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TECHNICAL MEMORANDUM

STATISTICAL ANALYSIS OF NONSTATIONARY STRUCTURAL RESPONSE UNDER FEEDBACK CONDITIONS

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ABSTRACT

A method for analyzing augmented systems of structures, controls, and other subsystems is presented in this paper. The systems have to be linear but may be time varying and under nonstationary excitation. The analysis results in equations for solving the covariance matrix of the state of the system. The state of the system is a vector which may contain translational and rotational motions of the structures as well as other variables of the system and their derivatives. Computational considerations involved in the solution of these equations are also discussed.

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FROM: I. Y. Bar-Itzhack
S. N. Hou

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TECHNICAL MEMORANDUM

INTRODUCTION

The analysis of structural vibrations caused by random excitation has received general attention for the past two decades. Structural engineers have benefited from the early mathematical work on random processes and its application by electrical engineers to Information Theory and Control Theory. Such adaptation of knowledge to the structural problem has advanced the art appreciably. However, in order to avoid confusion in terminology and concept, restating and extending the principles from one field to another field has been an essential task.

The works of Wiener,[1,2,3] Lee and Wiener,[4,5,6] Khinchin,[7] Titchmarsh,[8] and others opened the Wienerian era. Structural engineers adopted Wienerian concepts, which are widely used in the so-called "Wiener-Filter" to analyze linear structures under stationary random excitation. The works of Crandall[9] and especially Crandall and Mark[10] were the main contributions in this direction. The highlights of the Wienerian approach are as follows:

1. The spectral density function of a random and at least weakly stationary* process is the Fourier transform of the autocorrelation function of the process divided by 2π.

2. The spectral density function of the filtered output equals the product of the square of the absolute value of the frequency response function of the filter and the spectral density function of the excitation of this filter.

Since the Wienerian approach analyzes the system in the frequency domain, it has limited power; specifically, it cannot treat cases where (a) the excitation forces are nonstationary, and (b) the system, although linear, is time varying. Faced with these restrictions, structural engineers have tried to solve special cases where certain assumptions could be made,[11] but all the work was confined to Wienerian concepts and hence to frequency domain analysis, which still is the state of the art in structural engineering.

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*Other popular terms for a weakly stationary process are "stationary in the wide sense," "covariance stationary," or "second order stationary" random process.

In the beginning of the sixties a break-through was achieved by systems engineers, which was mostly due to the work of Kalman,[12,13] Bucy[14] and their common work.[15] Although many other investigators dealt with the same problems, it was Kalman and particularly the so-called "Kalman-Filter" technique that spread the interest in Kalmanian concepts in the very same way that the Wiener-Filter spread the Wienerian concepts. Although elements needed for the Kalmanian approach are already known to structural engineers,[16] the technique itself has yet to be applied. Once again it is time for the structural engineer to introduce new concepts taken from systems engineering to obtain nonstationary random responses.

The Kalmanian approach rids itself of the limitations involved with the Wienerian approach by being confined to only time domain analyses, and by using ensemble averages. This way neither ergodicity nor transformation to the frequency domain is necessary. For these reasons the Kalmanian approach is suitable for treating nonstationary excitation of time varying structures, whether they stand by themselves, or whether they constitute a subsystem in a feedback system. Moreover, even when the excitation is stationary and the system is constant, the Kalmanian approach offers certain computational advantages, especially when the system includes a lightly damped structure.

In most cases, we are mainly interested in the response intensity at certain points in the structures due to random excitations at some other points. We need to know not only the mean value of the response, which can be easily obtained, but also its variance about the estimated level. The problem of finding the mean response is a trivial one because it is a deterministic problem. Thus in our analysis we will concentrate mainly on the solution for the variance of the response, or more specifically, we will deal with the time solution of the covariance matrix of the system.

STATE VARIABLE APPROACH TO STRUCTURAL DYNAMICS

A basic requirement in the Kalmanian approach is the state variable representation of a linear system. Such an idea was first introduced in 1936 by A. M. Turing. However, its wide application in the control field was initiated in the forties by the Russian scientists M. A. Aizerman, A. A. Feldbaum, A. M. Letov, A. I. Lure, and others. A sufficient discussion of the state variable approach can be found in most of the modern control books.

The behavior of a linear physical system can usually be described by a mathematical model, which consists of a set of linear ordinary differential equations. The number of equations should be sufficient to describe the interesting properties of the system, and they can be of any order. This set, however, can be transformed into a first order matrix differential equation, which takes the following form:

$$\{\dot{x}\} = [A(t)]\{x\} + [B(t)]\{F\} \quad . \tag{1}$$

The vector $\{x\}$ is the state vector or the response of the system, whereas the vector $\{F\}$ is the input or the excitation of the system. The matrix $[A(t)]$ is the system matrix, and the matrix $[B(t)]$ is the input or the excitation matrix. Finally, it should be added that the state variable representation of a system is not unique. In this paper the linear system will include the structures as well as control loops and parasitic feedback loops, although any other conceivable linear subsystems can be included.

As an example, we will now show the derivation of a state variable representation of a system including structures, velocity feedback loops, and position feedback loops.

During a relatively short interval of time, this system will have the following linear equation of motion in matrix form:

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = [G_f]\{f\} + [G_u]\{u\} + [G_v]\{\dot{u}\} \quad , \tag{2}$$

where

$\{u\}$ is a vector containing the motion of nodes in their physical coordinates,

$\{f\}$ is a vector containing the time functions of the loads (i.e., engine thrust) at certain nodes,

$[m]$ is the mass matrix of the structure,

$[c]$ is the damping matrix of the structure,

$[k]$ is the stiffness matrix of the structure,

$[G_f]$ is the load coefficient matrix,

$[G_u]$, $[G_v]$ are the feedback coefficient matrices of $\{u\}$ and $\{\dot{u}\}$ respectively.

Rewriting the above equation as

$$\left[\begin{array}{c|c} m & 0 \\ \hline 0 & I \end{array}\right] \left\{\begin{array}{c} \ddot{u} \\ \hline \dot{u} \end{array}\right\} + \left[\begin{array}{c|c} c & k \\ \hline -I & 0 \end{array}\right] \left\{\begin{array}{c} \dot{u} \\ \hline u \end{array}\right\} = \left\{\begin{array}{c} G_f\ f \\ \hline 0 \end{array}\right\} \quad (3)$$

$$+ \left[\begin{array}{c|c} G_v & G_u \\ \hline 0 & 0 \end{array}\right] \left\{\begin{array}{c} \dot{u} \\ \hline u \end{array}\right\} \quad ,$$

where [I] is the identity matrix, and denoting

$$\{y\} = \left\{\begin{array}{c} \dot{u} \\ \hline u \end{array}\right\}$$

$$[A_y] = \left[\begin{array}{c|c} m^{-1} & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} G_v - c & G_u - k \\ \hline I & 0 \end{array}\right] = \left[\begin{array}{c|c} m^{-1}(G_v - c) & m^{-1}(G_u - k) \\ \hline I & 0 \end{array}\right]$$

$$[B_y] = \left[\begin{array}{c|c} m^{-1}G_f & 0 \\ \hline 0 & I \end{array}\right]$$

$$\{F\} = \left\{\begin{array}{c} f \\ \hline 0 \end{array}\right\}$$

Eq. (3) can be written in the form of Eq. (1) as the following first-order differential equation:

$$\{\dot{y}\} = [A_y]\{y\} + [B_y]\{F\} \quad . \quad (4)$$

Eq. (4) is a state variable representation of the dynamic system given by Eq. (2). Alternatively, Eq. (2) may be expressed in a modal coordinate form and for this case too, a state variable representation can be developed.

The size of the matrices $[A_y]$ and $[B_y]$ in Eq. (2) is (2n) x (2n), where n is the number of the degrees of freedom of the system. It is obvious that Eq. (4) will become impractical, if not impossible, to handle when the system is a large one. In such case the use of modal coordinates may be advantageous by considering only a limited number of modes hence reducing the size of the matrices handled. This, however, may be done provided the inclusion of only a finite and a relatively small number of modes gives an accurate enough description of the physical system. The switch to modal coordinates can be done as follows. Let

$$\{u\} = [\phi]\{q\} \tag{5}$$

$$\{x\} = \left\{ \begin{array}{c} \dot{q} \\ \hline q \end{array} \right\} \tag{6}$$

where

$[\phi]$ and $\{q\}$ are respectively the mode shape matrix and the modal coordinate vector of the system without feedback, {x} is a vector containing the new state variables q and $\dot{q}$.

Substituting Eq. (5) into Eq. (2) and premultiplying each term by $[\phi]^T$:

$$\left([\phi]^T[m][\phi]\right)\{\ddot{q}\} + \left([\phi]^T[c][\phi]\right)\{\dot{q}\} + \left([\phi]^T[k][\phi]\right)\{q\}$$
$$= [\phi]^T\left([G_f]\{f\} + [G_v \vdots G_u]\left[\begin{array}{c|c}\phi & 0\\ \hline 0 & \phi\end{array}\right]\left\{\begin{array}{c}\dot{q}\\ \hline q\end{array}\right\}\right) \quad . \qquad (7)$$

Owing to the characteristics of $[\phi]$, we can define the following diagonal matrices:

$$[M] = [\phi]^T[m][\phi]$$
$$[K] = [\phi]^T[k][\phi] \qquad (8)$$
$$[\zeta] = [\phi]^T[c][\phi]$$

provided that the damping matrix [c] can also be decoupled to modal damping.

Thus, using Eq. (8), Eq. (7) can be further expressed as:

$$[M]\{\ddot{q}\} + [\zeta]\{\dot{q}\} + [K]\{q\}$$
$$= [\phi]^T[G_f]\{f\} + [\phi]^T[G_v \vdots G_u]\left[\begin{array}{c|c}\phi & 0\\ \hline 0 & \phi\end{array}\right]\left\{\begin{array}{c}\dot{q}\\ \hline q\end{array}\right\} \quad . \qquad (9)$$

Following the passage from Eq. (2) to Eq. (3), Eq. (9) is transformed into:

$$\left[\begin{array}{c|c} \diagdown M \diagdown & 0 \\ \hline 0 & I \end{array}\right] \left\{\begin{array}{c} \ddot{q} \\ \hline \dot{q} \end{array}\right\} + \left[\begin{array}{c|c} \diagdown \zeta \diagdown & \diagdown K \diagdown \\ \hline -I & 0 \end{array}\right] \left\{\begin{array}{c} \dot{q} \\ \hline q \end{array}\right\}$$

$$= \left[\begin{array}{c|c} \phi^T G_f & 0 \\ \hline 0 & \phi \end{array}\right] \left\{\begin{array}{c} f \\ \hline 0 \end{array}\right\} + \left[\begin{array}{c|c} \phi^T G_v \phi & \phi^T G_u \phi \\ \hline 0 & 0 \end{array}\right] \left\{\begin{array}{c} \dot{q} \\ \hline q \end{array}\right\} . \tag{10}$$

By defining:

$$[A_x] = \left[\begin{array}{c|c} \diagdown M^{-1} \diagdown & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} \phi^T G_v \phi - \lceil\zeta\rfloor & \phi^T G_u \phi - \lceil K \rfloor \\ \hline I & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} M^{-1}(\phi^T G_v \phi - \lceil\zeta\rfloor) & M^{-1}(\phi^T G_u \phi - \lceil K \rfloor) \\ \hline I & 0 \end{array}\right] ,$$

$$[B_x] = \left[\begin{array}{c|c} M^{-1}\phi^T G_f & 0 \\ \hline 0 & \phi \end{array}\right] \tag{11}$$

and using Eq. (6), Eq. (10) becomes:

$$\{\dot{x}\} = [A_x]\{x\} + [B_x]\{F\} . \tag{12}$$

This is a state variable representation of the system given by Eq. (2), which after being solved yields the vector {u} by the use of Eqs. (6) and (5). It is evident that in generating $[A_x]$ and $[B_x]$ for Eq. (12), we can include a limited number of modes, such that the size of the system equation is reduced and therefore is easier to handle.

The solution of Eq. (1) is known to be

$$\{x\} = [\psi(t,t_o)]\{x(t_o)\} + \int_{t_o}^{t} [\psi(t,\xi)][B(\xi)]\{F(\xi)\}d\xi \quad , \qquad (13)$$

where t_o is the initial time at which the state vector $\{x(t_o)\}$ is known, and $[\psi]$, the state transition matrix, satisfies the following differential equation:

$$\frac{d}{dt} [\psi(t,t_o)] = [A(t)][\psi(t,t_o)] \quad , \qquad (14)$$

with the initial condition

$$[\psi(t_o,t_o)] = [I] \quad .$$

The solution of Eq. (14) is known to be the Neumann series:[17]

$$[\psi(t,t_o)] = [I] + \int_{t_o}^{t} [A(\xi_1)]d\xi_1 + \int_{t_o}^{t} [A(\xi_2)] \int_{t_o}^{\xi_2} [A(\xi_1)]d\xi_1 d\xi_2$$

$$+\cdots+ \int_{t_o}^{t} [A(\xi_n)] \int_{t_o}^{\xi_n} [A(\xi_{n-1})] \int_{t_o}^{\xi_{n-1}} \cdots \int_{t_o}^{\xi_2} [A(\xi_1)]d\xi_1 d\xi_2 \cdots d\xi_n$$

$$+\cdots \quad . \qquad (15)$$

When the matrix [A] is a time invariant matrix, the integral series in Eq. (15) is reduced to the following series:

$$[\psi(t,t_o)] = [\psi(t-t_o)] = [I] + [A](t-t_o) + \frac{[A]^2}{2!}(t-t_o)^2 + \frac{[A]^3}{3!}(t-t_o)^3 + \cdots + \frac{[A]^m}{m!}(t-t_o)^m + \cdots \triangleq e^{[A](t-t_o)} \quad . \tag{16}$$

A number of papers have been written on the computation of Eq. (16) and the errors associated with the use of the truncated series.[18-25]

THE STATE COVARIANCE MATRIX

The covariance matrix of the state of the system at time t described by Eq. (1) is defined as:

$$[P^*(t)] \triangleq [\mathrm{Cov}(\{x(t)\},\{x(t)\}^T)] \triangleq [E(\{x(t)\}\{x(t)\}^T)] - [E(\{x(t)\})E(\{x(t)\}^T)] \quad . \tag{17}$$

A more explicit expression for the covariance matrix is the following one:

$$[\mathrm{Cov}(\{x(t)\},\{x(t)\}^T)]$$

$$= \begin{bmatrix} \mathrm{Cov}(x_1(t),x_1(t)) & \mathrm{Cov}(x_1(t),x_2(t)) & \cdots & \mathrm{Cov}(x_1(t),x_n(t)) \\ \mathrm{Cov}(x_2(t),x_1(t)) & \mathrm{Cov}(x_2(t),x_2(t)) & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(x_n(t),x_1(t)) & \cdots & \cdots & \mathrm{Cov}(x_n(t),x_n(t)) \end{bmatrix}$$

The diagonal elements, which are the variances of each state variable, give a measure of the fluctuation of the response about their mean values. The off-diagonal elements indicate the amount of correlation between two corresponding state variables at the same time t. Notice that the state variables $x_1(t)$, $x_2(t)$, $\cdots$, $x_n(t)$ may be translational displacement, angular rotation, pressure, or any other variable of the system and their first or higher order derivatives.

The computation of the last term in Eq. (17) is simple, and involves only deterministic quantities, since by applying the expectation operation to Eq. (13) one obtains:

$$E(\{x(t)\}) = [\psi(t,t_o)]E(\{x(t_o)\}) + \int_{t_o}^{t} [\psi(t,\xi)][B(\xi)]E(\{F(\xi)\})d\xi \quad , \tag{18}$$

where $E(\{x(t_o)\})$ as well as $E(\{F(\xi)\})$ are known deterministic quantities. We shall, therefore, concentrate on the computation of $[P(t)]$, defined by

$$[P(t)] \triangleq [E(\{x(t)\}\{x(t)\}^T)] \quad . \tag{19}$$

The passage from $[P(t)]$ to the covariance matrix $[P^*(t)]$ using Eqs. (17) to (19) is immediate. It is shown in Appendix I that if the noise vector $\{F(t)\}$ and the initial state vector $\{x(t_o)\}$ are uncorrelated (and this is the case we are usually dealing with) then:

$$\boxed{\begin{aligned}[P(t)] &= [\psi(t,t_o)][P(t_o)][\psi(t,t_o)]^T \\ &+ \int_{t_o}^{t}\int_{t_o}^{t} [\psi(t,\xi)][B(\xi)][E(\{F(\xi)\}\{F(\rho)\}^T)][B(\rho)]^T[\psi(t,\rho)]^T d\xi d\rho \ .\end{aligned}} \tag{20}$$

If, in addition, the noise vector $\{F(t)\}$ is a white noise vector, then as shown in Appendix I:

$$\boxed{\begin{aligned}[P(t)] &= [\psi(t,t_o)][P(t_o)][\psi(t,t_o)]^T \\ &+ \int_{t_o}^{t} [\psi(t,\xi)][B(\xi)][Q(\xi)][B(\xi)]^T[\psi(t,\xi)]^T d\xi\end{aligned}} \tag{21}$$

where [Q(t)] is the intensity of the unbiased* noise covariance matrix defined by:

$$[E(\{F(t_1)\}\{F(t_2)\}^T)] = [Q(t_1)]\delta(t_1-t_2) \quad ,$$

and $\delta(\cdot)$ is the Dirac delta function.

In Appendix II it is shown that the following differential equation is equivalent to the integral expression for [P(t)] given in Eq. (21):

$$\boxed{\begin{aligned}\frac{d}{dt}[P(t)] &= [A(t)][P(t)] + [P(t)][A(t)]^T \\ &+ [B(t)][Q(t)][B(t)]^T \quad .\end{aligned}} \tag{22}$$

This equation is a special case of the well known Riccati equation

A SIMPLE EXAMPLE

To illustrate the correspondance between this method and the previously known method, which is based on frequency domain analysis, let us consider a case that will be useful in the next section.

---

*In this paper, an unbiased quantity will mean a quantity whose mean has been removed.

Consider a linear system whose set of differential equations is expressed in matrix form as follows:

$$\{\dot{x}\} + [A]\{x\} = \{h\} \quad . \tag{23}$$

Let [A] be the diagonal matrix:

$$[A] = \begin{bmatrix} \beta_1 & & & & 0 \\ & \beta_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \beta_n \end{bmatrix}$$

and let {h} be a white noise vector whose autocorrelation matrix is of the form:

$$[R_{hh}(\tau)] = \begin{bmatrix} 2\beta_1\sigma_1^2 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & 2\beta_n\sigma_n^2 \end{bmatrix} \cdot \delta(\tau) \quad . \tag{24}$$

Fourier transforming Eq. (23) yields:

$$(j\omega[I] + [A])\{X\} = \{H\}$$

or

$$\{X\} = (j\omega[I] + [A])^{-1}\{H\} \quad . \tag{25}$$

Since [A] is a diagonal matrix, Eq. (25) becomes:

$$\{X\} = \begin{bmatrix} \frac{1}{j\omega+\beta_1} & & & \\ & \frac{1}{j\omega+\beta_2} & & \\ & & \ddots & \\ & & & \frac{1}{j\omega+\beta_n} \end{bmatrix} \{H\} \quad . \tag{26}$$

By definition, the matrix in Eq. (26) is the transfer function of the system, since it transfers the input vector {H} into the output {X}. Using the Wienerian approach, the power spectrum of the output vector {X} is given by

$$[S_{xx}(\omega)] = \begin{bmatrix} \frac{1}{\omega^2+\beta_1^2} & & & \\ & \frac{1}{\omega^2+\beta_2^2} & & \\ & & \ddots & \\ & & & \frac{1}{\omega^2+\beta_n^2} \end{bmatrix} [S_{hh}(\omega)] \quad , \tag{27}$$

and since {h} is a white noise vector, we obtain in correspondence to Eq. (24):

$$[S_{hh}(\omega)] = \frac{1}{2\pi}\begin{bmatrix} 2\beta_1\sigma_1^2 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 2\beta_n\sigma_n^2 \end{bmatrix} . \qquad (28)$$

Substituting Eq. (28) in Eq. (27) yields:

$$[S_{xx}(\omega)] = \frac{1}{2\pi}\begin{bmatrix} \frac{2\beta_1\sigma_1^2}{\omega^2+\beta_1^2} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \frac{2\beta_n\sigma_n^2}{\omega^2+\beta_n^2} \end{bmatrix} ,$$

and multiplying $[S_{xx}(\omega)]$ by 2π yields the Fourier transform of the autocorrelation matrix of $\{x\}$, which, when inverse transformed results in the following matrix[27]:

$$[R_{xx}(\tau)] = \begin{bmatrix} \sigma_1^2 e^{-\beta_1|\tau|} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \sigma_n^2 e^{-\beta_n|\tau|} \end{bmatrix} . \qquad (29)$$

In particular for $\tau = 0$,

$$[R_{xx}(0)] = \begin{bmatrix} \sigma_1^{\,2} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \sigma_n^{\,2} \end{bmatrix} . \tag{30}$$

Let us now consider this system in the time domain and apply Eq. (I-8) from Appendix I to obtain the same results. Rewriting Eq. (23) in the form of Eq. (1):

$$\{\dot{x}\} = [-A]\{x\} + \{h\} \quad ,$$

we obtain the following transition matrix:

$$[\psi(t_b,t_a)] = [\psi(t_b-t_a)] = \begin{bmatrix} e^{-\beta_1(t_b-t_a)} & & & & \\ & e^{-\beta_2(t_b-t_a)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{-\beta_n(t_b-t_a)} \end{bmatrix} ,$$

since [A] is a constant matrix. Assume that at t_o, the initial time of the process, the covariance matrix $[P(t_o)]$ has the form:

$$[P(t_o)] = \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{bmatrix} .$$

From Eq. (24) it is obvious that

$$[Q(\xi,\rho)] \triangleq [E(\{h(\xi)\}\{h(\rho)\}^T)] = \begin{bmatrix} 2\beta_1\sigma_1^{\;2} & & & \\ & 2\beta_2\sigma_2^{\;2} & & \\ & & \ddots & \\ & & & 2\beta_n\sigma_n \end{bmatrix} \cdot \delta(\xi-\rho) \quad .$$

Using Eq. (I-8) one obtains

$$[P(t_1,t_2)] = \begin{bmatrix} p_1 e^{-\beta_1(t_1+t_2-2t_o)} & & \\ & \ddots & \\ & & p_n e^{-\beta_n(t_1+t_2-2t_o)} \end{bmatrix}$$

$$+\int_{t_o}^{\min(t_1,t_2)} \begin{bmatrix} 2\beta_1\sigma_1^2 e^{-\beta_1(t_1+t_2-2\xi)} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 2\beta_n\sigma_n^2 e^{-\beta_n(t_1+t_2-2\xi)} \end{bmatrix} d\xi$$

After performing the integration,

$$[P(t_1,t_2)] = \begin{bmatrix} (p_1-\sigma_1^2)e^{-\beta_1(t_1+t_2-2t_o)} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & (p_n-\sigma_n^2)e^{-\beta_n(t_1+t_2-2t_o)} \end{bmatrix}$$

$$+ \begin{bmatrix} \sigma_1^2 e^{-\beta_1|t_2-t_1|} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \sigma_n^2 e^{-\beta_n|t_2-t_1|} \end{bmatrix} . \qquad (31)$$

The first matrix on the right hand side of this equation is the transient part of $[P(t_1,t_2)]$ and is due to the propagation of the initial matrix $[P(t_o)]$. The second part is the steady-state part of $[P(t_1,t_2)]$ and is stationary. The stationary part is indeed equal to $[R_{xx}(\tau)]$ given by Eq. (29), and using Eq. (21) rather than Eq. (I-8) to compute $[P(t,t)]$ will result in a steady-state matrix equal to $[R_{xx}(0)]$, as given by Eq. (30). Note that the power spectral density method yields only the steady-state part of the autocorrelation function of a stationary process.

WHITENING OF CORRELATED EXCITATION

When the excitation vector $\{F(t)\}$ is a white noise vector, Eqs. (21) or (22) can be used to compute $[P(t)]$. These equations are by far easier to handle than Eq. (20), but unfortunately the white noise concept is a nice mathematical concept that does not exist physically. In many cases, however, the covariance matrix of the excitation vector can be broken into a sum of elementary covariance matrices. The matrices can be described as the covariance matrices of state vectors of special systems, which are excited with corresponding white noise vectors. These special systems are called "shaping filters,"[16,28] which can be added to the original system by modifing [A], [B], {x}, and {F} in Eq. (1). In such a modified state variable differential equation, the modified excitation vector $\{F\}$ is now a white noise vector and therefore Eqs. (21) or (22) can be used to find $[P(t)]$.

As an example, consider the system described by Eq. (4):

$$\{\dot{y}\} = [A_y]\{y\} + \{B_y\}\{F\} \quad . \tag{32}$$

Let us assume that the excitation vector $\{F\}$ consists of elements which are Markov processes, that is, the covariance matrix of $\{F\}$ is:

$$[E(\{F(t_1)\}\{F(t_2)\}^T] = \begin{bmatrix} \sigma_1{}^2 e^{-\beta_1|t_2-t_1|} & 0 & \cdots & 0 \\ 0 & \sigma_2{}^2 e^{-\beta_2|t_2-t_1|} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \sigma_n{}^2 e^{-\beta_n|t_2-t_1|} \end{bmatrix} \quad . \tag{33}$$

Consider now the following special system:[26]

$$\{\dot{F}\} = [A_s]\{F\} + \{W_s\} \quad , \tag{34}$$

where

$$[A_s] = \begin{bmatrix} -\beta_1 & & & & \\ & -\beta_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & -\beta_n \end{bmatrix} , \qquad (35)$$

and $\{W_s\}$ is a white noise vector whose covariance matrix is given by:

$$[E(\{W_s(t_1)\}\{W_s(t_2)\}^T)] = \begin{bmatrix} 2\beta_1\sigma_1^2 & & & & \\ & 2\beta_2\sigma_2^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 2\beta_n\sigma_n^2 \end{bmatrix} \cdot \delta(t_1 - t_2) . \qquad (36)$$

This special system is identical to the system treated in the preceeding chapter; hence the covariance matrix of $\{F\}$ is given by Eq. (31). If we choose:

$$[P_o] = [E(\{F(t_o)\}\{F(t_o)\}^T)] = \begin{bmatrix} \sigma_1^2 & & & & \\ & \sigma_2^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \sigma_n^2 \end{bmatrix} , \qquad (37)$$

then the covariance matrix $\{F\}$ is the one given by Eq. (33).

The initial system given in Eq. (32) and the shaping filter given in Eq. (34) can be combined to form the following augmented system:

$$\begin{Bmatrix} \dot{y} \\ \hline \dot{F} \end{Bmatrix} = \left[\begin{array}{c|c} A_y & B_y \\ \hline 0 & A_s \end{array}\right] \begin{Bmatrix} y \\ \hline F \end{Bmatrix} + \begin{Bmatrix} 0 \\ \hline W_s \end{Bmatrix} \tag{38}$$

Letting

$$\{Z\} = \begin{Bmatrix} y \\ \hline F \end{Bmatrix} \cdot \qquad [A] = \left[\begin{array}{c|c} A_y & B_y \\ \hline 0 & A_s \end{array}\right] \qquad \{W\} = \begin{Bmatrix} 0 \\ \hline W_s \end{Bmatrix} ,$$

then Eq. (38) can be written as:

$$\{\dot{Z}\} = [A]\{Z\} + \{W\} \quad .$$

This equation is in the form of Eq. (1) and the excitation vector {W} is a white noise vector. Therefore either Eq. (21) or Eq. (22) can be used to find [P(t)]. Moreover we see that we eliminated the matrix [B(t)] in Eq. (1), which further simplifies the computation of [P(t)]. The penalty for these simplifications is the increase in the size of the A matrix, which we can partly relieve as will be shown later.

The preceeding example can be now generalized as follows. Let the initial system be given by

$$\{\dot{y}\} = [A_y]\{y\} + [B_y]\{f_y\} \quad . \tag{39}$$

We assume that

$$\{f_y\} = [G_1]\{g_1\} + \cdot\ \cdot\ \cdot + [G_m]\{g_m\} \quad , \tag{40}$$

where all the matrices may be time varying and the vectors are functions of time. We assume that the vectors $\{g_i\}$, $i=1,2\cdots,m$, are the output vectors of shaping filters whose inputs are white noise vectors. The state variable equations of these shaping filters are:

$$\begin{aligned} \{\dot{g}_1\} &= [A_{s1}]\{g_1\} + \{W_1\} \\ \{\dot{g}_2\} &= [A_{s2}]\{g_2\} + \{W_2\} \\ &\ \vdots \\ \{g_m\} &= [A_{sm}]\{g_m\} + \{W_m\} \end{aligned} \quad , \tag{41}$$

where $\{W_i\}$, $i=1,2,\cdots,m$, are white noise vectors. The combination of Eqs. (39)-(41) yields the final system:

$$\begin{Bmatrix} \dot{y} \\ \hline \dot{g}_1 \\ \hline \dot{g}_2 \\ \hline \vdots \\ \hline \dot{g}_m \end{Bmatrix} = \left[\begin{array}{c|c|c|c|c} A_y & B_yG_1 & B_yG_2 & ---- & B_yG_m \\ \hline 0 & A_{s1} & 0 & & 0 \\ \hline 0 & 0 & A_{s2} & & 0 \\ \hline & & & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & A_{sm} \end{array}\right] \begin{Bmatrix} y \\ \hline g_1 \\ \hline g_2 \\ \hline \vdots \\ \hline g_m \end{Bmatrix} + \begin{Bmatrix} 0 \\ \hline W_1 \\ \hline W_2 \\ \hline \vdots \\ \hline W_m \end{Bmatrix} \quad . \tag{42}$$

The following quantities

$$
\begin{aligned}
& m \\
& [G_i] \\
& [A_{si}] \\
& [E(\{W_i(t_1)\}\{W_i(t_2)\}^T)] \\
& [E(\{g_i(t_o)\}\{g_i(t_o)\}^T)] \qquad i=1,2,\cdots,m
\end{aligned}
\tag{43}
$$

are determined such that the desired covariance matrix of $\{f_y\}$ is obtained. We note that $\{g_i(t_1)\}$ and $\{g_j(t_2)\}$ for $i \neq j$ are uncorrelated; therefore using Eq. (40) it can be shown that the covariance matrix of $\{f_y\}$ is given by:

$$
[E(\{f_y(t_1)\}\{f_y(t_2)\}^T)]
$$

$$
= \sum_{i=1}^{m} [G_i(t_1)][E(\{g_i(t_1)\}\{g_i(t_2)\}^T)][G_i(t_2)]^T \quad . \tag{44}
$$

The covariance matrices of $\{g_i\}$, $i=1,2,\cdots,m$, used in Eq. (44) can be found using Eq. (I-8).

We can see how the preceeding example follows from the general case by realizing that Eqs. (35)-(37) and the choice of $m=1$ and $[G_1] = [I]$ determine the quantities of Eq. (43), which result in the covariance matrix given by Eq. (33), and the augmented system given by Eq. (38).

If we assume that $m = 1$, $[G_1] = I$, $[A_{s1}] = 0$, $[E(\{W_1(t_1)\}\{W_1(t_2)\}^T)] = 0$, and

$$[E(\{g_1(t_o)\}\{g_1(t_o)\}^T)] = \begin{bmatrix} q_1 & & & & & \\ & q_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & q_n \end{bmatrix} , \tag{45}$$

then the covariance matrix of $\{f_y\}$ is the same one given in Eq. (45). If on the other hand $[E(\{g_1(t_o)\}\{g_1(t_o)\}^T)] = [0]$ and

$$[E(\{W_1(t_1)\}\{W_1(t_2)\}^T)] = \begin{bmatrix} q_1 & & & & \\ & q_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & q_n \end{bmatrix} \cdot \delta(t_1-t_2) ,$$

then

$$[E(\{f_y(t_1)\}\{f_y(t_2)\}^T)] = \begin{bmatrix} q_1 & & & & \\ & q_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & q_n \end{bmatrix} \cdot \min(t_1,t_2)$$

The last case is known as the random walk excitation vector.

COMPUTATIONAL CONSIDERATIONS

In most cases, an analytic solution to the state covariance matrix cannot be found and one has to resort to numerical techniques. The main difficulty in the use of the integral equations (I-5), (I-6), (20) and (21) to evaluate the state covariance matrix comes from the need of using $[\psi(t,t_o)]$, the state transition matrix, which is practically impossible to evaluate for time varying systems (see Eq. 15). In addition, if the excitation vector cannot be "whitened" through shaping filters, the simpler equation (Eq. 21) cannot be used, and Eq. (20) must be solved (if the initial state is correlated with the noise then Eq. I-5 must be used).

When the excitation is indeed a white noise vector, uncorrelated with the initial state vector, Eq. (22) can be used successfully because $[\psi(t,t_o)]$ is not needed. Several methods to solve this equation are known[29,30,31] of which integration[30] is the most straight forward one. It should be noted that when the system is time invariant and the excitation is stationary and further we are interested only in the steady-state covariance matrix, the derivative of $[P(t)]$ vanishes and we are left with an algebraic matrix equation for which there are known solutions.[29] Some computer programs are available for the solution of this case.[32,33]

If for the time varying system discussed here, where the excitation is a white noise vector uncorrelated with the initial state, one wants to use the integral equation (21), then in order to overcome the difficulty in the evaluation of $[\psi(t,t_o)]$ one has to divide the time of interest (t_o,t) into time increments, and assume an invariant system within the increments. Then the transition matrix can be evaluated using Eq. (16), where it is a function of the time increment only. The size of the time increment is mostly a function of the rate of change of the system; that is, the increment has to be small enough for the system to be considered constant through the increment. However, in cases where [Q] changes faster than [A], a step size smaller than the one necessary for assuming a constant [A] will enable us to assume a piece-wise constant [Q], which will facilitate the computation. One is forced to use this method if mode shapes are chosen to describe the structures.

Using this method one computes the value of $[P(t_{n+1})]$, the covariance matrix of the state vector at the time t_{n+1}, from the value of this matrix at the time t_n. In the development of Eq. (21) we used the assumption

$$[E(\{F(t)\}\{x(t_o)\}^T)] = 0 \quad \text{for} \quad t>t_o \quad . \qquad (46)$$

Similarly, it is necessary in the present case that:

$$[E(\{F(t)\}\{x(t_n)\}^T)] = 0 \quad \text{for} \quad t>t_n \tag{47}$$

in order to use the Eq. (21), which becomes

$$[P(t)] = [\psi(t,t_n)][P(t_n)][\psi(t,t_n)]^T + \int_{t_n}^{t} [\psi(t,\xi)][B(\xi)][Q(\xi)][B(\xi)]^T[\psi(t,\xi)]^T d\xi \quad . \tag{48}$$

As will be now shown, the latter requirement (Eq. 47) results from the first one (Eq. 46). This can be concluded as follows. Using Eq. (13) we can write for $t>t_n>t_o$

$$\begin{aligned}
[E(\{x(t_n)\}\{F(t)\}^T)] &= \Big[E\Big([\psi(t_n,t_o)]\{x(t_o)\}\{F(t)\}^T + \int_{t_o}^{t_n} [\psi(t_n,\xi)][B(\xi)]\{F(\xi)\}d\xi\{F(t)\}^T\Big)\Big] \\
&= [\psi(t_n,t_o)][E(\{x(t_o)\}\{F(t)\}^T)] + \int_{t_o}^{t_n} [\psi(t_n,\xi)][B(\xi)][E(\{F(\xi)\}\{F(t)\}^T)]d\xi \quad .
\end{aligned} \tag{49}$$

For a white noise excitation vector

$$[E(\{F(\xi)\}\{F(t)\}^T)] = [Q(\xi)]\delta(\xi-t) \quad .$$

Therefore the last part of Eq. (49) becomes

$$\int_{t_o}^{t_n} [\psi(t_n,\xi)][B(\xi)][Q(\xi)]\delta(\xi-t)d\xi = 0 \quad ,$$

since $t>t_n$. Using this result and Eq. (46), Eq. (49) is reduced to

$$[E(\{x(t_n)\}\{F(t)\}^T)] = 0 \quad .$$

Therefore in order to use Eq. (48) to find the state covariance matrix, it is sufficient that the excitation is a white noise vector and this vector is uncorrelated with the initial state vector. Since we are interested in [P] at the end of the interval following t_n, we write Eq. (48) as

$$[P_{n+1}] = [\psi_n][P_n][\psi_n]^T + [H_n] \quad , \qquad (50)$$

where:

$$[P_{n+1}] = [P(t_{n+1})] \quad ,$$

$$[\psi_n] = [\psi(t_{n+1},t_n)] \quad ,$$

$$[H_n] = \int_{t_n}^{t_{n+1}} [\psi(t_{n+1},\xi)][B(\xi)][Q(\xi)][B(\xi)]^T[\psi(t_{n+1},\xi)]^T d\xi \quad ,$$

and from Eq. (16)

$$[\psi(t_{n+1},t_n)] = \left[\exp\left([A(t_n)]\cdot(t_{n+1}-t_n)\right)\right] \quad .$$

Eq. (50) provides an iterative equation for the solution of [P(t)].

If the system is time invariant then Eq. (50) is quite easy to use. For a constant interval of time the $[\psi]$ matrix becomes constant; hence at every new step the only new computation necessary for applying Eq. (50) is the evaluation of $[H_n]$. If, in addition, the input is stationary then $[H_n]$ is also a constant matrix, and the use of Eq. (50) is further simplified. For this case it may, sometimes, be advantageous to solve Eq. (22).[29]

Finally, it is advisable to use matrix partitioning whenever possible. In many cases the matrices consist of sub-matrices that occupy only a part of the matrix array (for example see Eqs. 11 and 42). The use of partitioning decreases the computer storage requirement and speeds up the computation.

CONCLUSIONS

It has been shown in this paper how to operate in the time domain in order to obtain the covariance matrix of the state vector of a system that consists of structures, linear feedback, and any other linear subsystems. Being a time domain operation, it enables us to analyze time varying linear systems excited by nonstationary inputs. It has been shown that if the input is a white noise vector, or can be expressed as a filtered white noise vector, then the equations take a simpler form. However, by no means is the excitation limited to this case. A general approach for "whitening" a correlated noise vector is suggested; however, more work is needed in this direction.

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Attachments
References
Appendix I
Appendix II

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APPENDIX I: THE INTEGRAL EXPRESSION FOR THE STATE COVARIANCE MATRIX[28]

Define $[P(t_1,t_2)]$, the general covariance matrix of the unbiased state vector as:

$$[P(t_1,t_2)] = [E(\{x(t_1)\}\{x(t_2)\}^T)] \quad . \qquad \text{(I-1)}$$

Substituting the expressions for $\{x(t_1)\}$ and $\{x(t_2)\}^T$ as given by Eq. (13) into Eq. (I-1) yields:

$$[P(t_1,t_2)] = \Big[E\Big(\Big\{[\psi(t_1,t_o)]\{x(t_o)\} + \int_{t_o}^{t_1} [\psi(t_1,\xi)][B(\xi)]\{F(\xi)\}d\xi\Big\}$$

$$\cdot \Big\{[\psi(t_2,t_o]\{x(t_o)\} + \int_{t_o}^{t_2} [\psi(t_2,\rho)][B(\rho)]\{F(\rho)\}d\rho\Big\}^T\Big)\Big] \quad .$$

Performing the indicated transposes and multiplications, we obtain:

$$[P(t_1,t_2)] = \Big[E\Big([\psi(t_1,t_o)]\{x(t_o)\}\{x(t_o)\}^T[\psi(t_2,t_o)]^T$$

$$+ [\psi(t_1,t_o)]\{x(t_o)\} \int_{t_o}^{t_2} \{F(\rho)\}^T[B(\rho)]^T[\psi(t_2,\rho)]^T d\rho$$

$$+ \int_{t_o}^{t_1} [\psi(t_1,\xi)][B(\xi)]\{F(\xi)\}d\xi\{x(t_o)\}^T[\psi(t_2,t_o)]^T$$

$$+ \int_{t_o}^{t_1} [\psi(t_1,\xi)][B(\xi)]\{F(\xi)\}d\xi \int_{t_o}^{t_2} \{F(\rho)\}^T[B(\rho)]^T[\psi(t_2,\rho)]^T d\rho\Big)\Big] . \tag{I-2}$$

Rearranging the positions of the integrals in Eq. (I-2) yields:

$$\begin{aligned}[P(t_1,t_2)] = \Big[E\Big(&[\psi(t_1,t_o)]\{x(t_o)\}\{x(t_o)\}^T[\psi(t_2,t_o)]^T \\ &+ \int_{t_o}^{t_2} [\psi(t_1,t_o)]\{x(t_o)\}\{F(\rho)\}^T[B(\rho)]^T[\psi(t_2,\rho)]^T d\rho \\ &+ \int_{t_o}^{t_1} [\psi(t_1,\xi)][B(\xi)]\{F(\xi)]\}\{x(t_o)\}^T[\psi(t_2,t_o)]^T d\xi \\ &+ \int_{t_o}^{t_1}\int_{t_o}^{t_2} [\psi(t_1,\xi)][B(\xi)]\{F(\xi)\}\{F(\rho)\}^T[B(\rho)]^T \\ &\cdot [\psi(t_2,\rho)]^T d\rho d\xi\Big)\Big] . \end{aligned} \tag{I-3}$$

Defining

$$[C(t_o,t_1)] \triangleq E(\{x(t_o)\}\{F(t_1)\}^T) \tag{I-4}$$

$$[Q(t_1,t_2)] \triangleq E(\{F(t_1)\}\{F(t_2)\}^T) ,$$

and using the linearity properties of the expectation operation, which allow interchange between the expectation and integration operators, one may write Eq. (I-3) as:

$$[P(t_1,t_2)] = [\psi(t_1,t_o)][P(t_o)][\psi(t_2,t_o)]^T$$

$$+ \int_{t_o}^{t_2} [\psi(t_1,t_o)][C(t_o,\rho)][B(\rho)]^T[\psi(t_2,\rho)]^T d\rho$$

$$+ \int_{t_o}^{t_1} [\psi(t_1,\xi)][B(\xi)][C(t_o,\xi)]^T[\psi(t_2,t_o)]^T d\xi$$

$$+ \int_{t_o}^{t_1}\int_{t_o}^{t_2} [\psi(t_1,\xi)][B(\xi)][Q(\xi,\rho)][B(\rho)]^T[\psi(t_2,\rho)]^T d\rho d\xi \ . \qquad \text{(I-5)}$$

In our case, $\{x(t_o)\}$ and $\{F(t)\}$ are uncorrelated for all t greater than and equal to t_o, then from Eq. (I-4)

$$[C(t_o,t)] = 0$$

and Eq. (I-5) reduces to:

$$[P(t_1,t_2)] = [\psi(t_1,t_o)][P(t_o)][\psi(t_2,t_o)]^T$$

$$+ \int_{t_o}^{t_1}\int_{t_o}^{t_2} [\psi(t_1,\xi)][B(\xi)][Q(\xi,\rho)][B(\rho)]^T[\psi(t_2,\rho)]^T d\rho d\xi \ . \qquad \text{(I-6)}$$

Eq. (20) is immediately obtained from Eq. (I-6) by letting $t_1 = t_2 = t$.

When $\{F(t)\}$ is a vector of unbiased white noise processes, then by definition the covariance matrix of $\{F\}$ is expressed as:

$$[E(\{F(t_1)\}\{F(t_2)\}^T)] = [Q(t_1)]\delta(t_1-t_2) \quad \text{(I-7)}$$

where $\delta(\cdot)$ is the Dirac delta function. Substituting Eq. (I-7) into Eq. (I-6) clearly yields:

$$[P(t_1,t_2)] = [\psi(t_1,t_o)][P(t_o)][\psi(t_2,t_o)]^T$$

$$+ \int_{t_o}^{\min(t_1,t_2)} [\psi(t_1,\xi)][B(\xi)][Q(\xi)][B(\xi)]^T[\psi(t_2,\xi)]^T d\xi \quad \text{(I-8)}$$

from which Eq. (21) is obtained by letting $t_1 = t_2 = t$.

APPENDIX II: THE DIFFERENTIAL EQUATION OF THE STATE COVARIANCE MATRIX[34]

Noting that $[\psi(t,\xi)] = [\psi(t,t_o)][\psi(t_o,\xi)]$, we can write

$$\int_{t_o}^{t} [\psi(t,\xi)][B(\xi)][Q(\xi)][B(\xi)]^T[\psi(t,\xi)]^T d\xi$$

$$= [\psi(t,t_o)] \int_{t_o}^{t} [\psi(t_o,\xi)][B(\xi)][Q(\xi)][B(\xi)]^T[\psi(t_o,\xi)]^T d\xi\, [\psi(t,t_o)]^T$$

$$= [\psi(t,t_o)][J(t)][\psi(t,t_o)]^T \quad , \qquad \text{(II-1)}$$

where

$$[J(t)] \triangleq \int_{t_o}^{t} [\psi(t_o,\xi)][B(\xi)][Q(\xi)][B(\xi)]^T[\psi(t_o,\xi)]^T d\xi \quad . \qquad \text{(II-2)}$$

Substituting Eq. (II-1) into Eq. (21) yields

$$[P(t)] = [\psi(t,t_o)]\Big([P(t_o)] + [J(t)]\Big)[\psi(t,t_o)]^T \quad . \qquad \text{(II-3)}$$

Differentiation of this equation results in

$$[\dot{P}(t)] = [\dot{\psi}(t,t_o)]\Big([P(t_o)] + [J(t)]\Big)[\psi(t,t_o)]^T$$

$$+ [\psi(t,t_o)][\dot{J}(t)][\psi(t,t_o)]^T$$

$$+ [\psi(t,t_o)]\Big([P(t_o)] + [J(t)]\Big)[\dot{\psi}(t,t_o)]^T \quad . \qquad \text{(II-4)}$$

Recall from the matrix differential equation that the transition matrix satisfies Eq. (14), namely

$$[\dot{\psi}(t,t_o)] = [A(t)][\psi(t,t_o)] \quad .$$

Substituting this equation into Eq. (II-4), yields

$$[\dot{P}(t)] = [A(t)][\psi(t,t_o)]\Big([P(t_o)] + [J(t)]\Big)[\psi(t,t_o)]^T$$

$$+ [\psi(t,t_o)][\dot{J}(t)][\psi(t,t_o)]^T$$

$$+ [\psi(t,t_o)]\Big([P(t_o)] + [J(t)]\Big)[\psi(t,t_o)]^T[A(t)]^T \quad . \qquad \text{(II-5)}$$

Using Eq. (II-3), Eq. (II-5) may be written as

$$[\dot{P}(t)] = [A(t)][P(t)] + [P(t)][A(t)]^T$$

$$+ [\psi(t,t_o)][\dot{J}(t)][\psi(t,t_o)]^T \quad ; \qquad \text{(II-6)}$$

and using Eq. (II-2), it is easy to show that

$$[\psi(t,t_o)][\dot{J}(t)][\psi(t,t_o)]^T$$

$$= [\psi(t,t_o)][\psi(t_o,t)][B(t)][Q(t)][B(t)]^T[\psi(t_o,t)]^T[\psi(t,t_o)]^T$$

$$= [I][B(t)][Q(t)][B(t)]^T[I] = [B(t)][Q(t)][B(t)]^T . \qquad \text{(II-7)}$$

From Eq. (II-6) and (II-7), it is obvious that

$$\frac{d}{dt}[P(t)] = [A(t)][P(t)] + [P(t)][A(t)]^T + [B(t)][Q(t)][B(t)]^T \quad .$$

This is Eq. (22). We note that the assumptions made in this appendix are those made in the derivation of Eq. (21), namely:

a) $\{F(t)\}$ is an unbiased white noise vector, hence

$$[E(\{F(t_1)\}\{F(t_2)\}^T)] = [Q(t_1)]\delta(t_1-t_2) \quad .$$

b) The initial state vector $\{x(t_o)\}$ and the noise vector $\{F(t)\}$ are uncorrelated for all time t; hence $[C(t_o,t)]$ as defined in Eq. (I-4) vanishes.