \Delta t_M$ is positive for retrograde orbits. The cone base angle θ_M is thus an explicit function of time dependent variables (d, Δd, and Δt_M), orbit geometric variables (a, i, and ε), launch azimuth (α) and launch site position variable ($\Delta\rho$). Figure 3 shows the respective angles relative to the cone base inertial point P and sun position.

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*The sign of ξ is established by the corresponding sign of tan α. Figures 2 and 3 show positive ξ corresponding to $90>\alpha>0$ and $270>\alpha>180$.

Knowing θ permits specification of the unit normal to the cone generator. The unit normal to the cone generator is also the unit normal to the orbital plane and is defined by the angles i, e and θ in the inertial coordinate system. With $\theta = 0$, this vector lies in the x-z plane and is*

$$\overline{N} = -(si\ ce + ci\ se)\ \overline{i} + (si\ se - ci\ ce)\ \overline{k}$$

For $\overline{N}$ not in the x-z plane ($\theta > 0$)

$$\overline{N} = -(si\ ce\ c\theta + ci\ se)\ \overline{i} - si\ s\theta\overline{j} + (si\ se\ c\theta - ci\ ce)\ \overline{k} \quad , \qquad (3)$$

The components of these two unit normal vectors are determined by inspection from Figure 4.**

IV. <u>ANGLES BETWEEN ECLIPTIC AND ORBITAL PLANES AND ORBITAL PLANE AND SOLAR VECTOR</u>

The angle (σ) between two planes is also the angle between their respective normals. For $90 \geq i > (90-e)$ the orbital plane unit normal $\overline{N}_M$ can point either north or south of the ecliptic plane dependent upon θ. For smaller orbital inclinations, $\overline{N}_M$ always points south of the ecliptic plane.

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*The notation s( ) and c( ) is used for sine and cosine functions.

**The components of $\overline{N}$ in the inertial system can also be determined from

$$\begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix} = \begin{bmatrix} ce & 0 & se \\ 0 & 1 & 0 \\ -se & 0 & ce \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -si \\ 0 \\ -ci \end{pmatrix}$$

Considering the unit normals between the orbital and ecliptic planes at time t_M and using the vector dot product yields,

$$c\sigma = -\bar{N}_M \cdot \bar{k}, \qquad \bar{N}_M \text{ points south of ecliptic plane}$$

$$c\sigma = \bar{N}_M \cdot \bar{k}, \qquad \bar{N}_M \text{ points north of ecliptic plane}$$

$$c\sigma = |si\ se\ c\theta_M - ci\ ce| \ , \ 90 \geq i \geq 0 \ , \qquad (4)$$

At mission date t_M, the unit solar pointing vector from the cone vertex is $\bar{S}_M = -c\gamma_M\bar{i} - s\gamma_M\bar{j}$.* The angle ($\beta$) between the orbital plane and solar vector at date t_M is obtained from the vector dot product.

$$\bar{S} \cdot (-\bar{N}_M) = c\ (\beta + 90)$$

$$s\beta = c\gamma_M\ (si\ ce\ c\theta_M + ci\ se) + s\gamma_M\ si\ s\theta_M^{**} \qquad (5)$$

V. <u>SUN ANGLE ENVELOPES</u>

For due east/west launch azimuths and certain launch times at the start of each season the initial β is a function of only the angles i and e. Table I evaluates β at the beginning of each season with orbit insertion times that are multiples of six hours from midnight.

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* Appendix A discusses the difference between the mean sun motion used in the analysis and the real sun motion.

**This equation in different format is derived using coordinate transformations in Reference (5).

Table 1

INITIAL SUN LINE-ORBITAL PLANE ANGLE (β) AT FIRST DAY OF SEASON FOR DIFFERENT INSERTION TIMES AND DUE EAST/WEST LAUNCH AZIMUTHS

Orbital Insertion Time, Hrs.	Winter Solstice $\gamma_I = 0°$	Vernal Equinox $\gamma_I = 90°$	Summer Solstice $\gamma_I = 180°$	Autumnal Equinox $\gamma_I = 270°$
0	$-(i-e)$	$-i$	$-(i+e)$	$-i$
6	$\sin^{-1}(ci\ se)$	0	$-\sin^{-1}(ci\ se)$	0
12	$i+e$	i	$i-e$	i
18	$\sin^{-1}(ci\ se)$	0	$-\sin^{-1}(ci\ se)$	0

Two envelope curves with the angles from Table 1 are shown in Figure 5 for $(90-e) > i > 0$ and due east/west launch azimuths. Figure 5(a) shows β as a function of insertion time at the winter and summer solstices when absolute maximum and minimum β angles for a given inclination occur with noon and midnight insertions respectively.* Equations of the two bounding curves are obtained from (5) with γ equal to 0° and 180°

$$s\beta = \pm c\theta cesi \pm seci$$

The +/- signs apply respectively to the winter/summer solstice boundaries. Because θ is measured from the inertial point P and the cone is revolving about the sun, this angle will vary for the same time at successive dates. At the beginning of the winter, spring, summer and autumn seasons the magnitudes of θ corresponding to local noon are 0°, 90°, 180°, and 270°

*Midnight at the launch site occurs first after winter solstice at $\gamma=(1-\Delta\rho)\ \omega_E$ or $\Delta\rho\omega_E$ for locations respectively east or west of the winter solstice midnight meridian.

respectively. The β vs. launch time curve for the equinox positions is also shown in Figure 5(a). The equation of this curve is obtained from (5) with γ equal to 90° and 270°

$$s\beta = \pm s\theta \; si$$

The +/- signs apply respectively to the vernal/autumnal equinox positions and the curves coincide.

Figure 5(b) shows the envelope of β angle vs. date of orbital insertion. Local β extrema occur within this envelope as functions of θ and γ. From (5) the cone base angle (θ_M) which yields an extremum value of β at any given time of the year for a corresponding ecliptic plane angle γ_M is

$$\theta_M = \tan^{-1} (\tan \gamma_M / \cos e)$$

This is the same result as (1) and thus indicates that maximum and minimum initial β angles for any time of the year occur with orbital insertions at noon and midnight respectively with due east/west launch azimuths.

The β angle vs. date envelope, Figure 5(b), and initial β for due east/west launch azimuths at noon may be synthesized without (5). An equatorial observer at noon sees the effect of the earth's rotation about the sun as a change in the sun declination angle (ψ_o). By ignoring the slight ellipticity of the earth's orbit the noon declination is $\psi_o = -\sin^{-1} (\sin e \cos\gamma)$ where northerly declinations ($90° < \gamma < 270°$) are positive. Figure 6 shows the noon sun declination as a function of time and the angle γ.*

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*Latitude determination from the earth can also be accomplished with the aid of Figure 6. If the noon sun inclination from the local horizon (ϕ) is measured, the latitude is $\lambda = \psi_o + 90 - \phi$. At any earth location the illuminated parallel of latitude arc for noon declination ψ_o is $2 \cos^{-1} (\tan\lambda \tan\psi_o)$. No illumination occurs at $\lambda > 90 - \psi_o$. Appendix A presents a correction to the sun declination angle to compensate for the difference in mean solar and apparent times.

In the previous discussion β is negative when the orbital noon position is below the ecliptic plane. In an equatorial orbit $\beta = -\psi_\circ$ at noon. The β vs. date envelope for any orbital inclination $<(90 - e)$ is simply established by adding $\pm i$ to the $-\psi_\circ$ curve.

VI. INITIAL β ANGLE

With due east/west launches at a longitude where the noon sun declination is $\psi_\circ$, the initial β for a noon insertion is simply the difference $(i - \psi_\circ)$. For other arbitrary launch azimuths the initial β for local noon insertion is

$$\beta_\circ = \sin^{-1}[\sin\alpha\sin(\lambda-\psi_\circ)]$$

Orbital insertion, in general, will not occur at noon. With perseverance and several spherical triangles the initial β is determined for orbital insertion at a general time and date. For a date with noon declination $\psi_\circ$* and insertion at a time whose corresponding local time angle is B, the initial β for due east/west launch azimuths is

$$\beta_\circ = \sin^{-1}(-sic\psi_\circ cB - cis\psi_\circ)$$

VII. TIME BETWEEN SUCCESSIVE β ANGLE EXTREMA

The angular velocity of the terminal side of θ in the cone base relative to the sun consists of two parts. One component is the uniform angular velocity (ω_θ) relative to the cone axis. The other component $(\dot{\gamma}_\theta)$ corresponds to the angular velocity of the solar pointing vector. Although ω_E is constant in the ecliptic plane for the model, the corresponding rate $\dot{\gamma}_\theta$ in the cone base is position dependent upon γ. This variability results from the cone axis inclination. Since the angular velocity of the orbital plane about the cone's lateral surface is not uniform relative to the sun, the time from any date to the next β extremum is also γ dependent. The angular velocity ratio derived from (1) is

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*The launch site sun declination at orbital insertion is $\psi = \sin^{-1}(-\sin\psi_\circ \cos B)$. Appendix B presents a spherical trigonometry derivation for the general initial β angle.

$$\dot{\gamma}_\theta/\omega_E = ce/(c^2ec^2\gamma + s^2\gamma) \quad , \tag{6}$$

Maximum/minimum absolute values of this ratio are 1.090/0.917 and these occur first at γ angles of 0°/90°. At four times during the year $\dot{\gamma}_\theta = \omega_E$ and the first date occurs at $\gamma = 43°46'$. Based on (1) and (6) the average angular velocity for the mission duration Δt_M is

$$\overline{\dot{\gamma}}_\theta = \omega_E[\tan^{-1}(\tan\gamma_M/ce) - \tan^{-1}(\tan\gamma_I/ce)]/(\gamma_M-\gamma_I) \quad , \tag{7}$$

When (7) is written in the form

$$(\gamma_{\theta M}-\gamma_{\theta I})/\overline{\dot{\gamma}}_\theta = (\gamma_M-\gamma_I)/\omega_E$$

the time equality on the average between corresponding angles generated in the cone base and ecliptic planes is evident.

If a β extremum occurs at angle γ_e, the angle of the next extremum $(\gamma_e + \omega_E\Delta t_e)$ is of interest. For posigrade orbits the rotation direction of ω_θ is opposite that of $\dot{\gamma}_\theta$. The cone base angle corresponding to the angle $\omega_E\Delta t_e$ traveled in the ecliptic to the next extremum is $\tan^{-1}[\tan(\omega_E\Delta t_e)/\cos e]$. The cone base radius corresponding to γ_e will regress an angle $\omega_\theta\Delta t_e$ to the next extremum. Since the angle in the cone base between the two β extrema at the current position (γ_e) is 180°, it follows that

$$180 = \tan^{-1}[\tan(\omega_E\Delta t_e)/\cos e] + \omega_\theta\Delta t_e$$

Neglecting the cone axis inclination provides an explicit approximate solution for the time between successive β extrema.

$$\Delta t_e \simeq 180/(\omega_\theta \pm \omega_E) \quad , \tag{8}$$

The negative sign of ω_E in (8) applies for retrograde orbits. If γ_M = 90° and γ_I = 0° in (7), $\bar{\dot{\gamma}}_\theta = \omega_E$ and (8) is exact.

VIII. TIME FROM AN ARBITRARY DATE TO NEXT β ANGLE EXTREMA

With (8) the approximate times between an arbitrary date and the next β maximum and minimum are readily obtained. At some date the angle locating the orbital plane in the cone base measured from midnight is B + ξ = D. Table 2 lists the approximate times after orbital insertion to the next β angle extrema as functions of D and Δt_e.

Table 2

APPROXIMATE TIMES FROM ORBITAL INSERTION TO β ANGLE EXTREMA

	Orbit	D < 180° *	D > 180°
Time to Next β Maximum	Posigrade	$(1+D/180)\Delta t_e$	$(D/180-1)\Delta t_e$
	Retrograde	$(1-D/180)\Delta t_e$	$(3-D/180)\Delta t_e$
Time to Next β Minimum	Posigrade	$D\Delta t_e/180$	$D\Delta t_e/180$
	Retrograde	$(2-D/180)\Delta t_e$	$(2-D/180)\Delta t_e$

The accuracy of these approximations increases with the following conditions: D ≃ 90° and 270°; missions where the time midway between orbital insertion and the extremum approximately coincides with a midseason date; circular orbits of low altitude and low inclination for which $\bar{\dot{\gamma}}_\theta/\omega_\theta \simeq 0.1$.

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*D = B + ξ = 360 (Δρ + Δd) + ξ

The corresponding exact equations for the next β extrema contain time implicitly. At some arbitrary time t_M the orbital plane is located by the angle D measured from midnight in the cone base. If the time to the next β extremum is Δt_{Me}, the corresponding easterly angular shift of the midnight position in the cone base to the extremum date is $E = \tan^{-1}[\tan(\Delta t_{Me}\omega_E)/\cos e]$. The times to the next extrema from D are established by evaluating the angles in the cone base to either the local noon or midnight positions at date $t_M + \Delta t_{Me}\,\omega_E$ with due consideration of the rotation direction of ω_θ. During time Δt_{Me}, the orbital plane rotation to the next extrema is $F = \omega_\theta \Delta t_{Me}$. The exact equations for these times are listed in Table 3.

Table 3

EQUATIONS FOR DETERMINATION OF TIME FROM AN ARBITRARY DATE TO THE FOLLOWING β ANGLE EXTREMA

	Orbit	D < 180°	D > 180°
To Next β Maximum	Posigrade	D-E+180=F *	D-E-180=F
	Retrograde	-D+E+180=F	-D+E+540=F
To Next β Minimum	Posigrade	D-E=F	D-E=F
	Retrograde	-D+E+360=F	-D+E+360=F

Ignoring the inclination of the cone axis with respect to the ecliptic plane normal reduces the exact equations of Table 3 to their corresponding approximate counterparts of Table 2.

$*(B+\xi)-\tan^{-1}[\tan(\Delta t_{Me}\omega_E)/\cos e]+180=\omega_\theta \Delta t_{Me}$

IX. ORIENTATION AND MOTION IN THE ORBITAL PLANE

Spacecraft motion in the orbital plane and the secular perturbation which rotates the orbit in its plane are considered here. At orbital insertion the unit cone generator ($\overline{G}_I$) to which the orbital plane is tangent is in the direction of the perigee radius. This vector from the vertex along the cone's lateral surface corresponds to a base angle θ and is defined from the geometry of Figure 4.

$$\overline{G} = (\text{si se} - \text{ci ce c}\theta)\ \overline{i} - \text{ci s}\theta\overline{j}$$

$$+ (\text{si ce} + \text{ci se c}\theta)\ \overline{k}\ , \qquad (9)$$

Figure 7 is a view normal to the orbital plane at the date t_M. The angle in the cone base between tangent positions of the orbital plane at insertion and mission interest times is $\Delta\theta = \omega_\theta \Delta t_M$. The angle ($\Delta\theta$ cos i) in the orbital plane between the traces of generators $\overline{G}_I$ and $\overline{G}_M$ is the angle of the developed lateral conical surface for a cone base angle $\Delta\theta$. The total angular displacement in the orbital plane measured from the perigee during time Δt_M is the sum of the true anomaly (ν) and the perturbed apsidal angular displacement ($\omega_\nu \Delta t_M$) of the orbit in its plane. With circular orbits, $\nu = \Delta t_M \mu^{1/2}/r^{3/2}$. With elliptical orbits the mean anomaly is the product of the mean motion ($\mu^{1/2}/a^{3/2}$) and Δt_M. Kepler's equation must be solved using the mean anomaly to determine ν for elliptical orbits. The angular velocity (ω_ν) of the orbit in its plane is induced by the orbital plane component of the perturbing force from the earth's equatorial bulge.(3)(4)

$$\omega_\nu \simeq 5(R_E/a)^{3.5}\ (5\cos^2 i - 1)/(1-\varepsilon^2)^2\ , \text{ deg/day}$$

For 63.5° > i > 116.5° the perigee rotation direction is that of increasing ν. For 63.5° < i < 116.5° the perigee rotation direction is that of decreasing ν. No rotation of the orbit in its plane occurs for i=63.5°, 116.5°.

At date t_M the spacecraft radius vector ($\overline{r}$) is displaced from the perigee unit vector $\overline{G}_I$ an angle ($\nu \pm \omega_\nu \Delta t_M$). The angular displacement of $\overline{r}$ from the unit vector $\overline{G}_M$ at date t_M is $\nu \pm \omega_\nu \Delta t - \Delta\theta ci$. The components of the unit vector $\overline{R}$ in the direction $\overline{r}$ are obtained from:

$$\overline{R} \cdot \overline{G}_M = \cos\,(\nu \pm \omega_\nu \Delta t_M - \Delta\theta ci) = G_{Mx}R_x + G_{My}R_y + G_{Mz}R_z$$

$$\overline{R} \cdot \overline{N}_M = 0 = N_{Mx}R_x + N_{My}R_y + N_{Mz}R_z$$

$$1 = R_x^2 + R_y^2 + R_z^2$$

The appropriate sign of $\omega_\nu \Delta t_M$ is determined from the magnitude of i as discussed above. Figure 7 shows the perigee rotation in the direction of increasing ν. Since the components of $\overline{G}_M$ and $\overline{N}_M$ are determined from (9) and (3) in terms of θ, the geometry of the system at any time is established.

X. ACKNOWLEDGMENT

Mr. B. D. Elrod thoroughly reviewed the text and found areas where the analysis was either imprecise or unclear. His effort and suggestions are appreciated.

1022-JWP-mef

J. W. Powers

Attachments
- References
- List of Symbols
- Appendices A-B
- Figures 1-8

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List of Symbols

a	Semi-major axis, elliptical orbit
B	Angle in cone base corresponding to launch site local time at insertion, 360 $(\Delta\rho + \Delta d)$
c( )	Cosine of angle ( )
d	Number of integer mean solar days since winter solstice at launch site to orbital insertion
D	Angle measured to the east in cone base from midnight to orbital plane tangent position, $B+\xi$
E	Angle in cone base between local midnight positions for an arbitrary date and a β angle extremum date
F	Angle of orbit plane rotation in cone base from an arbitrary date to next β angle extremum
Δd	Fractional portion of a day at launch site to orbital insertion
e	Angle between ecliptic and equatorial planes ($\approx 23°27'$), also inclination of cone axis to normal from ecliptic plane
$\bar{G}$	Unit vector from cone vertex along cone generator to which orbital plane is tangent
i	Angle between equatorial and orbital planes, also cone base angle
$\bar{i}$	Unit Vectors: Winter Solstice Direction
$\bar{j}$	Unit Vectors: Vernal Equinox Direction
$\bar{k}$	Unit Vectors: $\bar{i} \times \bar{j}$
$\bar{N}$	Unit vector normal to orbital plane
$\bar{r}$	Orbit position vector of spacecraft, $r\bar{R}$
$\bar{R}$	Unit vector in direction $\bar{r}$
R_E	Radius of earth
s( )	Sine of angle ( )
$\bar{S}$	Unit solar pointing vector from cone vertex in ecliptic plane

t	Time, days
Δt	Time difference
x	Coordinate Axes: Winter Solstice Direction
y	Coordinate Axes: Vernal Equinox Direction
z	Coordinate Axes: Normal to Ecliptic Plane
α	Launch azimuth measured from north
β	Angle between orbital plane and solar vector
γ	Angle measured in ecliptic plane between winter solstice and solar vector directions
$\dot{\gamma}$	Angular velocity of solar pointing vector in cone base
ε	Orbit eccentricity
θ	Angle in cone base locating orbital plane
$\Delta\theta$	Angular difference in cone base between times of mission interest and orbital insertion, $\theta_M - \theta_I$
λ	Earth latitude
μ	Earth gravitational constant
ξ	Angle measured in cone base from launch meridian to orbital plane tangent position
$\Delta\rho$	Local time difference between launch site and midnight meridians at winter solstice, fractional portion of a day
σ	Angle between orbital and ecliptic planes
ν	Angle, true anomaly
ϕ	Angle between horizon and sun measured on earth at local noon
ψ	Sun declination angle
ω	Angular velocity

Subscripts

e	β angle extremum to next β extremum conditions
I	Insertion conditions
M	Mission interest conditions
Me	Mission interest to β angle extremum conditions
o	Local noon or initial conditions
x,y,z	Coordinate directions
θ	Plane of cone base
ν	Plane of orbit

APPENDIX A

COMPENSATION FOR THE VARYING ANGULAR VELOCITY OF THE EARTH ABOUT THE SUN

The previous analysis assumes the earth to be in a circular orbit about the sun. This appendix provides an adjustment for the effect of this assumption in the sun declination and β angle equations. The procedure is the inverse of that used to convert from sundial (apparent) time to clock (mean) time.

A small time dependent periodic change occurs in the earth's angular velocity about the sun as a result of orbit eccentricity ($\simeq 1/60$). If winter solstice (22 December) and perihelion (5 January) are both assumed to occur simultaneously at the time origin, the angular velocity of the earth to a good approximation is

$$\omega \simeq \omega_E (1 + \varepsilon \cos \gamma)^2/(1 - \varepsilon^2)^{3/2}$$

Maximum and minimum values of ω from the above equation are 1.019 and 0.945 deg/day respectively as compared with the mean value (ω_E) of 0.986 deg/day.

An observer on the cone base will notice that the noon sun is either fast or slow by his clock (mean solar) for all but a maximum of four days in the year. This time difference is a maximum of approximately 16 minutes and is caused by the combined effects of the variable earth angular velocity about the sun and the obliquity of the ecliptic. This time difference between the real and mean suns at noon for any date can be determined from the ephemeris equation of time or the analemma.

For a date when the ecliptic plane mean sun angle measured from winter solstice is γ, the corresponding cone base angle measured from the inertial point P is

$$\gamma_\theta = \tan^{-1} (\tan \gamma/\cos e), \qquad (1)$$

The angular difference ($\Delta\gamma_\theta$) between the real and mean noon positions is readily calculated from the equation of time for any date. One minute of time equals fifteen minutes of arc in the cone base. The cone base angle from P to the true noon position for the ecliptic plane angle γ is thus $\gamma_\theta \pm \Delta\gamma_\theta$. The +/- signs apply respectively for dates when the real sun lags/leads the mean noon. From (1) the new ecliptic plane angle (γ') corresponding to the cone base angle $\gamma_\theta \pm \Delta\gamma_\theta$ and the real sun position is determined.

$$\gamma' = \tan^{-1} \{\cos e \tan [\tan^{-1}(\tan \gamma/\cos e) \pm \Delta\gamma_\theta]\}, \qquad (1A)$$

Use of this angle in (5) yields a more accurate determination of β.

In section V the following equation for noon sun declination is given.

$$\psi_o = -\sin^{-1} (\sin e \cos \gamma)$$

This equation also assumes a uniform angular velocity for the earth about the sun. With use of (1A) in the above equation a more precise sun declination is obtained.

APPENDIX B

DETERMINATION OF INITIAL ANGLE BETWEEN THE SOLAR VECTOR AND ORBITAL PLANE AT INSERTION

In Section VI the initial β angle is given for both general launch azimuths at noon and due east/west launch azimuths for a general time of the day. This Appendix develops the initial β angle for an arbitrary date and time of the year and an arbitrary launch azimuth.

Figure 8 shows the portion of the spherical earth's lune between local noon and 6 p.m. which is north of the ecliptic plane. This surface represents a day during the fall or winter seasons when the equator is north of the ecliptic plane with orbital insertion between noon and 6 p.m. All arcs shown on the surface are those of great circles.

Points a to f on the surface define certain spherical right triangles. Combinations of these three letters define angles between the planes shown and two letters with the origin location, o, define interior angles with vertices at o. With this notation the following are defined:

Launch site position, a

Sub-Solar point, e

Launch site latitude, λ = (aob)

Local noon declination, $-\psi_o$ = (doe)*

Launch azimuth, α = (bac)

Angle between local noon and launch site meridians at insertion, B-180 = (bod)

Initial angle between solar vector and orbital plane, $-\beta_o$ = (eof)*

Orbit inclination, i=(acb)

Angle between noon direction and ascending node, $B+\xi-270$=(cod)

From spherical triangle III, (cef)

$$s(-\beta_o) = s[i-(dce)]s(coe)$$

---

* As shown in Fig. 8 ψ_o and β_o are negative as defined in Sec. V.

From spherical triangle II, (cde)

$$c(dce)s(coe) = c\psi_o c(B+\xi)$$

$$s(dce)s(coe) = -s\psi_o$$

Substituting the above values from spherical triangle II in the equation from triangle III

$$s\beta_o = -[sic\psi_o c(B+\xi) + cis\psi_o], \tag{1B}$$

For due east/west launch azimuths $\alpha=90°$, $\xi=0$ and the general β angle equation (1B) reduces to the last equation of Section VI.

$$s\beta_o = -(sic\psi_o cB + cis\psi_o), \tag{2B}$$

With insertion at noon B=180° and (1B) reduces to

$$s\beta_o = sic\psi_o c\xi - cis\psi_o, \tag{3B}$$

Equation (3B) is equal to but of different form to the first equation of Section VI. With B=180° in (2B) and $\xi=0$ in (3B), both equations reduce to the expected result

$$\beta_o = i - \psi_o$$

The above equations yield correct signs for β with negative values when the orbital noon position is below the ecliptic plane. The following general rules for checking the sign of β_o apply for east/west launch azimuths.

- Initial β will always be positive for insertions between 6 a.m. and 6 p.m. during the fall and winter seasons
- Initial β will always be negative for insertions between 6 p.m. and 6 a.m. during the spring and summer seasons

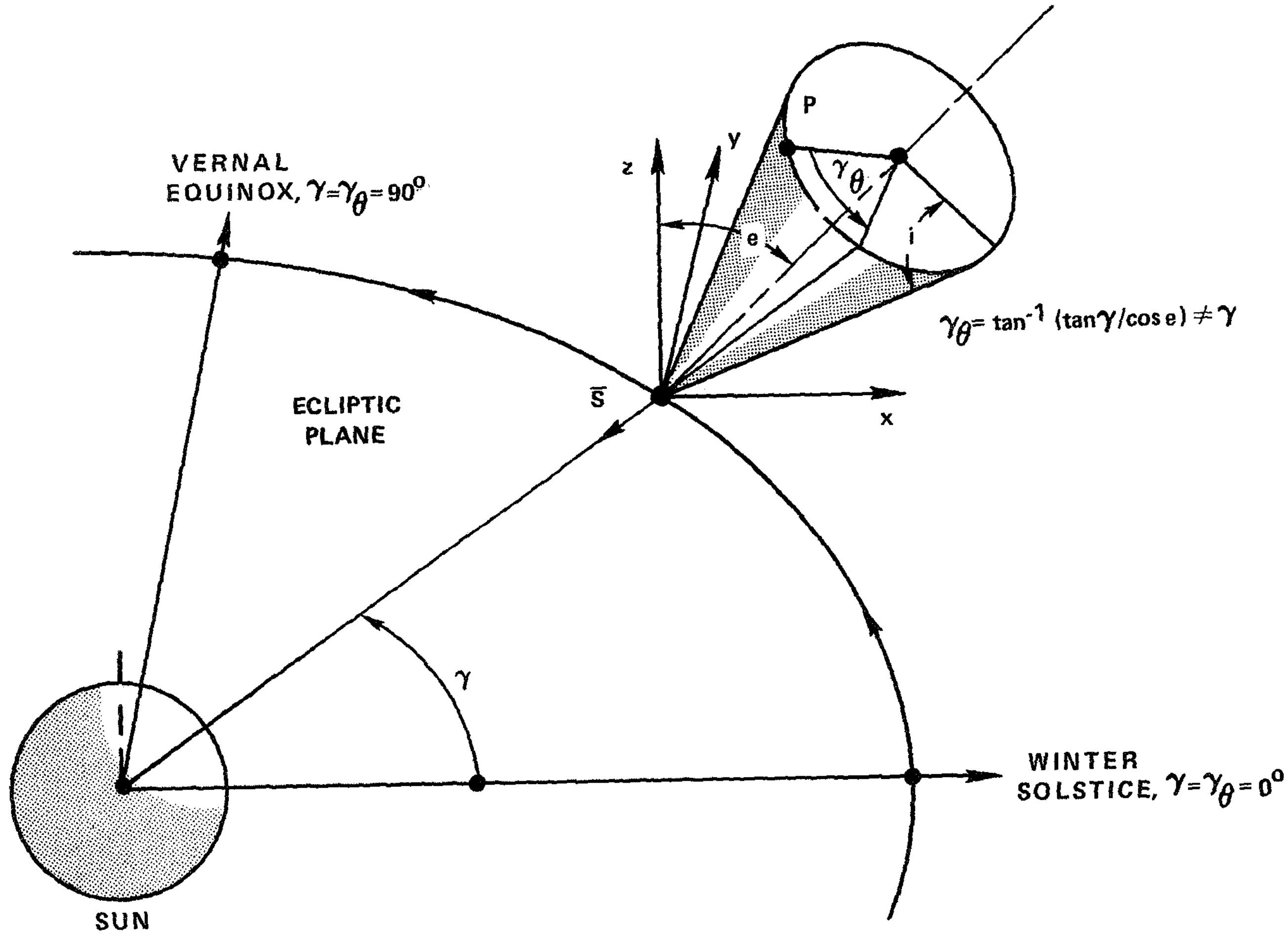


FIGURE 1- INERTIAL CONE MODEL & COORDINATE SYSTEM

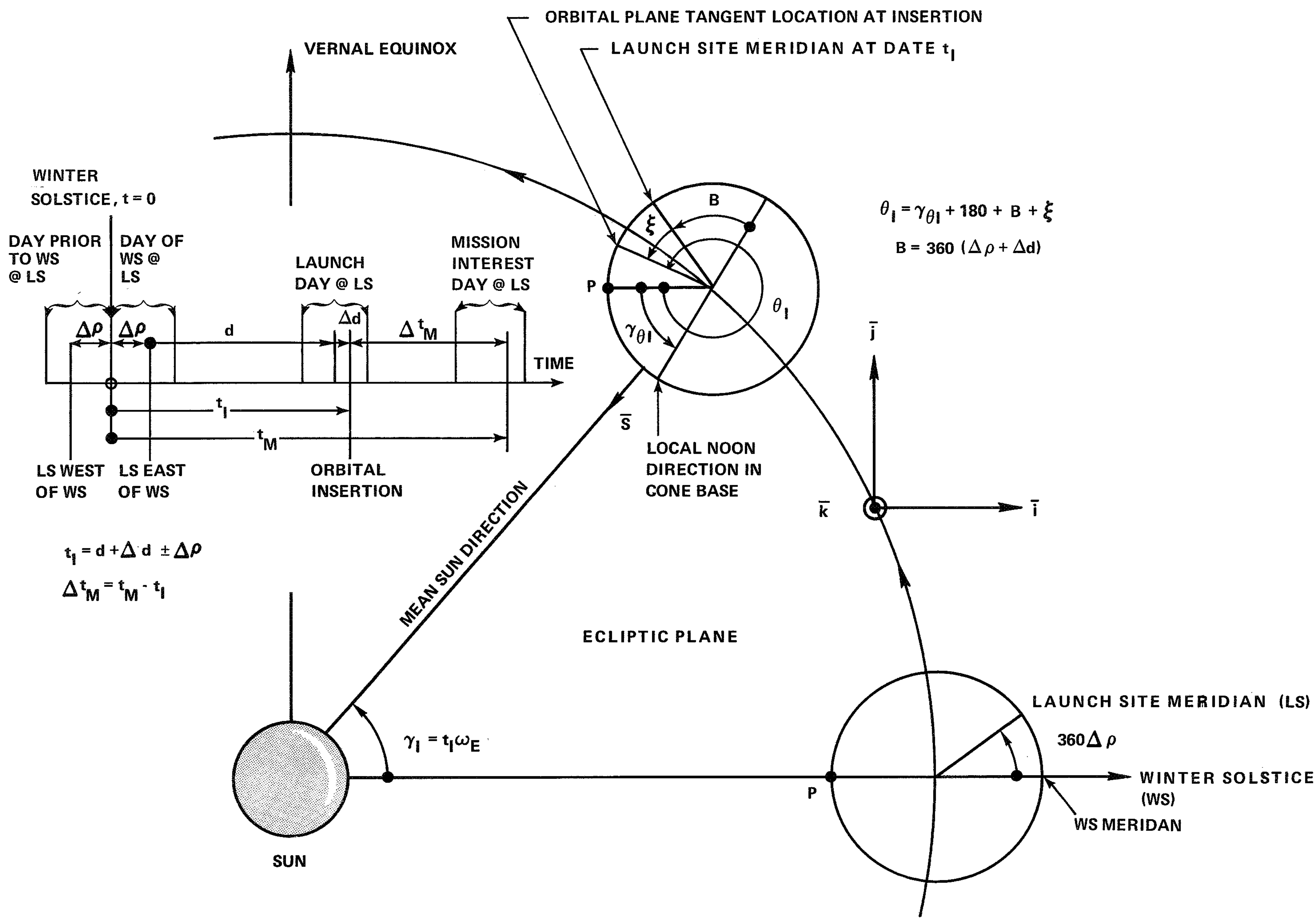
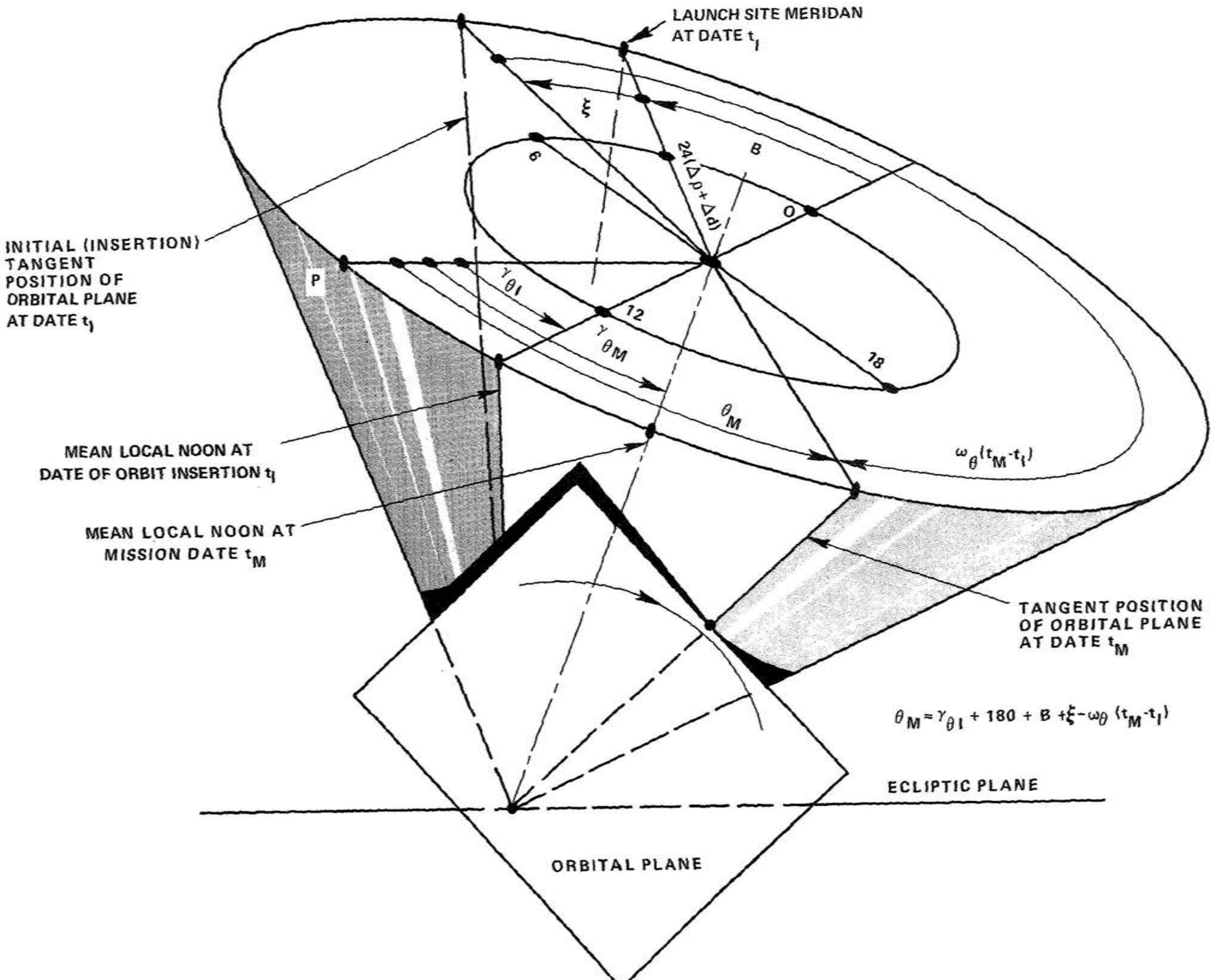


FIGURE 2- CONE BASE ORIENTATION AT WINTER SOLSTICE AND ORBITAL INSERTION DATE t_I. INCLINATION OF CONE TO $\bar{k}$-AXIS NOT SHOWN.

FIGURE 3 - VIEW OF INERTIAL CONE FROM CENTER OF SUN DURING A DAY OF WINTER SEASON
($0 < \gamma < 90$)



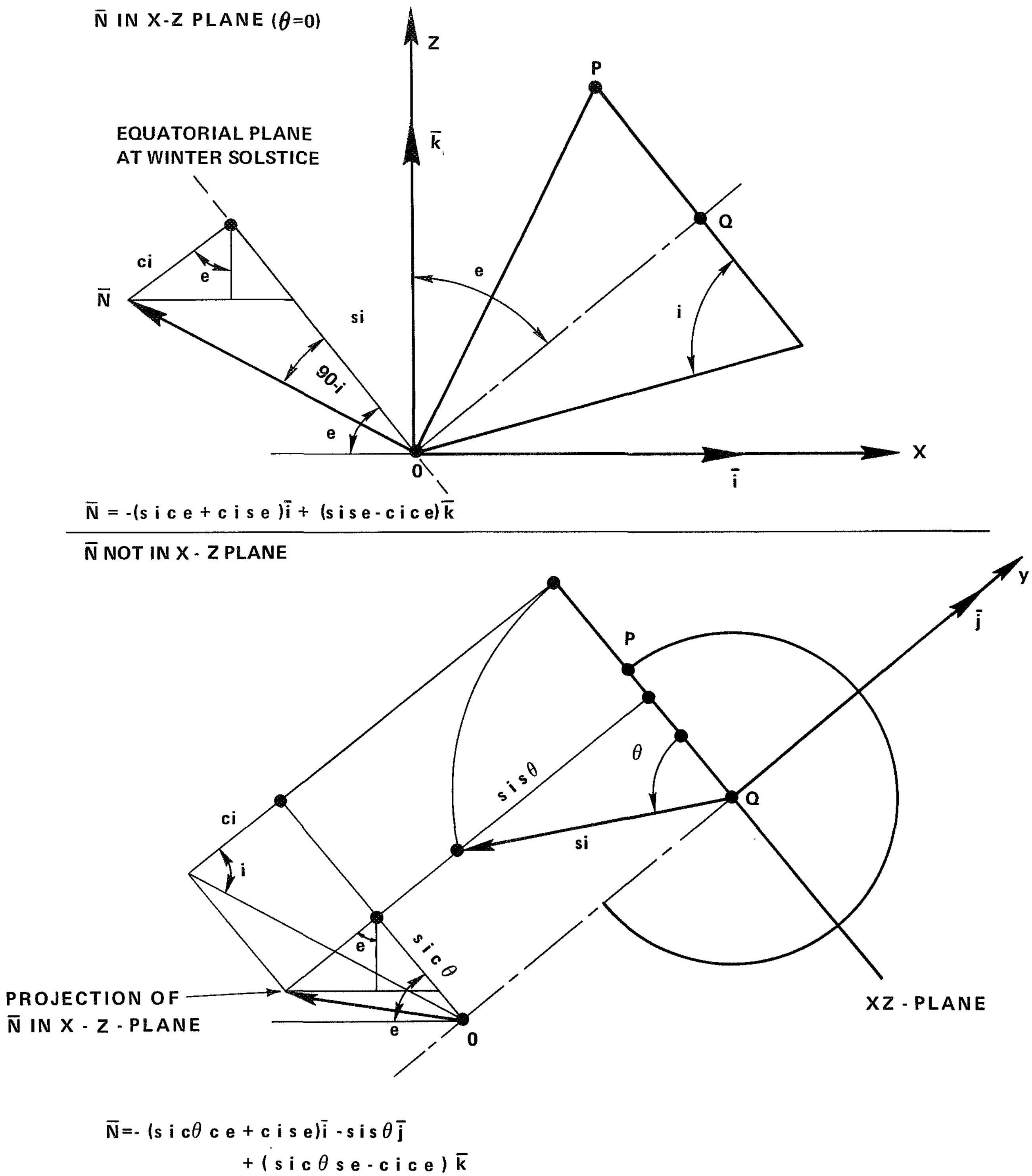


FIGURE 4- GEOMETRY FOR DETERMINATION OF UNIT VECTORS NORMAL TO CONE GENERATOR

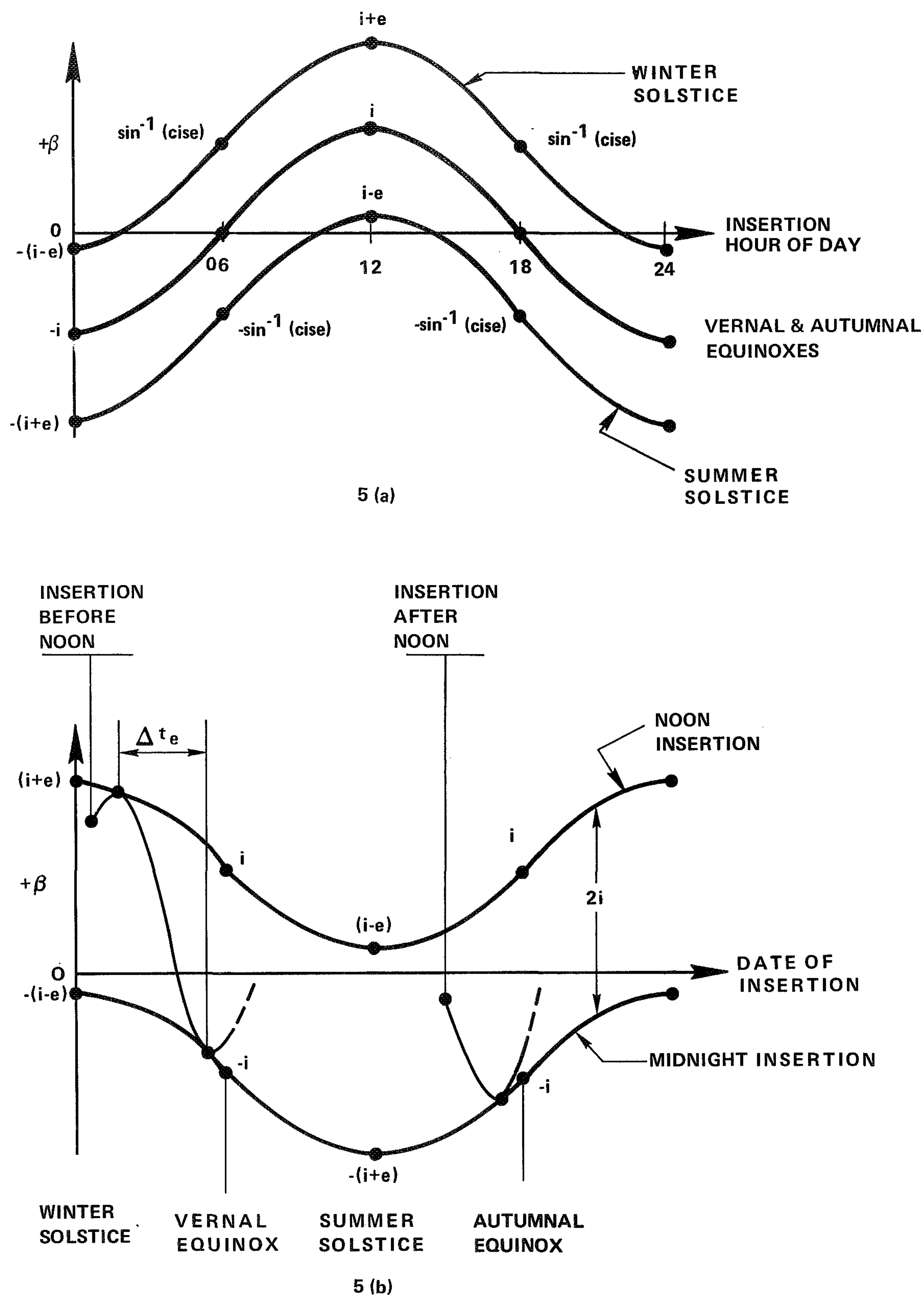


FIGURE 5- β ANGLE ENVELOPES AS FUNCTIONS OF INSERTION HOUR & DATE. DUE EAST/WEST LAUNCH AZIMUTHS, $(90 - e) > i > 0$

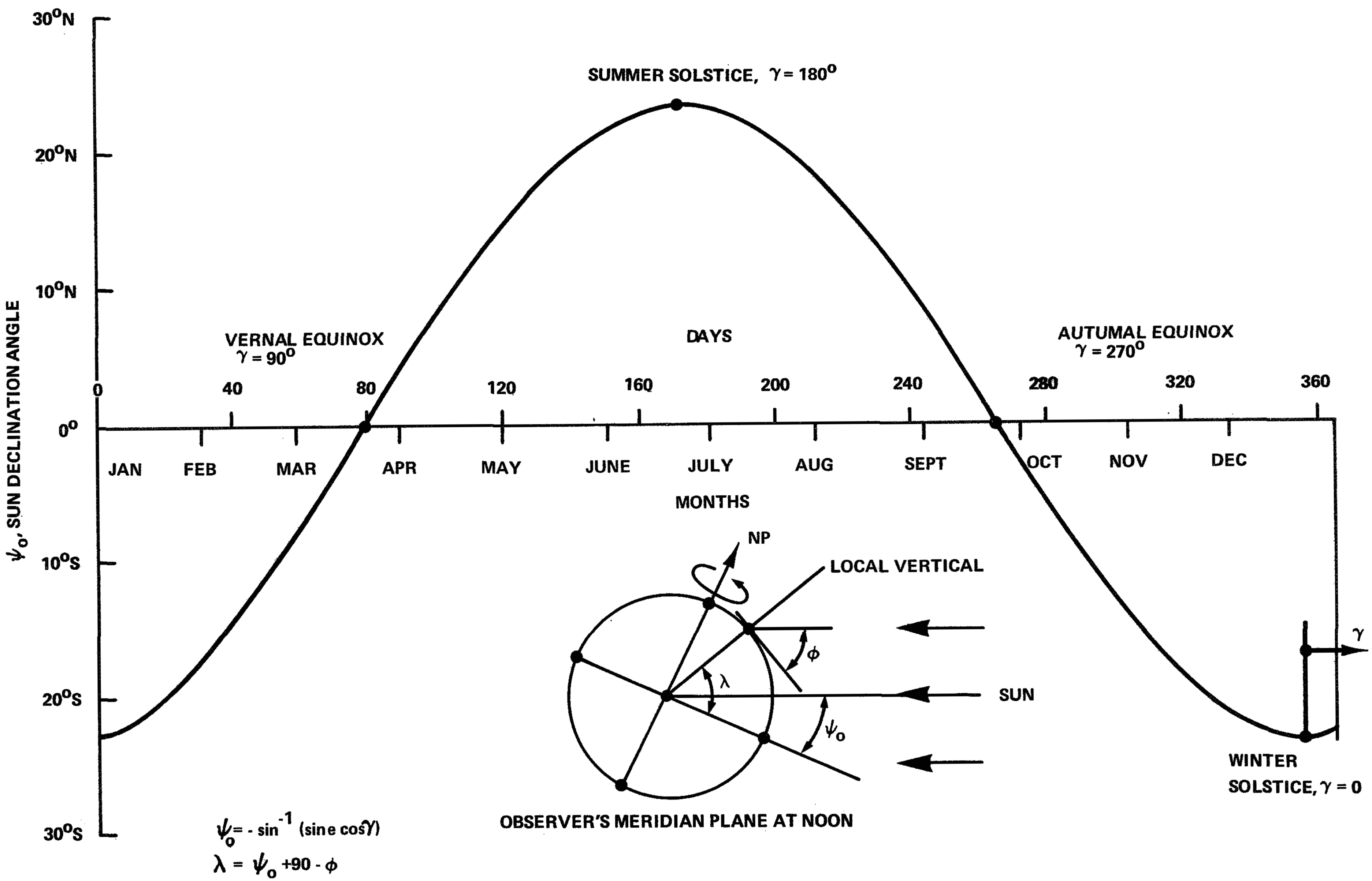


FIGURE 6 - SUN DECLINATION AT LOCAL NOON AS A FUNCTION OF DATE OF YEAR

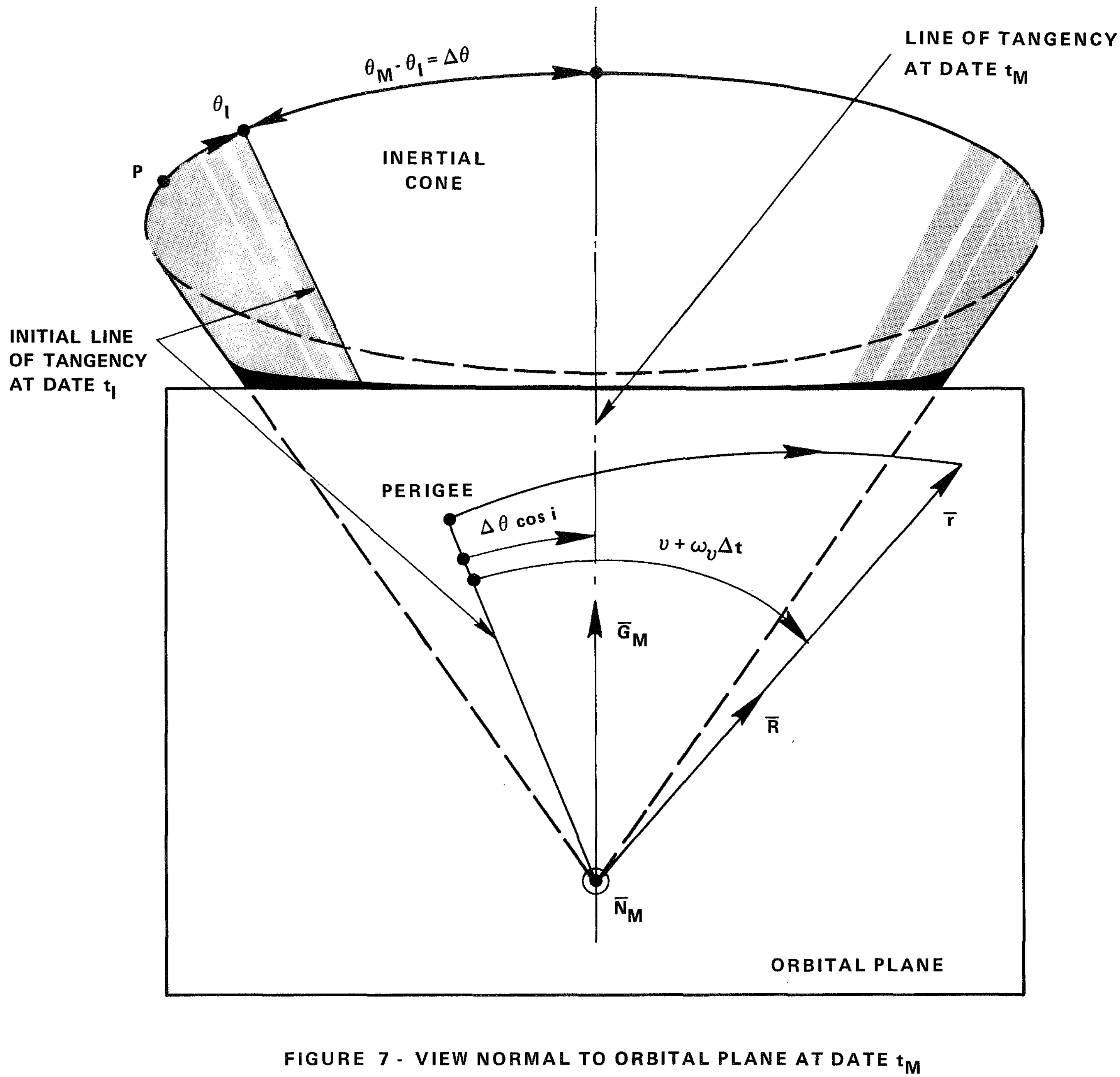


FIGURE 7 - VIEW NORMAL TO ORBITAL PLANE AT DATE t_M

$63.5^o > i > 116.5^o$

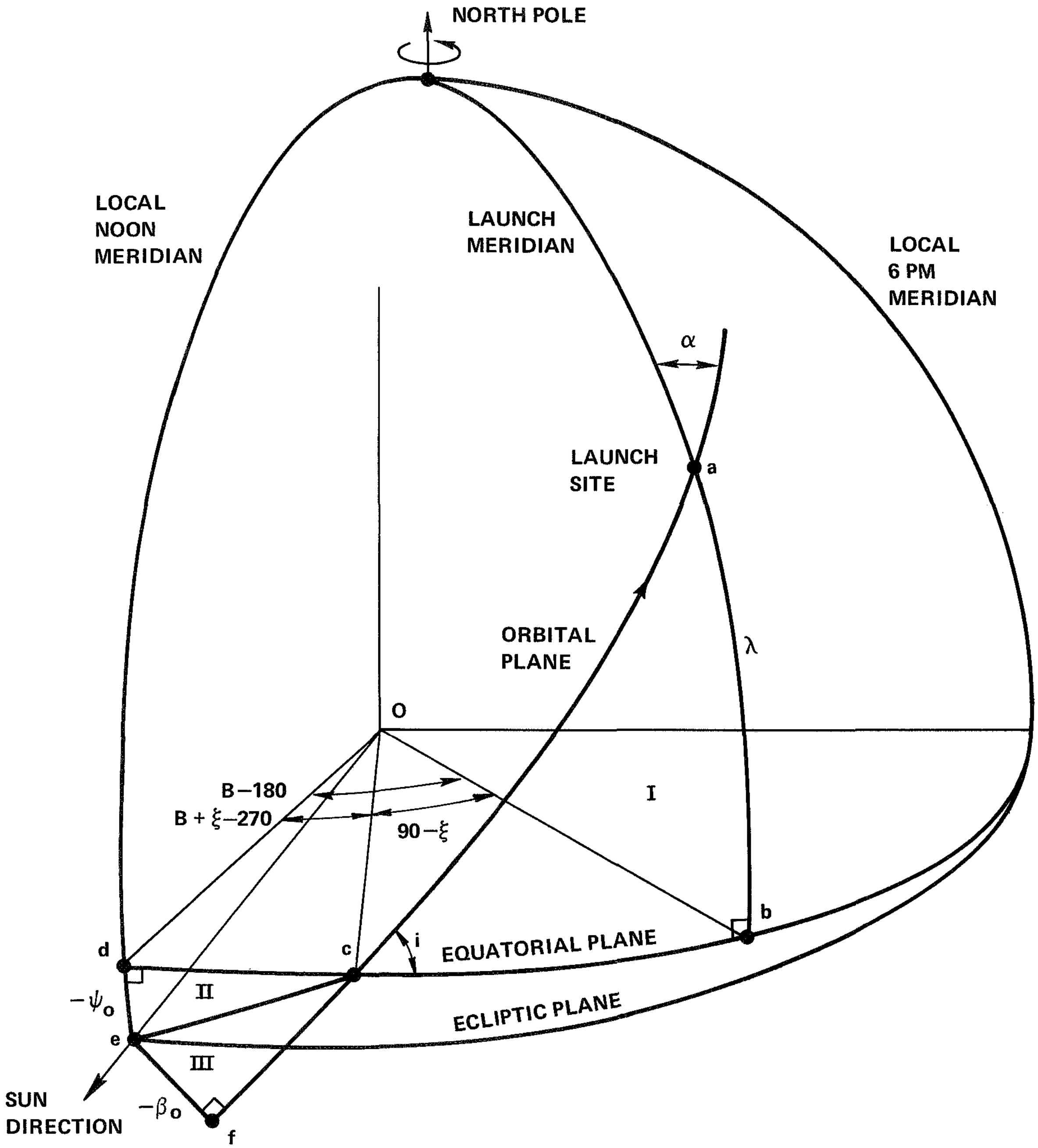


FIGURE 8. SPHERICAL GEOMETRY FOR DETERMINATION OF INITIAL GENERAL ANGLE BETWEEN SUN DIRECTION AND ORBITAL PLANE.