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TECHNICAL MEMORANDUM

INTRODUCTION TO THE PHYSICS OF WEIGHTLESSNESS

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ABSTRACT

This paper was stimulated by numerous studies of manned space experiments where there seemed need for a treatment of the "weightless environment" that was complete, physically oriented, and emphasizing analytical rather than numerical results.

The mathematical problem, which has been solved before, is to describe the motion of a mass point as seen from coordinates, local vertical or inertial, fixed in an orbiting spacecraft. This problem is exactly soluble, for small distances (kilometers), including gravity gradient and a constant drag acceleration. The apparent forces are described as they would be perceived by an observer on the spacecraft. The resulting trajectories are shown to be the sum of three elementary trajectories, each of which is geometrically simple. Combining analysis and geometry, it is easy to visualize most space experiment or sub-satellite problems intuitively before confirming with numerical analysis.

In low orbit, weightlessness is a good description for times approaching a minute. Gravitational accelerations on objects differ by about 10^{-7}g per meter of separation, and motion in response to this difference should be quite clear in a few minutes and large in an orbit (∿90 min.). Drag acceleration, significant for Skylab (10^{-8}g), can dominate (10^{-5}g) low altitude shuttle flights near the solar maximum. A capability may be desirable for low thrust systems to compensate for drag.

When drag is small, the local vertical orientation appears superior to the inertial one for zero-G space laboratories. Since particles along the track line co-orbit precisely, the local vertical laboratory has many sites with equivalent acceleration environments and can support more operations at once. It is clearly desirable that shuttle or station laboratory areas include the center of gravity and for local vertical, the track line.

Some practical problems are insensitive to these accelerations. For example, astronaut work aids fall in two classes. The first includes rugged devices to close the force system between astronaut and work piece. The second includes devices to prevent objects from drifting away due to the small forces. Sophisticated analysis is unnecessary.

For sub-satellites, the elementary trajectories permit a complete enumeration of possible bound orbits. Three distinct classes are described; with this flexibility, the capability for supporting such satellites from a shuttle seems clearly established.

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TECHNICAL MEMORANDUM

1.0 INTRODUCTION

This paper reviews the motion of free objects within or near spacecraft in circular orbit. Understanding this area is important in dealing with zero-g experiments; it is also important in dealing with subsatellites which might be deployed by a shuttle sortie mission and later retrieved.

The text describes the forces -- gravity and drag are included -- and typical motions. The general solution is shown to be the sum of three geometrically simple trajectories. These are described as seen by observers on either inertial or local vertical space vehicles. In the appendices, the equations of motion are derived and the solution worked out. The text concludes with some applications.

The mathematical problem apparently arose first in studies of rendezvous. Roberson reports that the closed-form solutions for small displacements of the second body (tens of miles) appear in "several" of the references reviewed in his 1963 review paper.(1) Englar presents an "irreducibly simple" derivation in Reference 2. There have been a number of applications to specific aspects of both the zero-g experiment(3)(4) and the subsatellite problem.(5)

The present paper was stimulated by numerous experiments program studies where there seemed need for a treatment of the "weightless environment" that was complete, physically oriented, and that emphasized analytical rather than numerical results. Section two describes the forces, with emphasis on the assumptions and limitations of the treatment. Section three shows the solutions for particle trajectories. Representation with the three elementary trajectories makes geometrical interpretation easy. The applications in section four comprise a series of notes on a variety of problems stimulated by the analysis.

The treatment is intended to be instructive and complete. A compact, vector analytical geometrical formalism is employed.

2.0 THE FORCES

2.1 General

To understand the motion of a small body near a spacecraft, we must describe the larger forces and typical resulting motions. This gives an intuitive base upon which more precise work can be done. These forces and motions are quite different from those we are familiar with on Earth and study in elementary physics. The objective is to describe the orbital phenomena at the level of detail corresponding to knowledge of the downward force of gravity, friction, and of low speed, near-parabolic trajectories. In the remainder of this section, the forces and equations of motion are described.

2.2 Gravity in Free Fall

A space vehicle spends most of its time in free fall, with the Newtonian acceleration,

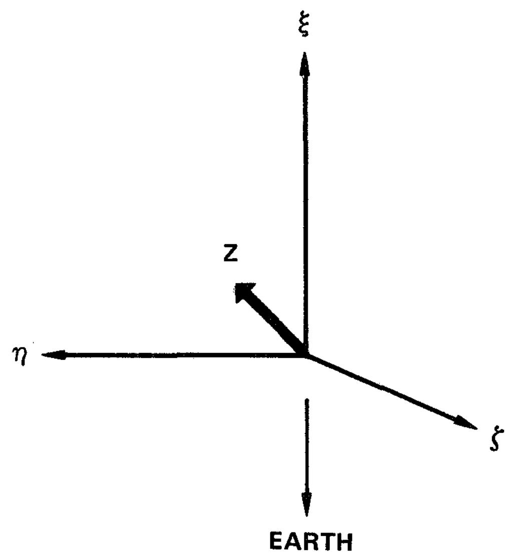
$$\ddot{R} = - \frac{R(GM)}{|R|^3} \qquad (1)$$

(Capital letters are used for vectors and linear operators, with the single exception of the gravitational parameter (GM). R is the radius from the center of the Earth.)

An observer on this vehicle measuring a test object is unable to measure this gross acceleration. He and the objects fall together. In the simple case, the vehicle is inertially oriented; the axes are at rest relative to the stars. Then, provided attitude motions and vibrations are small enough, the observer will detect small accelerations depending on the displacement of the test object from the center of gravity (C.G.). These are in part due to gravity gradient and can be calculated by expanding (1) in Taylor series. If the C.G. is at the vector distance D from the earth, and the object is at $Z = (R-D)$, then to first order in $|Z|/|D|$,

$$\ddot{R} = -\frac{D(GM)}{|D|^3} + \frac{(GM)}{|D|^3}\left[-Z + 3D\frac{(Z\cdot D)}{|D|^2}\right]. \qquad (2)$$

With the large, first term unobservable, the remainder, $\ddot{Z}$, corresponds to a restoring force proportional to Z and a divergent force in the radial direction, proportional to the radial component of Z.



This is particularly clear in matrix notation. Consider the simplest case, that of a vehicle in a linear orbit, falling radially towards the Earth. A sounding rocket trajectory is a good example, an Apollo translunar trajectory a less precise example. Choose the coordinate frame (ξ,η,ζ) with ξ radial. The observable acceleration of of a test mass at a point Z, due to gravity, is:

$$\ddot{Z} = \frac{GM}{|D|^3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} Z \qquad (3)$$

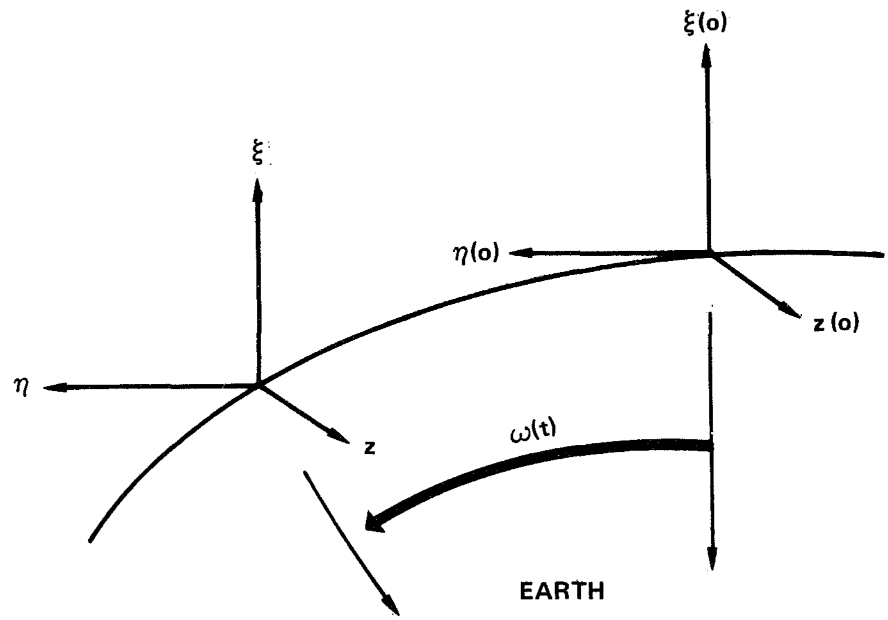
For segments of the linear orbit where D is reasonably constant this is a complete equation of motion; further, it separates and the solutions are hyperbolic functions in ξ and trigonometric ones in η and ζ.* The solutions show that there should be no recollision problem for objects jettisioned from a sounding rocket, for instance. The equations become somewhat more complicated in circular orbit. The matrix (3), however, always represents the instantaneous acceleration field due to gravity gradient.

2.3 "Inertially Oriented" Space Station

A space vehicle in circular orbit travels at an angular velocity ω. The centripetal acceleration, $-\omega^2 D$, is supplied by gravity, so

$$\frac{(GM)}{|D|^3} = \omega^2 \tag{4}$$

For an inertially oriented station, the axes are fixed relative to the stars. (ξ, η, z) is a right handed system such that at zero time ξ lies along the radius and η along the positive velocity vector or track. The letter z distinguishes this case from



*Let $GM/|D|^3 = \alpha$: then

$$\xi(t) = \xi(0) \cosh(\sqrt{2}\alpha t) + \frac{\dot{\xi}(0)}{\sqrt{2}\alpha} \sinh(\sqrt{2}\alpha t)$$

$$\eta(t) = \eta(0) \cos(\alpha t) + \frac{\dot{\eta}(0)}{\alpha} \sin(\alpha t)$$

$$\zeta(t) = \zeta(0) \cos(\alpha t) + \frac{\dot{\zeta}(0)}{\alpha} \sin(\alpha t)$$

the last and points to the orbital pole. At time zero, the observable gravity forces are given by equation (3), with the substitution (4):

$$\ddot{Z}(0) = \omega^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} Z(0) \qquad (5)$$

At time t, the space station has moved ωt around the Earth. To the observer within, it is the force field which rotates. Define $A(\omega t)$: the operator

$$\begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & +\cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (6)$$

which rotates a vector counter clockwise in the orbital plane by (ωt). Then the acceleration of a test mass at Z, observed in the inertial station at time t, is:

$$\ddot{Z}(t) = \omega^2 A(\omega t) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} A(-\omega t)\ Z(t), \qquad (7)$$

where the standard method of transforming an operator has been applied.* Equation (7) can be read from right to left as: take the measurement (ξ, η, z) of position, Z; transform to the instantaneous local vertical coordinate system; apply equation (5); go back to inertial coordinates.

What (7) says, again, is that there is a generalized "Hookes law" force on a particle, divergent in the radial direction but restoring in the tangential plane. This force describes all <u>gravitational</u> deviations of particle motion from a straight line. Additional terms, F/m, can be added, to account for other forces. Drag will be covered below.

The accelerations are small. The magnitude of ω^2 in

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*(7) is derived in Appendix A. Note that there are no Coriolis or centrifugal terms; Z is an inertial system with a rotating acceleration field.

low orbit is 1.3×10^{-6} per second squared.* An observer on such a laboratory will measure accelerations of $\sim 10^{-6}$ m/s^2 for every meter of displacement from the laboratory C.G. This is 10^{-7} the gravitational acceleration at the Earth's surface.

Even with these small accelerations, a particle starting from rest can move substantially in a few minutes. To order of magnitude, the distance s is,

$$s = \frac{1}{2} at^2 \sim \frac{1}{2} \left[(1 \text{ or } 2)\omega^2 |Z| \right] t^2 \sim (\omega t)^2 |Z| . \qquad (8)$$

If the station moves one radian along its orbit ($\omega t = 1$), $\sim$ 15 min of time, the test object can move a distance comparable to its initial separation from the C.G.

2.4 "Local Vertical" Space Station

The local vertical station rotates to keep a constant orientation relative to radius and track. Forces like gravity and drag have constant direction and are more simply described. On the other hand, centrifugal and Coriolis effects are introduced and are of comparable or larger size.

A measurement of position in local vertical coordinates (x,y,z) is labelled V. At time zero, the coordinates coincide with the inertial set (ξ, η, z). Thereafter, x remains the outward radial and y the positive track. z, of course, remains the pole of the orbit.

If, as in the figure, V is a constant local-vertical vector, the corresponding inertial vector, Z, is rotating counter clockwise, Thus,

$$Z = A(\omega t)V \text{ and conversely, } V = A(-\omega t)Z. \qquad (9)$$

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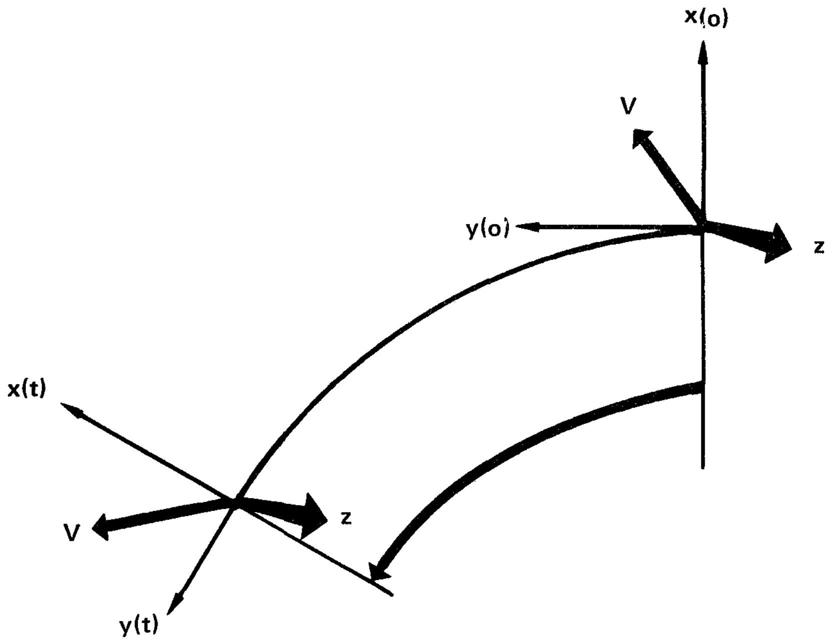
*Values of ω^2 are tabulated versus altitude for Earth and Moon in Appendix B. Approximately the same accelerations will be observed for low orbits around any planet, since for a near surface orbit around a planet of radius r, $\omega^2 = GM/r^3 \sim \frac{4}{3} \pi\rho G$, dependent only on planetary density, ρ. For the Moon at 100 km altitude, $\omega^2 = .79 \times 10^{-6}$ per (second)^2.



Considering the motion of free particles, since the observer is rotating, he will see most straight-line motions as curvilinear. Coriolis and centrifugal forces are defined to 'describe' what is in fact an unfortunate choice of laboratory conditions.*

The accelerations - derived more carefully in Appendix A-are

$$\omega^2 \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \end{pmatrix} V$$, the centrifugal component, and

$$2\omega \begin{pmatrix} 0&1&0 \\ -1&0&0 \\ 0&0&0 \end{pmatrix} \dot{V}$$, the velocity dependent Coriolis term.

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*These forces on a free particle are said to be "fictitious"; they can be made to disappear by giving the observer a counter-rotating chair. When spacecraft walls constrain the particle to move with the coordinate system, real forces must be exerted to do so.

The local vertical equations of motion for a free particle then include the gravity gradient, (5), added to these, and any additional forces, F/m.

$$\ddot{V} = \omega^2 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} V + 2\omega \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{V} + \frac{F}{m} \qquad (10)$$

Note that the centrifugal component has cancelled the restoring force in track. This suggests that for low acceleration space laboratories, where a number of zero-G experiments are to be performed, the local vertical orientation is better than the inertial. There are more equivalent low acceleration sites.*

2.5 Drag

Atmospheric drag is the next significant force (ignoring spacecraft motions). At low altitudes, it is larger than gravity gradient. Appendix B includes a plot of atmospheric density versus altitude and solar activity, plus the conventional expressions for drag in terms of spacecraft velocity and drag coefficients. Drag varies with time; it can vary by a factor of 2-3 around an orbit, being strongest shortly after orbital noon. At four or five hundred kilometer altitudes, drag can be a hundred times more severe at times of intense solar activity near solar maximum than it is at solar minimum.

For simplicity drag is modelled as a constant acceleration in the positive track direction,**

$$F/m = \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} . \qquad (11)$$

This is the correct sign for the motion of particles within the station; it is the spacecraft, not the particles, which is being dragged. For sub-satellites, the sign may be plus or minus depending on the relative drag coefficients of parent and daughter vehicles.

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*Coriolis accelerations can be large if the specimens move. This can be remedied by mounting sensitive experiments on inertial tables which counter-rotate at orbital rate.

**Solutions in the appendix are carried out for a general in-plane acceleration (ℓ,d,0).

For Skylab, drag acceleration will be about 10^{-8}g (see Appendix B; Skylab flies near solar minimum at an altitude near 435km.). Drag is small relative to gradient. For the shuttle, drag accelerations can range from 10^{-8}g near space station altitude (500km) to nearly 10^{-5}g at 100 nautical miles. Early shuttles will fly near solar maximum. The accelerations are very sensitive to spacecraft attitude, but can be typically larger than gravity gradient.

2.6 Other Forces

A number of other factors affect the motion of particles. Light pressure and higher terms in the Earth's field are believed the next external factors in size. They have smaller effects than the variation in drag and spacecraft motion, which already limit the accuracy of the treatment.

(a) Light pressure. Light pressure is smaller than gravity gradient forces, but can be comparable with drag. The momentum of a photon is E/C, the energy/speed of light. For perfect reflection, twice this momentum is imparted.

$$\text{light pressure} \sim 2\,\frac{\text{power density}}{\text{speed of light}} \sim 2\,\frac{1400\ \text{w/m}^2}{3\text{x}10^8\ \text{m/s}} \sim 10^{-5}\ \text{N/m}^2 \qquad (12)$$

Near Skylab altitudes, for solar minimum when drag is weak, the drag pressure is slightly larger, $\sim .6\text{-}3 \times 10^{-5}$ N/m^2 (from Figure B-1).

(b) Higher terms in the gravitational potential. The Earth's equatorial bulge (J_2 term) contributes a gravitational force about 10^{-3} of the $1/r^2$ term; the next higher moments are of order 10^{-6}. The contribution to gravity gradient is a few parts in 10^{-3} of equation (5) and is not significant for the present treatment.

(c) Spacecraft motion. Estimates of the effect of spacecraft motions are attached in Appendix D. Major maneuvers are excluded. Provided attitude excursions are kept small, as by Control Moment Gyros, the limiting factor appears to be the shift in spacecraft C.G. (2 or 3 cm) caused by astronaut translation. The motion of a free particle relative to the spacecraft is uncertain by a few centimeters, a reasonable error size.

3.0 TYPICAL TRAJECTORIES

In Section 3, typical particle motions are described. Just as projectile motion on earth can be expressed (grossly) as a combination of linear and parabolic motion, particle motion near a spacecraft can be expressed as a sum of circular, elliptical, linear, and parabolic motions. The analysis is done in Appendix C.

3.1 Perpendicular to the Orbital Plane

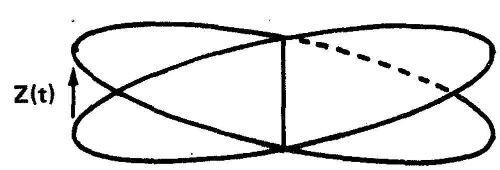
An important property of the differential equations (7) and (10) is that the component z, parallel to the orbital pole, decouples. The z equation is the familiar differential equation for simple harmonic motion.

$$\ddot{z} = - \omega^2 z \tag{13}$$

The solution has the form:

$$z(t) = z(0) \cos \omega t + \frac{\dot{z}(0)}{\omega} \sin \omega t. \tag{14}$$

This sinusoidal motion up and down from the plane corresponds physically to the particle being in an inclined orbit of the same period as the station. It is independent of the in-plane motions described later.



3.2 Solutions of the Equations - The Elemental Trajectories

The in-plane local vertical equation of motion is (from equations 10 and 11).

$$\ddot{V} = \omega^2 \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} V + 2\omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{V} + \begin{pmatrix} 0 \\ d \end{pmatrix} \quad (15)$$

In Appendix C this equation is solved by Laplace transformation.* The result will contain two vector initial conditions, conventionally the initial position, V(0), and velocity, $\dot{V}(0)$. Any linear combination of these will do, however, and Appendix C shows that a suitable choice separates the solution into three terms, each with a single constant, a simple time dependence, and a simple geometrical behavior. The resulting initial conditions are labelled E and C; the drag acceleration, d, controls the third term.

$$V(t) = \frac{2}{3}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} A(-\omega t) E + \left[1 - \frac{3}{2}\omega t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] C + \frac{d}{2\omega^2} \begin{pmatrix} 4\omega t \\ -3(\omega t)^2 \end{pmatrix} \quad (16)$$

In Appendix C, the inertial equivalent of (16) is obtained by rotation, that is, by multiplying (16) on the left by $A(\omega t)$ and simplifying. The separation of (16) into three terms is not, of course, affected, and the equations for these elementary trajectories are tabulated in Table 1.

The letters E and C are mnemonics. Relative to the spacecraft in its circular earth orbit, a particle in a pure E-trajectory is in an Elliptical orbit of the identical period. The E-term has no secular time dependence.

A particle in a C-trajectory is in a Circular orbit, of identical period if the initial radial component (C_x or C_ξ) is zero. Otherwise, it has a different period and drifts away.

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*For simplicity, the 2x2 matrices are kept explicit in the text, although in Appendix C they are represented as combinations of a small number of linear operators.

The third elementary trajectory depends only on the Drag acceleration, d, and always shows a secular time dependence (t and t^2).

The elements, and a certain range of deliberately chosen combinations are geometrically simple, and are summarized in Figure 1. The drawings on the left are for the inertially oriented station, those on the right for the local vertical station. Each drawing shows the initial condition vector, the locus of motion, and an indication of the sense of motion. Note that motions in the inertial station turn with the orbit, counterclockwise, while motions in local vertical turn against the orbit. The rest of section three describes the geometry of these trajectories.

The "general" trajectory is a combination of these. A few examples are given in section four. Spirals are typical of inertial trajectories (see Drag in Figure 1), looping motions, of local vertical.

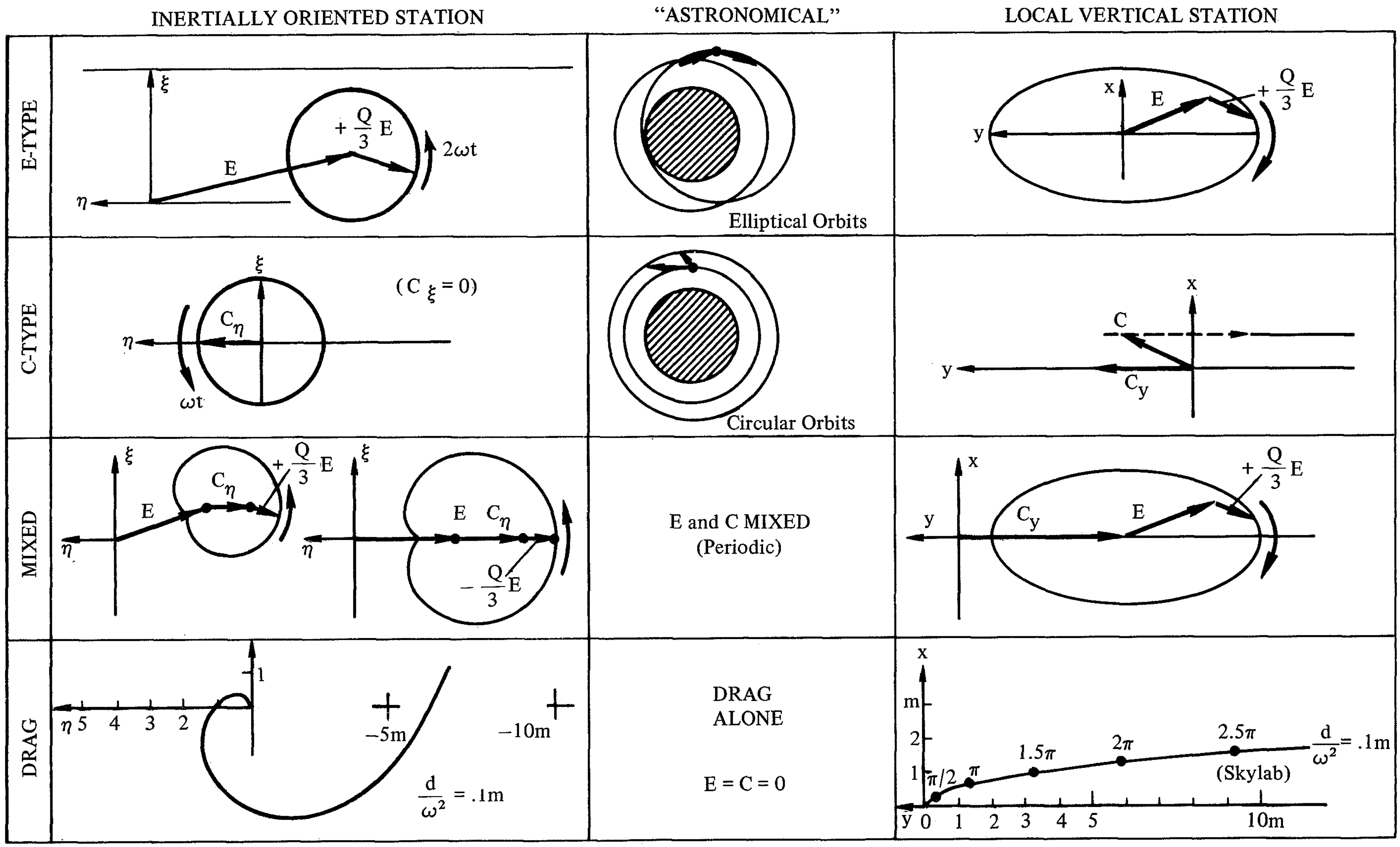


FIGURE 1 - SUMMARY OF ELEMENTAL TRAJECTORIES, SEEN IN THREE COORDINATE SYSTEMS

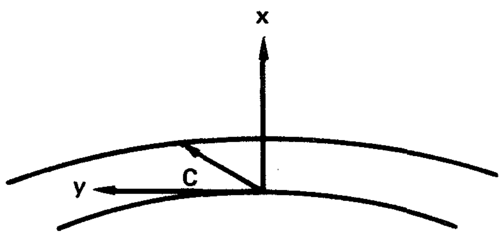
THE MATRIX $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ IS CALLED Q

TABLE I

SUMMARY OF EQUATIONS FOR ELEMENTAL TRAJECTORIES

	INERTIALLY ORIENTED STATION	LOCAL VERTICAL STATION
E-TYPE	$Z(t) = E + \frac{1}{3}A(2\omega t)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}E$	$V(t) = \frac{2}{3}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}A(-\omega t)E$ or, $\left[A(-\omega t) + \frac{1}{3}A(\omega t)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right]E$
C-TYPE	$Z(t) = A(\omega t)\left[C - \frac{3}{2}\omega t\begin{pmatrix}0\\ C_\xi\end{pmatrix}\right]$ or, $A(\omega t)\left[I - \frac{3}{2}\omega t\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]C$	$V(t) = C - \frac{3}{2}\omega t\begin{pmatrix}0\\ C_x\end{pmatrix}$ or, $\left[I - \frac{3}{2}\omega t\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]C$
DRAG	$Z(t) = A(\omega t)\,\frac{d}{2\omega^2}\begin{pmatrix}4\omega t\\ -3(\omega t)^2\end{pmatrix}$	$V(t) = \frac{d}{2\omega^2}\begin{pmatrix}4\omega t\\ -3(\omega t)\end{pmatrix}$

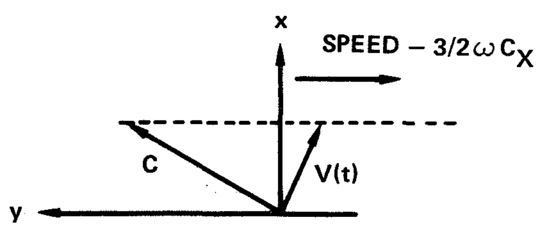
3.2.1 The C-Type Elemental Trajectory



C-type trajectories describe bodies moving in circular orbit around the Earth. From a local vertical space station also in circular orbit, we should see the bodies move at constant altitude.

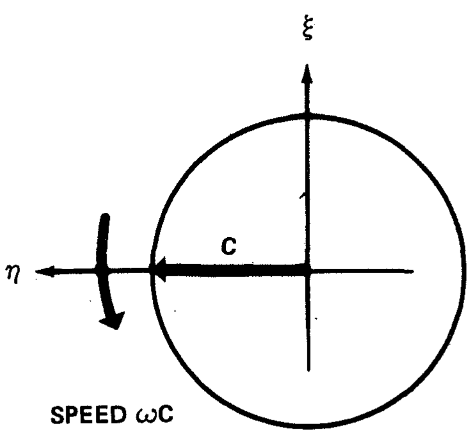
From Table I (or (16)), the C-local vertical elementary trajectory is:

$$V(t) = C - \frac{3}{2}\omega t \begin{pmatrix} 0 \\ C_x \end{pmatrix} \tag{17}$$



The initial position, C, can be chosen arbitrarily in the V-plane (x,y). If C_x is zero, the body lies on track and is stable there. Except for phase it is in an identical orbit with the station.

For a general C, the particle drifts. As drawn, it lies above the station orbit and drifts aft. The drift velocity must be given at insertion.

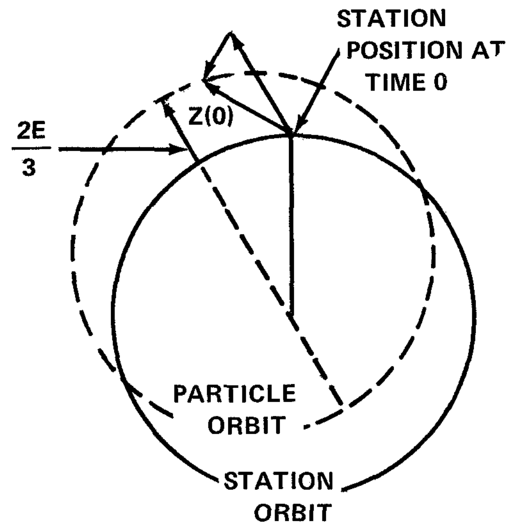


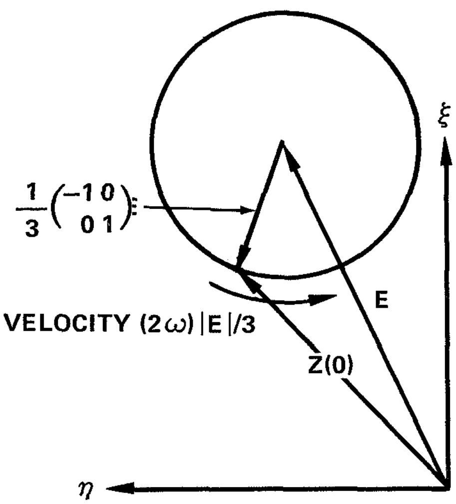
From an inertial station, an observer sees the track-line, like the Earth, travel around the station.

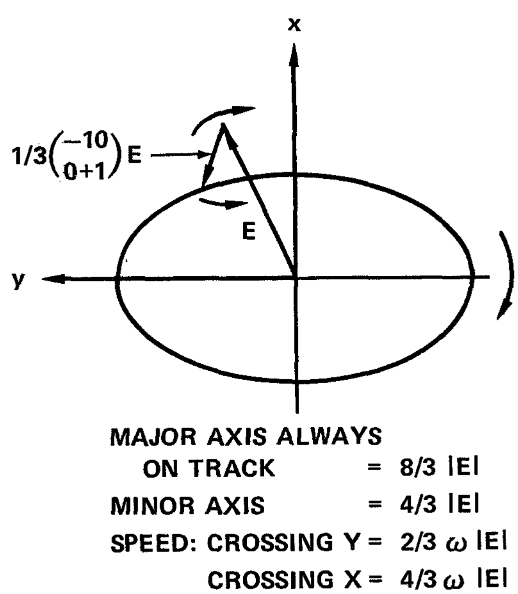
The inertial view of the stable particle on orbital track is a circle, <u>with</u> the orbit, at orbital rate. The particle must be inserted into this trajectory on track, with the speed ωC, as shown. The general C-trajectory is a spiral around the C.G. of radius $|V(t)|$.

$$Z(t) = A(\omega t)\left[C - \frac{3}{2}\omega t \begin{pmatrix} 0 \\ C_\xi \end{pmatrix}\right] \tag{18}$$

3.2.2 The E-Type Elemental Trajectory







E-Type trajectories describe bodies moving relative to the station, in elliptical orbits of semi-major axis and thus period equal to the station. The line of apses of the elliptical orbit lies along E. Apoapse is displaced outward by 2E/3.

As seen from the inertial station, the particle remains generally in one area. From Table I, the E-Inertial trajectory is:

$$Z(t) = E + \frac{1}{3}A(2\omega t)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}E \qquad (19)$$

This is a constant vector plus a second which rotates <u>with</u> the orbit at twice orbital rate. Construct the drawing as follows. Choose the arbitrary initial vector E, in the plane. The matrix $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ (called Q in Figure 1 (following page 12) and in the appendix) changes the sign of E_ξ; geometrically, it reflects E in the η axis. The sum locates the initial position Z(0). The point Z(t) then describes a circular path of radius $|E/3|$ around E as center.

The E-trajectory would be a good subsatellite orbit for an inertial station.

As seen from a local vertical station, the E-trajectory travels around the origin, <u>against</u> the orbit. In Table 1, the alternate form for the E trajectory can obtained from (19) by multiplying on the left by $A(-\omega t)$. Appendix C shows that the reflection matrix changes the sense of ω on commutation; matrix addition then gives

$$V(t) = \frac{2}{3}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}A(-\omega t)E \qquad (20)$$

$A(-\omega t)E$ describes a particle moving in a circle against the orbit. The matrix distorts the circle into a 1:2 ellipse* with major axis on track.

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*This may be seen in a special case.

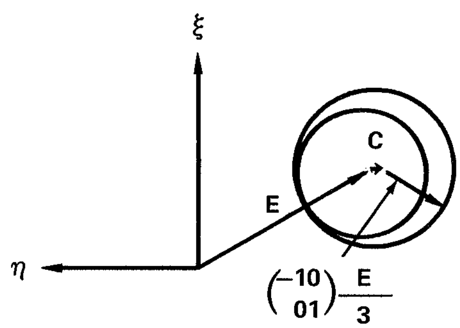
$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}A(-\omega t)\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}\cos\omega t\\ -\sin\omega t\end{pmatrix} = \begin{pmatrix}a\cos\omega t\\ -b\sin\omega t\end{pmatrix}$$

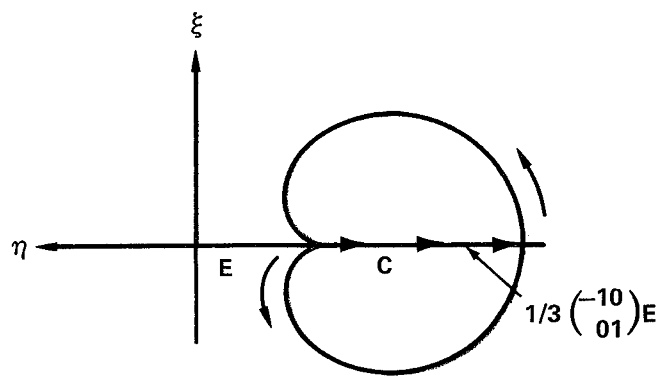
These satisfy $(x/a)^2 + (y/b)^2 = 1$.

3.3 Interesting Special Cases

As indicated above, the general inertial trajectory spirals, and the general local vertical trajectory tends to loop. There are some interesting special cases among the mixed E and C orbits, however. All are periodic--that is, C_x or C_ξ must be zero.

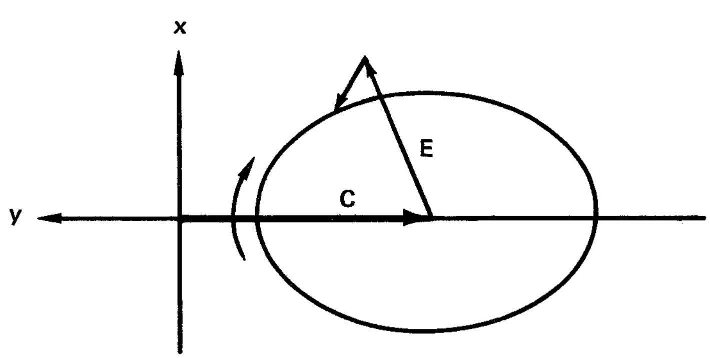
Looking at Figure 1, the inertial E and C trajectories are circles, the E-offset from the origin, with angular velocity 2ω, the C-centered on the origin with angular velocity ω. There is a continuous range between these. The figure on the left shows an arbitrary example. The interesting case is on the right; $|E| = 3/2|C_\eta|$.





The trajectory is a cardioid, and the particle comes to rest at the cusp. This fits the case of the particle re-leased from rest on the instantaneous track line. It is the only periodic orbit accessible to a particle started from rest in an inertial frame (the particle at rest at the C.G. is a degenerate case of this).

Looking at Figure 1 for the local vertical C and E trajectories, the most general periodic motion is simple. The E-type ellipse is displaced arbitrarily along the track by the vector C.

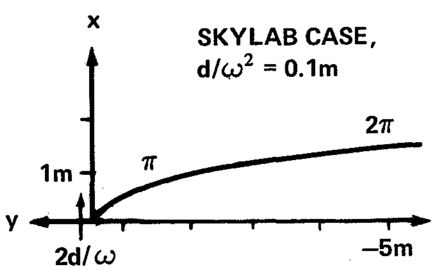


If C_x were positive, the center of the ellipse would drift to the right. In Earth orbit, drag makes it impossible to strictly realize any of these periodic orbits. Drag is covered in the next section.

3.4 Drag-Local Vertical: E = C = 0

Even for missions at Skylab altitudes, drag is an important force, resulting in translations approaching 10 meters per orbit. The elemental drag trajectory in local vertical is a parabola. The particle rises at constant velocity $2d/\omega$ and accelerates aft in track. From (16),

ELEMENTAL DRAG TRAJECTORY



$$V(t) = \frac{d}{2\omega^2}\begin{pmatrix} 4\omega t \\ -3(\omega t)^2 \end{pmatrix} = \begin{pmatrix} 2dt/\omega \\ -\frac{3}{2}\,dt^2 \end{pmatrix} \qquad (21)$$

The initial position is the origin. The initial velocity, $V(0)=2d/\omega$.

$$y = -\frac{3}{8}\,\frac{\omega^2}{d}\,x^2 \qquad (22)$$

The drawing is for a case like Skylab, for which the drag acceleration is $\sim 10^{-8}g$ ($10^{-7} m/s^2$) and the parameter $d/\omega^2 \sim 0.1$ meter. The particle rises vertically (velocity $\sim$0.2mm/s) and in one orbital period ($\omega t = 2\pi$) drifts about 6 meters aft. Physically, the Skylab is dropping and moving forward.*

3.5 Use of the Elemental Trajectories

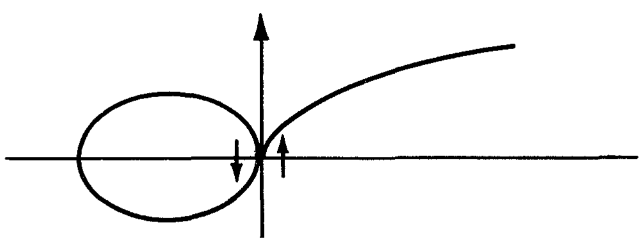
The drag trajectory illustrates the use of the elemental trajectories. These are particular solutions which can be added together to represent arbitrary initial conditions. The last figure demonstrates a surprising fact: the particle responds to an acceleration in the plus track direction by rising in radius and accelerating aft. This result is explicable as follows.

---

*As noted, these numbers are about right for a Shuttle in the late 70's at 500 km. Near 100 nautical miles (200 km), d/ω^2 can be 10 meters, and the displacement in one orbit, 600 meters! See Appendix B.

First, drag is a small force. The physical meaning of $d/\omega^2 \sim 0.1$ meter is that for distances from the C.G. greater than this, the gravity gradient accelerations $\omega^2 V$ exceed d. Thus, the last figure (5 meters full scale) is a picture of gravity motion in response to drag. The particle is rising into higher, C-type orbits, and drifting aft accordingly.

Secondly, the particle is not started from rest; the drag trajectory includes an initial, radial velocity, which in fact does not change. To obtain the solution for a particle started from the origin at rest, an appropriate 2:1 ellipse is added. By matching the position ($V(0)=0$) and velocity requirements, we find

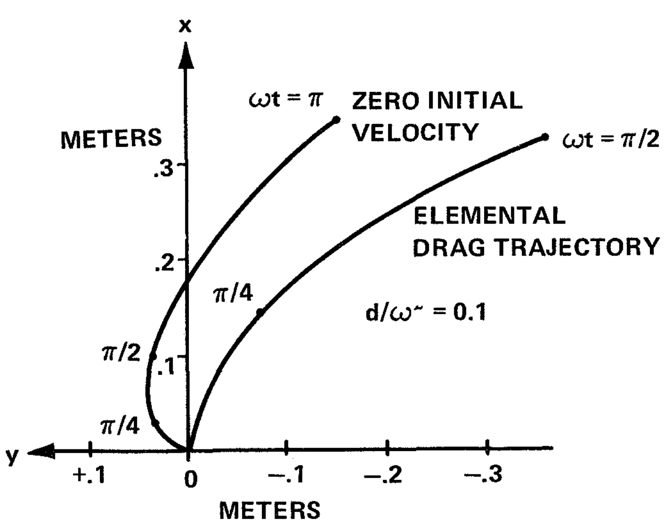


$E_y = -3\ d/\omega^2$, or .3m, and

$C_y = 4\ d/\omega^2$, or .4m

with the vertical components zero. Alternately, Appendix C has expressions for E & C in terms of initial conditions.

This gives the following qualitative results: The motion of the particle from rest starts slowly; the deviation from the drag trajectory is always leftward; the deviation in track will not exceed the major axis of the ellipse, or 0.8 meter; deviations in vertical will not exceed the semi-minor axis of the ellipse, or 0.2 meter.



Grossly, then, the drag trajectory is a good guide to how the particle started from rest will behave. The figure at left shows the detailed path for the first half orbit on a scale such that the initial, $s=\frac{1}{2}dt^2$ behavior can be seen.

4.0 APPLICATIONS

The vector equations of motion (7) and (10) and the elementary trajectories, summarized in Figure 1, give a basis for understanding forces and motion in orbit. Applications are presented in five areas: the general behavior of loose objects in spacecraft; work-aids; the acceleration environment for low-G experiments; trajectory design for sub-satellites; and rendezvous. Much of this, of course, is not new.

4.1 General Observations

This section deals with the general behavior of loose objects in a space station.

Weightlessness is a good description for times of about a minute. That is, if objects are observed casually, to a centimeter or so, they follow first law trajectories ($\ddot{R}=0$) for about a minute. To show this, calculate the time, t, for a displacement s assuming that, as in the Skylab crew quarters, the object is a distance r of about 10 meters from the center of gravity. Gravity gradient is the important force, since $r \gg d/\omega^2$.

$$t = \sqrt{\frac{2s}{a}} \sim \sqrt{\frac{s}{\omega^2 r}} \sim \sqrt{\frac{s}{r}}\ 10^3 \text{sec.}$$

For s of one cm and r of 10 meters, this is $t \sim 30$ seconds.*

Similarly, Coriolis accelerations are not important for gross motions in a short time. Suppose one astronaut throws an object to another in a local vertical station. Calculate the transverse displacement, s, for an object thrown a distance d in time t.

$s = \frac{1}{2}(2\omega v)t^2 \sim 10^{-3} dt$. The angular displacement,

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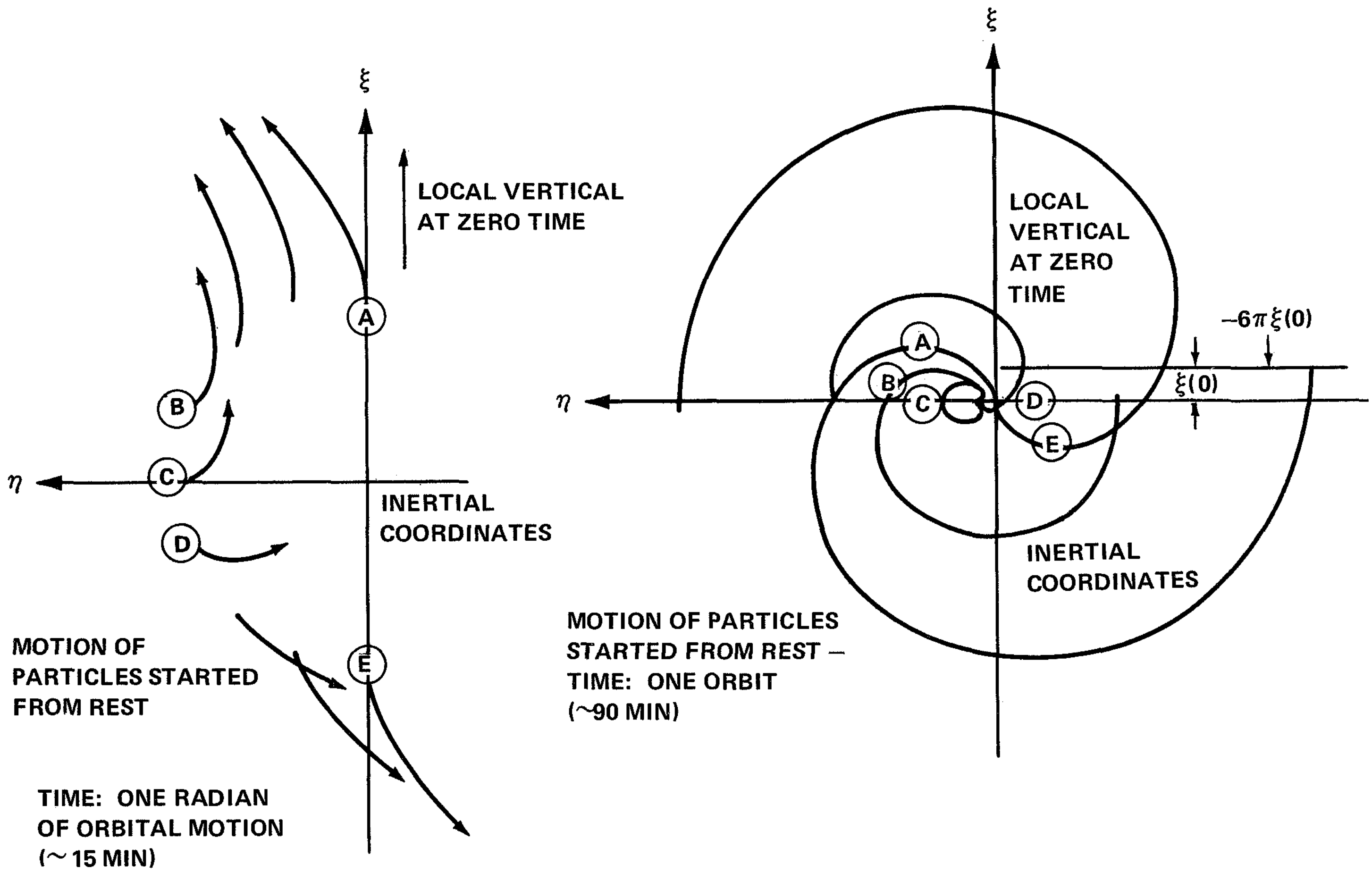
*Practically, objects are still within reach after moving ten cm; distance to the C.G. is less for vehicles like Apollo. Therefore, t's of several minutes are sensible. On the way to the Moon, ω^2 drops as the cube of the distance from Earth and weightlessness is an excellent description.

$s/d = 10^{-3}t$, is proportional to transit time and quite small (one degree) even for long slow throws (say, 10 m at 1/2m/s.).

Over longer times, objects will move with the acceleration patterns of the differential equations. Figure 2 shows the inertial acceleration pattern of equation (5) superposed on a Skylab silhouette. Two cases are shown, separated by a quarter orbit or 22 minutes. In the first, the long axis of the assembly lies near track and any object dropped in the CM tends to propagate down into the body of the workshop. Twenty minutes later the forcefield is reversed.

An object at rest starts with this acceleration field. In the inertial case, it then turns counterclockwise and (in general) spirals out until it meets an obstacle. The displacement after the first orbit is 6π times the initial radial ($\xi(0)$) distance from the C.G. Particles released near track are displaced less. The particle on-track describes a closed, cardioid orbit.



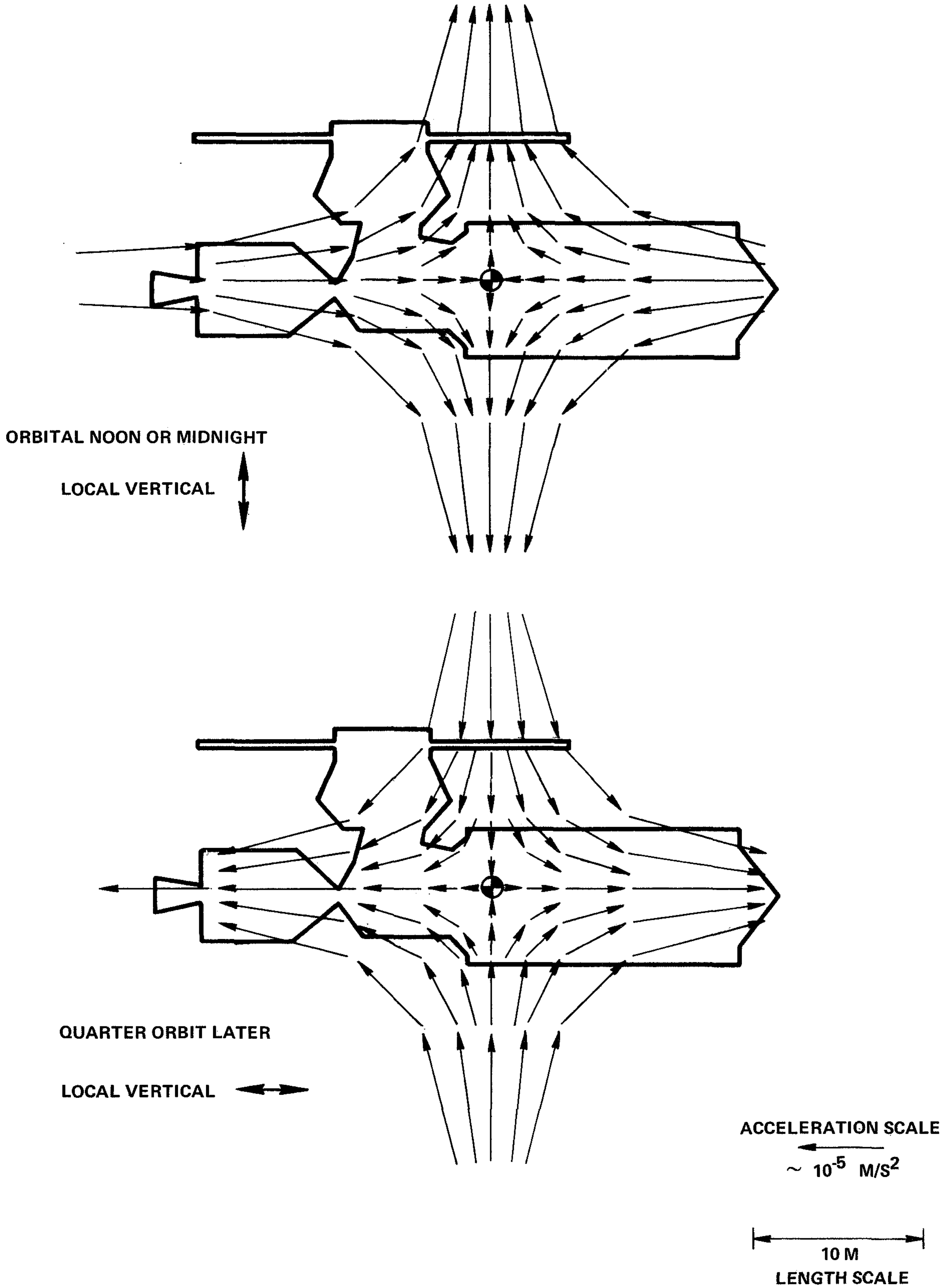
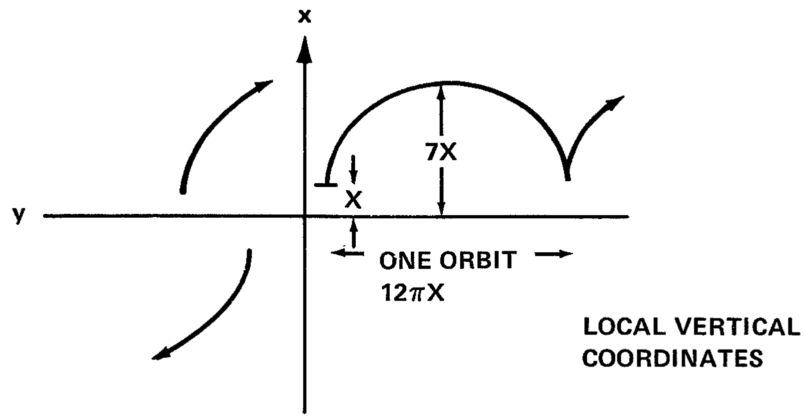


FIGURE – 2 ACCELERATION FIELD NEAR AN INERTIAL VEHICLE (SKYLAB) FOR ORBITAL NOON / MIDNIGHT AND A QUARTER ORBIT LATER. ARROW LENGTH IS PROPORTIONAL TO ACCELERATION AT THE ARROW TAIL. COMPONENT PERPENDICULAR TO THE PAPER IS RESTORING.



In a local vertical station, the first plane motion of an object is radially away from the track line. Coriolis turns the particle clockwise. The trajectory is a general C + E type. For an initial displacement x in the vertical, the elliptical component is large (minor axis 6x). The maximum excursions are shown in the figure.*

To summarize, "weightlessness" is a good description of free motion for the first minute or so. Displacements from first law behavior are typically under a centimeter. Subsequent motion is curved, with the orbit for inertial stations, against the orbit for local vertical stations. Displacements in the first radian of orbital motion (15 minutes) are comparable with the initial c.g. displacement, r. Displacements over an orbit are (excluding periodic motions) typically 20-40 times r, plus whatever is due to drag (5 or 6 m for Skylab).

4.2 Work Aids

There are two quite different physical problems for which work-aids must be devised, closing the force-loop and

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*This is the type orbit the Apollo lunar sub-satellite would follow if it were merely set loose in the Service Module bay, facing along a radius.

localization. These may not have been adequately defined in the past; and, judging at least by the space experiments which have been prepared in the last few years, there has been a tendency to assume the problems were mysterious and required sophisticated devices to solve them.

In short time periods, the astronaut may exert large forces and torques on a work-piece. The requirement is to close the system so that the net acceleration of the two is zero. Putting the Hasselblad between one's knees to work on it is a homely example. For larger jobs such as disassembling an electric motor or plumbing, there is no substitute for terrestrial tools like the vise which immobilize the piece conveniently relative to the spacecraft. Another device must be used to tie the astronaut to the spacecraft. Foot restraints are used on Skylab. The extreme example is a deep molded chair which clamps to the bench and couples the forces through thigh, knee, and back restraints. The critical point is that these work-aids are rugged, and transmit substantial forces. A good workshop on earth or in orbit has many kinds of clamps, vices, or restraints.

The second problem is to prevent objects from drifting away. The requirement is that a restraint system (or systems) shall bring objects of various sizes to rest from velocities of a few centimeters per second, and hold them neatly until they are wanted. Obvious methods, many used already on manned flights, include:

(a) elastic cords for larger objects.

(b) a pegboard with clips

(c) velcro (velcro of the right size probably will hold small screws and nuts)

(d) magnetized racks. At these force levels, some ferrite paint would probably be adequate for immobilizing objects.

(e) plastic bags and boxes, organized in shelves, or on the bench by (b) and (c).

(f) lazy susans, that is, rotating trays which hold parts by centrifugal force. (The rotation rate needs to be faster than orbital rate, but one rotation in a few minutes is fast enough.)

The problem of work in weightlessness is not hard. Astronaut experience shows it; the analysis here can add little more. It is only the projection of tools (and the man) into this apparently alien environment which is hard and requires high technology.

4.3 Experiment Design

The acceleration levels required for experiments vary widely. Current estimates for space biology term 10^{-4}g satisfactory; 10^{-6}g was observed to start convection in the Apollo 14 oxygen tanks. There are more stringent requirements in relativity experiments, for instance, but it is unlikely these would be flown on a manned vehicle. This section deals with the acceleration environment in low Earth orbit.

There is an irreducible stress within a specimen, set by the gravity gradient acceleration of order $\omega^2 s$, where s is the specimen size. The experiment sensitive to gradients of order $10^{-6}s$ must be flown at higher altitude.

Other stresses will be communicated to the specimen by its container, or by associated fields. For short times, the specimen may be left free. For longer times, it must be confined. Within limits, the larger the zone within which the specimen can drift, the smaller the control forces can be. In low Earth orbit, contributions to specimen motion, assuming an initial C.G. displacement r, are:

	Acceleration	Displacement	in 1 rad
Gravity Gradient*	$(1-3)r\omega^2$	$\sim$	$3r$
Initial Velocity (Δv)	0	$\sim$	$10^3\Delta v$
Coriolis (for ΔV)	$2\omega(\Delta v)$	$\sim$	$10^3\Delta v$
Drag	d	$\sim$	d/ω^2
Light Pressure	smaller than drag		
Vibration**	$\sim$(frequency)^2	(periodic) probably mm, max.	
Astronaut Motion**	?	several centimeters,** max.	
Attitude Change θ**		$\sim r\theta$ over attitude cycle	

There will be other, substantial sources of motion. A free floating molten drop can probably provide 10^{-8} accelerations by uneven evaporation. A moving animal can obviously pulse himself to a g or so.

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*Including the centrifugal forces in local vertical.

**The last three entries are discussed in Appendix E.

Considering the list, it appears that the contributions could be held individually near the limiting drag acceleration of a Skylab class vehicle, 10^{-8}g provided: displacement (<10 cm) is allowed to isolate the specimen from vibration and astronaut motion; work sites are located within $r \sim 10$cm of the C.G. for an inertial station or within $r \sim 10$cm of the trackline for a local vertical station; attitude motions are small (as with C.M.G. control); and initial velocities are small ($\Delta v < 0.1$ mm/s).

The average acceleration of the specimen -- or the control accelerations necessary to keep it near rest -- could optimistically be as small as 10^{-7}g, or 10^{-6}m/s^2.

It is doubtful that most experiments will require such low acceleration environments. A knowledge of the residual accelerations may however be important to an investigator. For Skylab, many experiments are located in the crew quarters at distances of 10 meters from the center of gravity. An experiment attached at such a point will see gravity gradient accelerations ranging from 10^{-5}m/sec^2 towards the C.G. to about twice that away from the C.G., alternating over periods of twenty minutes or so (see Figure 2). If the experiment is marginally sensitive to accelerations of this order (10^{-6}g), this shifting around could confuse the experimental results.

It is clearly desirable that the Space Shuttle and its Sortie module be configured so that the on-orbit C.G. and (for local vertical) track line lie within the experimental area. Configurations where the Sortie module is hinged and deployed outward have higher residual accelerations* and can not achieve the minimum acceleration environment.

At lower altitudes where the atmosphere is denser drag will dominate, providing in reasonable cases (Table B-1) nearly 10^{-4}m/s^2 (10^{-5}g). This is high enough to be significant for experiments involving convection in large volumes.

Accordingly, it would be desirable to conduct studies of systems which could provide a Shuttle this kind of acceleration. The force required is $\sim$10N or a few pounds force. Since drag is not constant, a servo controlled scheme would be required. The shuttle would "fly around" the experiment.

4.4 Subsatellite Orbits

There are at least three families of orbits in which subsatellites can be placed and subsequently retrieved. These are: the E-type orbit, the C-type orbit, and drag-type orbits.

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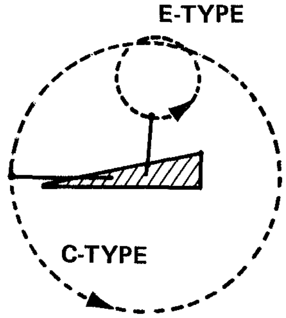
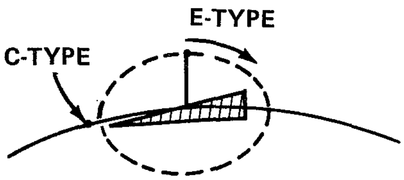
*e.g., if the shuttle flies nose down with the module along track, drag is highest. In other orientations, gravity gradient is high.



INERTIAL



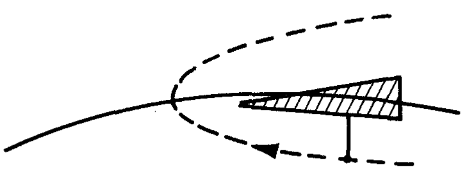
LOCAL VERTICAL

The relative drag of the subsatellite can be trimmed to equal that of the parent spacecraft (Appendix B) by using much heavier structure than is typical of automated satellites today.

The first order solution requires that the subsatellite be extended on a boom. To simplify the satellite, the device on the end of the boom imparts the initial velocity necessary to insert the satellite in an E- or C-type orbit. The required velocity is small, but must be precise.

A motion perpendicular to the plane $\dot{z}(0)$ could be added so that the satellite will not pass through the station wake.

If the subsatellite is to be kept visible, or within the field of a communications antenna, E-type trajectories are preferred for an inertial mission and C-type for a local-vertical mission.



LOCAL VERTICAL

C. O. Guffee pointed out several years ago[5] that drag could be employed to achieve a free return of a sub-satellite.* In the drawing, the parent vehicle has higher drag than the satellite. The satellite is deployed in a circular orbit below the parent, with appropriate forward motion and a small upward velocity $2d/\omega$. From the parent point of view, it drifts forward and upward in a drag-type parabola and is recaptured above the parent.

4.5 Rendezvous

Related techniques have been used to study Gemini and Apollo rendezvous. The elemental trajectories are easy to use in hand calculations and to see what is or is not possible.

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*Guffee's particular case is a mixed orbit and shows looping elliptical motion.

As an example, Appendix D describes a Hohmann transfer.

5.0 SUMMARY AND CONCLUSIONS

The equations of motion for a particle in or near an orbiting spacecraft have been presented, and the solutions written in a geometrically simple way. The gross motion can be described as a sum of three elementary ones, two due to Keplerian motion and one due to drag. The geometrical results are in Figure 1.

It is believed this material gives a basis for approximate understanding of both particle motion and the design of low-g experiments.

Weightless, unaccelerated behavior is a good description for at most a few minutes. Beyond this, motion due to gravity gradient and drag must be accounted for. Under favorable circumstances, low-g experiments may realize acceleration environments near these limiting values, approximately 10^{-6} to 10^{-7}g. The experiment must be isolated from spacecraft motions and vibrations.

As regards the supporting design of Shuttle or station experiment systems:

(a) three orbit classes are shown for co-orbiting satellites

(b) for low gravity laboratories, the local vertical attitude is promising, and deserves more careful study

(c) accordingly, it is desirable that space stations or Shuttle Sortie modules include the center of gravity of the shuttle and the track-line in a local vertical attitude

(d) drag is sufficiently large at low altitudes that a low thrust (10N) system capable of compensating for drag probably needs consideration.

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3. Mario H. Rheinfurth, MSFC, "Low Gravity Gradient Mechanics", Manufacturing Technology Unique to Zero Gravity Environment, pp. 181-196, NASA MSFC-Form 454, November 1, 1968.

4. "Appendix A - The Statics and Dynamics of a Material System Contained within an Orbiting Vehicle", Feasibility Studies of Promising Stability and Gravity (Including Zero-G) Experiments for Manned Orbiting Missions, EOS Report 7000-Final, January 1966.

5. C. O. Guffee, "A Method for Free Flight of the LM/ATM During AAP Missions", Memorandum for File, B67 1073, October 31, 1967.

6. W. W. Hough, "Required Artificial G Field for the Skylab Gravity Substitute Workbench", Memorandum for File B70 09086, September 30, 1970.

*A complete literature search was not attempted.

APPENDIX A

Equations of Motion

Appendix A leads to the vector equations of motion for a small particle in or near a spacecraft in circular orbit, with emphasis on the operational meaning of the terms. The procedure is to start with definitions in inertial space, where they are presumably well understood, then make the coordinate transformations to obtain derived results for local vertical and inertially oriented space stations.

Inertial Laboratory (R)

Measurements of Position, Velocity, and Acceleration

The observer in his laboratory measures the position of the object in some reasonable way. Conceptually, he can set out meter sticks along orthogonal axes, take Polaroid photographs in pairs (including an image of his stop-watch), and analyze these to obtain the three components of a vector position R. He can reduce successive measurements, R_1, R_2, R_3... to obtain estimates of velocity $\dot{R}$ and acceleration $\ddot{R}$.

The expected behavior of the object depends on the experimental conditions, and is codified in the descriptions of common forces, which can be added to give a resultant force and divided by particle mass to give a resultant acceleration.

$$\ddot{R}_{(\text{expected})} = \frac{1}{m}\sum F_i \qquad \text{(A-1)}$$

Some of these forces are convenient rules, like Galileo's observation that all objects fall with the same acceleration (9.8 m/s^2 to better than one percent, anywhere on Earth). Others are more general, and at least to experimental accuracy, are "laws". This includes, for instance, the Newtonian expression for the acceleration of a small body near a larger one:

$$\ddot{R} = -\frac{R_o(GM)}{|R_o|^3} \qquad \text{(A-2)}$$

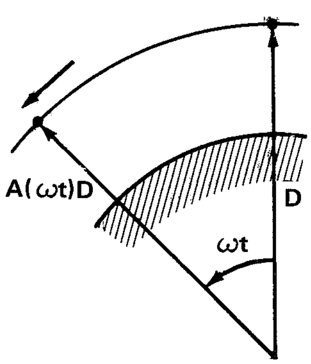
R_o is the radius vector from the center of the Earth to the particle, and (GM) a constant, the gravitational parameter of the Earth.

Most of these laws are most simply expressed in an inertial laboratory, unaccelerated and non-rotating as judged by observations of the fixed stars. The object of this appendix is just to make this common catalog of laws applicable to orbital flight, e.g., to express the measured accelerations as some correction terms plus an $\ddot{R}_o$, corresponding to the "inertial" catalog of forces.

Forces and Stresses

One caution is necessary. Often, as when a ball is accelerated by a bat, the force $F = mR$ is conveyed through the body by internal stress. This is not so for gravitation, which acts nearly equally on all mass elements of a body, or for the centrifugal and Coriolis forces which are accidents of the coordinate system. The freely falling body is almost unstressed; it is when the object is constrained from accelerating by lying on the Earth or resting on the walls of the space station that it is stressed.*

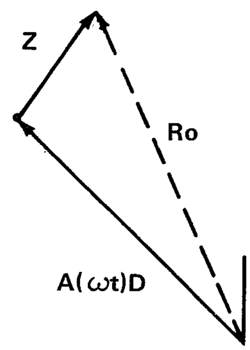
Laboratory System (Z) (The "inertially oriented" Space Station)



Consider now a laboratory which is inertially oriented, but accelerating in a circle around the Earth with angular velocity ω.

In inertial, astronomical coordinates, let the initial position be $R_o = D$. Define the rotation operator $A(\omega t)$ which rotates a vector counterclockwise around a chosen axis by an angle (ωt). Then at t, the station position is $A(\omega t)\ D$.

*This kind of distinction led Einstein in the General Theory of Relativity to try and make gravity a property of the coordinate system rather than an independent force.



The observer now sets up shop with meter stick and stop watch, and performs measurements of position, Z. A corresponding "Inertial" (R) measurement would be

$$R_o = Z + A(\omega t)\ D \qquad \text{(A-3)}$$

By differentiating, obtain a relationship between the measured values, Z, $\dot{Z}$, and $\ddot{Z}$ and the catalog of 'expected' behavior in terms of $\ddot{R}_o$. Properties of $A(\omega t)$ are developed in the note (see next page). In particular,

$$\dot{R}_o = \dot{Z} + \omega P\ A_p(\omega t)\ D \qquad \text{(A-4)}$$

$$\ddot{R}_o = \ddot{Z} - \omega^2\ A_p(\omega t)\ D \qquad \text{(A-5)}$$

Inverting,

$$\ddot{Z} = \omega^2\ A_p(\omega t)\ D + \ddot{R}_o \qquad \text{(A-6)}$$

NOTE:

In component form, with z the orbital pole, x along the radius, D, and y in track, $A(\theta)$ is a rotation matrix,

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) As written, A will rotate a vector counterclockwise in the direction of prograde motion as seen from the orbital pole. Its properties can be derived from this or the component representation.

(2) Two rotations around the same axis are additive

$$A(\theta_1)\ A(\theta_2) = A(\theta_1 + \theta_2)$$

(3) Rotations commute (from (2))

$$A(\theta_1)\ A(\theta_2) = A(\theta_2)\ A(\theta_1)$$

(4) There is an identity,

$$A(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

(5) And for each $A(\theta)$ there is an inverse $A^{-1}(\theta)$, so that $A^{-1}(\theta)\ A(\theta) = I$. $A^{-1}(\theta) = A(-\theta)$ and is the transpose of $A(\theta)$.

(6) The time derivative $\dot{A}(\theta) = \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\theta} = A_p(\theta + \pi/2)\dot{\theta}$

A_p is the plane rotation operator with no z component. Of course, $-\sin\theta = \cos(\theta + \pi/2)$, $\cos\theta = \sin(\theta + \pi/2)$.

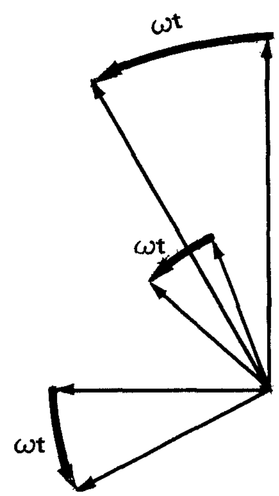
Note $A_p(\theta_1)\ A(\theta_2) = A_p(\theta_1)\ A_p(\theta_2)$

(7) Special values. $A_p(\pi/2)$ occurs often $= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For brevity, call it P for perpendicular. Note $P^2 = A_p(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_p$. Thus, the velocity associated with $A(\omega t)D$ is $\omega P\ A_p(\omega t)D$, perpendicular to $A(\omega t)D$ and of magnitude ωD, as it ought to be.

(8) $A_p(\theta) = I\cos\theta + P\sin\theta$ is used in Appendix C.

LABORATORY SYSTEM (V) (The "local vertical oriented" Space Station)

The origin of (V) coincides with the origin of (Z). At time zero, the axes coincide. The (V) laboratory axes then rotate as the station goes around the Earth. Call the measurements of position, V. Any measured object which appears to be at rest, is, from the (Z) or (R) point of view, in prograde rotation. That is, the corresponding Z are rotating,



$$Z = A(\omega t)\ V. \tag{A-7}$$

Substituting in (A-3),

$$R_o = A(\omega t)(V+D) \tag{A-8}$$

$$\dot{R}_o = A(\omega t)\ \dot{V} + \omega P A_p(\omega t)(V+D) \tag{A-9}$$

$$\ddot{R}_o = A(\omega t)\ddot{V} + 2\omega P A_p(\omega t)\dot{V} - \omega^2 A_p(\omega t)(V+D) \tag{A-10}$$

To find the (V) observer's expected accelerations, multiply (A-10) on the left by $A(-\omega t)$ and rearrange

$$\ddot{V} = \omega^2\ I_p\ (V+D) - 2\omega P\dot{V} + A(-\omega t)\ddot{R}_o \tag{A-11}$$

In the absence of force, he will see accelerations in the plane which are functions both of position V and velocity $\dot{V}$. These are the centrifugal and Coriolis forces.

GRAVITY GRADIENT

The primary force on an orbiting station is of course gravity, which supplies the centripetal force $\omega^2 D$ necessary to keep the vehicle in its circle.

$$\frac{GM}{|D|^3} = \omega^2 \tag{A-12}$$

Since laboratory and free objects fall together, no measurement made within the (Z) or (V) laboratory can detect the full gravitational acceleration of Equation A-2. On the other hand, variations in the gravitational field will be detectable, albeit over longer times. These can be estimated by expanding A-2 in Taylor series. A-2 was:

$$\ddot{R}_o = - \frac{R_o \ (GM)}{|R_o|^3} \tag{A-13}$$

Expressed in the measurables of the local vertical (V) system, this is (use [A- 8])

$$\ddot{R}_o = - \frac{A(\omega t)\,(V+D)\,GM}{|V+D|^3} \tag{A-14}$$

(the value of the scalar denominator is not affected by the $A(\omega t)$).

A Taylor series expansion for a vector F has the following form;

$$F(V) \sim F(0) + V\cdot\nabla F\Big|_{V=0} \tag{A-15}$$

Note that keeping only the first order in $|V|/|D|$, with D greater than 6000 km, gives a very useful approximation to $\ddot{R}_o$. ∇F is the direct product whose matrix representation would be, $(\nabla F)_{ij} = \frac{\partial}{\partial x_i} F_j$. Expanding, (A-16)

$$\ddot{R}_o \sim - A(\omega t)\,(GM) \left\{ \frac{D}{|D|^3} + V \cdot \left[\frac{I}{|D|^3} - 3\,\frac{\hat{D}\hat{D}}{|D|^3} \right] \right\}^* \tag{A-17}$$

---

*Note: if $V = (x,y,z)$, $\nabla V = I$, $\nabla|D+V| = \frac{D+V}{|D+V|} = \widehat{D+V}$, a unit vector. In (A-17), $V\cdot\hat{D}\hat{D} = (V\cdot\hat{D})\hat{D}$, a resultant vector in the radial direction.

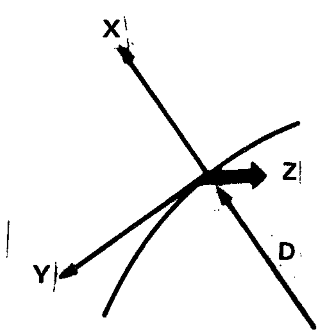
The first term is the 9 m/s^2 or so which supplies the centripetal acceleration to hold the station in orbit. The second term is a vector restoring force, tending to restore the test object to the origin. The third term represents the decrease in gravitational force with distance from the Earth. Its action is purely radial (in the D direction) and repels the test object from the tangent plane.

Equations of Motion - Local Vertical

To obtain the equations of motion, which describe the perceived motions of a particle under gravity gradient, with any peculiarities of the laboratory involved, (A-17) must be substituted for $\ddot{R}_o$. Taking the local vertical case, (A-11), and using $\omega^2 = GM/|D|^3$,

$$\ddot{V} = \omega^2 (I_p - I + 3\hat{D}\hat{D})\, V - 2\omega P\dot{V} \qquad \text{(A-18)}$$

Note here that the centrifugal terms from A-11 cancel all but the axial component of the restoring force gravity gradient term. The radial gradient and the velocity dependent Coriolis term remain. (Since gravity gradient does not include "all forces", a term $A(-\omega t)\ddot{R}'_o$ should be added.)



Note that the physics--rotation and gravity--gives observer (V) a strongly polarized view of the world. A component representation is natural. With x in local vertical, parallel to D; y along track; z parallel to the orbital pole; and V = (x,y,z)

$$\ddot{V} = \omega^2 \begin{pmatrix} 3x \\ 0 \\ -z \end{pmatrix} + 2\omega \begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} + A(-\omega t)\ddot{R}'_o. \qquad \text{(A-19)}$$

A more direct expression of A-18 is the matrix form used in the text, where $F/m = A(-\omega t)R_o$.

$$\ddot{V} = \omega^2 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} V + 2\omega \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{V} + A(-\omega t)\ddot{R}'_o \qquad \text{(A-20)}$$

In the solution outlined in Appendix C, the "other" acceleration $\ddot{R}'_o$ is taken as a constant, in-plane, local vertical vector,

$$A(-\omega t)\ddot{R}'_o = \begin{pmatrix} \ell \\ d \\ 0 \end{pmatrix} = L \qquad \text{(A-21)}$$

where the labels of the components are acronyms (lift and drag--although d acts in the positive track direction).

Note that Equations (A-20) and (A-21) comprise two coupled equations, in x and y, and a separate, z equation. This is a major simplification. The rest of the work can be carried out in two dimensions.

Equations of Motion: Inertially oriented

It is easiest to get the inertial equations by rotation of A-20. In two-dimensional form, (A-20) is*

$$\ddot{V} = \omega^2 \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} V - 2\omega P\dot{V} + L \qquad \text{(A-22)}$$

The relation between the V and the inertial, Z, family of vectors is:

$$V = A(-\omega t)\ Z \quad , \text{ from (A-7)}, \qquad \text{(A-23)}$$

$$\dot{V} = A(-\omega t)\ \dot{Z} - \omega P A(-\omega t)\ Z \qquad \text{(A-24)}$$

$$\ddot{V} = A(-\omega t)\ \ddot{Z} - 2\omega P A(-\omega t)\ \dot{Z} - \omega^2\ A(-\omega t)\ Z \qquad \text{(A-25)}$$

Substituting and multiplying on the left by $A(\omega t)$, the Coriolis term cancels:

$$\ddot{Z} = \omega^2 \left\{ A(\omega t) \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} A(-\omega t) - I \right\} Z + A(\omega t)L \qquad \text{(A-26)}$$

*Since all vectors are in-plane, we can omit the A_p subscripts for operators with zero third components.

This is simplified (using property (5), p. A-4).

$$\ddot{Z} = \omega^2 \left\{ A(\omega t) \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} A(-\omega t) \right\} Z + A(\omega t) L \,, \qquad \text{(A-27)}$$

which, with the z component added in again is the equation of the text (7).

APPENDIX B

Drag and Numerical Data

This appendix contains a resume of the drag equation, plus support for various numbers used in the text.

The drag acceleration on a vehicle of frontal area a and mass m is:

$$d = \frac{C_d}{2} \frac{\rho a\ v^2}{m} \qquad \text{(B-1)}$$

where ρ is atmospheric density, v is orbital velocity, and C_d is the drag coefficient. A unit frontal area sweeps out a mass (ρv) of atmosphere each second, or a momentum ρv^2. For free molecular flow, it is typically assumed that molecules attach themselves to the vehicle, thermalize, and re-emit with a much lower velocity v_t. Then, the net force on the vehicle is a little greater than $\rho a\ v^2$, and $C_D \sim 2$.

Under rare circumstances, such as a flat plate with clean surfaces at low angle of attack, molecules can re-emit specularly. C_D then is greater than two, and there can be substantial lift.

$C_D = 2$ is assumed

v^2 is reasonably constant in low Earth orbit (see Table B-2). Log v^2 is taken $\sim$ 7.8 with v^2 in m^2/s^2.

Atmospheric density is highly variable. As a result, the drag d varies widely and the calculation is not sensitive to uncertainty in C_D or v. Figure B-1 shows the logarithm of atmospheric density (kg/m^3) as a function of altitude. It is adapted from the 1966 supplement to the U.S. Standard atmosphere. It shows that density, ρ, is two or three times greater in the

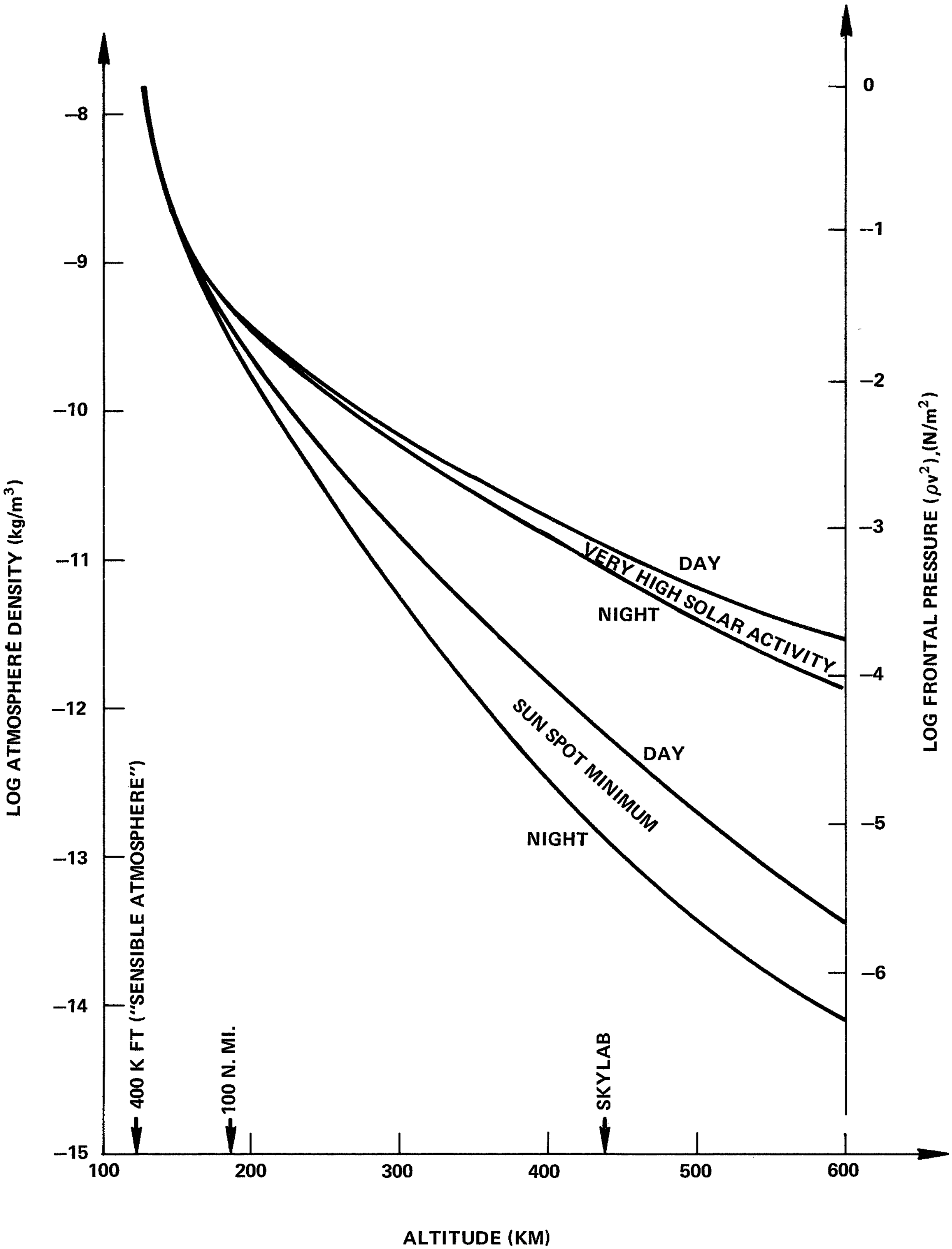


FIGURE B-1 - ATMOSPHERIC DENSITY AND DRAG PRESSURE VERSUS ALTITUDE sunlit part of the orbit than in the shaded portion. Density peaks shortly after orbital noon. Further, density is very sensitive to solar activity. Skylab will fly near solar minimum when drag is low. With current schedules, Shuttle and Station will fly near solar maximum when densities can be a hundred times greater.

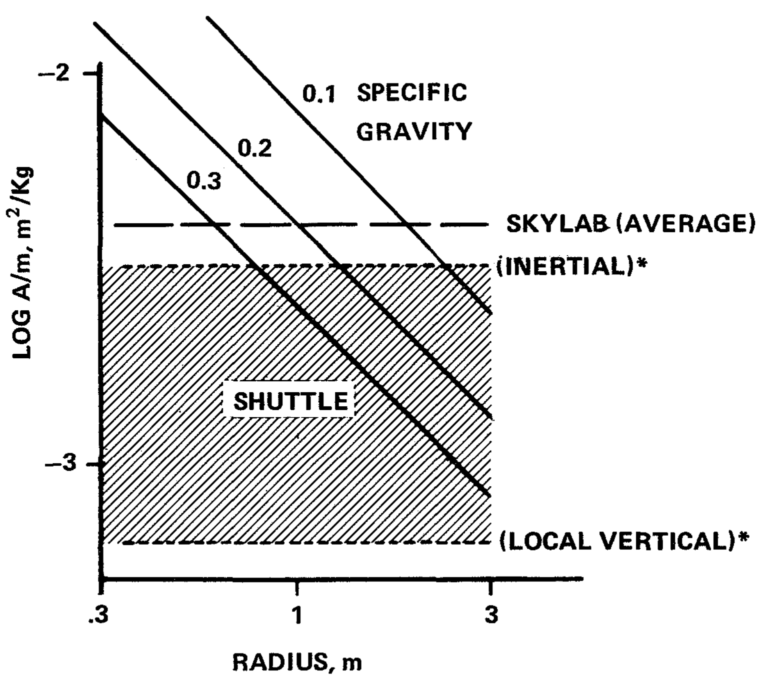
The right-hand scale of Figure B-1 shows the frontal pressure (newtons/m^2) on a vehicle with $C_D = 2$,

$$\log \rho v^2 = \log \rho + 7.8.$$

Area/mass ratios and typical vehicle drag accelerations are tabulated in Table B-1 for Skylab and Shuttle. A monolithic, nuclear powered space station would be comparable with the first entry under Shuttle ($d = 2 \times 10^{-7}$ m/s^2). Modular versions with solar power will have log A/m more like the Skylab.

Subsatellites will tend to higher A/m. Typical satellites today have bulk densities of (5-20 lb/ft^3, or a specific gravity of .1 to .3. For a sphere, for instance, A/m would be:

$$\frac{A}{m} = \frac{3}{4\rho R} = \frac{.75 \times 10^{-3}}{(\text{Sp. gravity})\ R} \quad m^2/\text{Kg}$$



*SEE NOTES TO TABLE B-1

Table B-1

Typical Drag Calculation for Skylab and Shuttle

Program	Years	Altitude km	Log Pressure (Fig.B-1), N/m^2	Area m^2	Mass kg	Log A/m	Log Drag	Drag 1 m/s^2
Skylab	1973	435 (235 nmi)	-4.5 to -5.2 sunspot minimum	320*	83,000*	-2.41	-6.9 to -7.6	$.6x10^{-7}$
Shuttle	1979 et seq	500 (270 nmi)	-3.5 to -3.7 sunspot maximum	100** local V.	$1.3x10^5$**	-3.1	-6.6 to -6.8	$2x10^{-7}$
		200 (108 nmi)	-1.7	400** inertial	$1.3x10^5$	-2.5	-4.2	$6x10^{-5}$

*Skylab data from MSFC via I. Hirsch of Bellcomm:

reference area: $79.46m^2$; mass (typical): 83,066kg; drag coefficient (accounting for solar panels, CSM's, etc); 8.1147. C_D was set = 2 and an average corresponding area calculated. The expected drag for the mission using the Modified Jacchia atmosphere is 10^{-8}g, or $\sim 10^{-7}$ m/sec^2.

**Approximate values as of NAR 180 day mid-term. Areas were estimated as: head on: $110m^2$, transverse: $400m^2$; from below, $760m^2$. The tabulated values are a minimum area "best case" and a reasonable worst case. (It was not assumed Shuttle would ever fly broadside around an orbit.)

It appears possible by using very heavy structure, which is permissible in the shuttle era, to trim sub-satellites to A/m's identical with a parent vehicle.

Table B-2 shows that orbital velocity does not vary much with vehicle altitude. Values of ω^2 for the gravity gradient calculations are also tabulated.

Table B-2

ω^2 and v Versus Altitude*

	Moon	Earth		
Altitude (km)	ω^2 (s^{-2})	ω^2 (s^{-2})	v(m/s)	log v^2
100	$.79 \times 10^{-6}$	1.47×10^{-6}	7.84×10^3	7.789
200	.67	1.40	7.78	7.782
400	.50	1.28	7.67	7.769
600	.38	1.17	7.56	7.757
800	.30	1.08	7.45	7.744
1000	.24	.99	7.35	7.733

*(GM)Earth=$3.98 \times 10^{14} m^3/s^2$, (GM)Moon=$4.902 \times 10^{12} m^3/s^2$, Equatorial Earth radius=6.378×10^6m, Moon radius=1.738×10^6m.

APPENDIX C

Solution

This appendix shows a method of solution of the differential equations, leading to the elemental trajectories of the text. The initial condition vectors, E and C, are expressed in terms of initial coordinates and momenta.

The z equation

It is important that the in-plane motion decouples from that perpendicular to the orbital plane. The z equation is separable, and recognized in the text as the equation for simple harmonic motion.

The equations in the plane: introduction

The components of local-vertical equation (A-20) and (A-21) form a set of linear, simultaneous, second order differential equations. There are well known methods of solution. The result will be the sum of a solution to the homogeneous equation (lift and drag forces zero) and a particular solution of the inhomogeneous equation. The solution to the homogeneous equation will depend on four scalar or two vector integration constants.

The Laplace transformation method is used to solve the local vertical equation, obtaining closed form solutions in terms of the time and the initial conditions. That is, the integration constants are $V(0)$ and $\dot{V}(0)$. The simpler representation in E and C is then "recognized". The case of a constant in-plane force L is solved, although it is clear that solutions can be obtained for forces that vary in time, for instance, models of solar pressure or a cyclic drag force.

The local vertical equation in the plane is:

$$\ddot{V} = \omega^2 \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} V - 2\omega P\dot{V} + L \qquad \text{(C-1)}$$

In the next section enough properties of queer matrices such as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are defined to permit algebraic manipulation.

Definitions and Algebra

In Appendix A it was noted that physics in zero-G gave a strongly polarized view of the world. The gravity gradient force and rotation result in operators like $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$ which are sometimes awkward to handle algebraically. In particular, $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse and has no simple rule of commutation with $A(\omega t)$.

For compactness, define:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \qquad \text{(C-2)}$$

X can be read, "select the x component of the vector..." X and Y will be used below when they cause no particular trouble.

A better set when algebra may involve the $A(\omega t)$ is:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ the unit matrix, and}$$

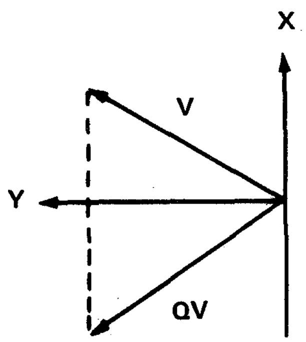
$$Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ the queer matrix.} \qquad \text{(C-3)}$$

$$I = X + Y \quad , \quad Q = -X + Y , \qquad \text{(C-4)}$$

$$X = \frac{1}{2}(I-Q) \quad , \text{ and } \quad Y = \frac{1}{2}(I+Q) . \qquad \text{(C-5)}$$

Q appears in the solutions of the equations, as is quite evident on Figure 1 (following page 12).

<u>Properties of Q</u>



(1) Q reflects a vector in the y axis.

(2) Q is its own inverse: QQ = I.

(3) Trace Q = 0 and determinant Q = -1.

(4) PQ =-QP

$$PQ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix};$$

$$QP = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix};$$

Q.E.D.

(5) Q does not commute with $A(\omega t)$, but
$QA(\omega t) = A(-\omega t)Q$.

(a) Q commutes with the unit matrix:
QI = IQ.

(b) $A(\theta) = \cos\theta I + \sin\theta P$. (Note (8) on p. A-4)

Then $QA(\theta) = (\cos\theta I - \sin\theta P)Q$
$= A(-\theta)Q$, Q.E.D.

<u>Properties of X and Y</u>

(6) X and Y do not commute usefully with the $A(\omega t)$. However, from (4) above, and (C-5), PX = YP and conversely.

(7) XY = 0; XX = X; XI = X; YY = Y, etc.

(8) Combinations of X and Y have an inverse,

$(aX + bY)^{-1} = (bX + aY)/ab$, $a \neq o$, $b \neq o$.

(9) $[aX + bY + cP]^{-1} = [bX + aY - cP]/(ab + c^2)$

$a \neq 0$
$b \neq 0$

<u>LAPLACE TRANSFORM SOLUTION</u>

The differential equation (C-1) becomes

$$\ddot{V} = 3\omega^2 XV - 2\omega P\dot{V} + L \qquad \text{(C-6)}$$

The Laplace transformation of a function V(t) is defined:

$$V(s) = \int_0^\infty e^{-st} V(t)\, dt \qquad \text{(C-7)}$$

Familiar Transforms, obtained directly or by integration by parts are:

Transform	Function
a/s	constant = a
$1/s^2$	t
$2/s^3$	t^2
$\omega/(s^2+\omega^2)$	$\sin\omega t$
$s/(s^2+\omega^2)$	$\cos\omega t$
$sV(s) - V(0)$	$\dot{V}(t)$ } V(0) is the value of V(t) at t=0, etc.
$s^2V(s) - sV(0) - \dot{V}(0)$	$\ddot{V}(t)$
$\dfrac{sI + \omega P}{(s^2 + \omega^2)}$	$A(\omega t) = I \cos\omega t + P \sin\omega t$

The transformed version of (C-6) is

$$s^2V(s) - sV(0) - \dot{V}(0) = 3\omega^2 XV(s) - 2\omega sPV(s) + 2\omega PV(0) + L/s. \qquad \text{(C-8)}$$

This equation can be solved for V(s).

$$[s^2(X+Y) - 3\omega^2 X + 2\omega sP]V(s) = (s+2\omega P)V(0) + \dot{V}(0) + L/s. \qquad \text{(C-9)}$$

Use property (9) to find the inverse of the operator on the left; left multiply, obtaining:

$$V(s) = \frac{(s^2 - 3\omega^2 Y - 2\omega sP)((s+2\omega P)V(0)+\dot{V}(0)+L/s)}{s^2(s^2 + \omega^2)} \qquad \text{(C-10)}$$

The numerator of C-10 is:

$$s^3 V(0) + s^2\dot{V}(0) + s[\omega^2(4X+Y)V(0) - 2\omega P\dot{V}(0) + L]$$
$$- [6\omega^3 YPV(0) + 3\omega^2 Y\dot{V}(0) + 2\omega PL] - 3\omega^2 YL/s. \qquad \text{(C-11)}$$

Use expansion in partial fractions to express these terms as transforms of tabulated functions. The expansions are:

$$\frac{1}{s(s^2 + \omega^2)} = \frac{1}{\omega^2 s} - \frac{s}{\omega^2(s^2 + \omega^2)} \qquad \text{(C-12a)}$$

$$\frac{1}{s^2(s^2 + \omega^2)} = \frac{1}{\omega^2 s^2} - \frac{1}{\omega^2(s^2 + \omega^2)} \qquad \text{(C-12b)}$$

$$\frac{1}{s^3(s^2 + \omega^2)} = \frac{1}{\omega^2 s^3} - \frac{1}{\omega^4 s} + \frac{s}{\omega^4(s^2 + \omega^2)} \qquad \text{(C-12c)}$$

Terms can now be collected:

Transform	Function	Coefficient	
$sI/(s^2+\omega^2)$	$\cos \omega t$	$-3XV(0)+2P\dot{V}(0)/\omega - (X+4Y)L/\omega^2$	(C-13a)
$\omega P/(s^2+\omega^2)$	$\sin \omega t\ P$	$6XV(0)-(4X+Y)P\dot{V}(0)/\omega + 2L/\omega^2$	(C-13b)
I/s	(constant)	$(4X+Y)V(0)-2P\dot{V}(0)/\omega + (X+4Y)L/\omega^2$	(C-13c)
$\omega P/s^2$	$(\omega t)P$	$-6XV(0)+3XP\dot{V}(0)/\omega - 2L/\omega^2$	(C-13d)
ω^2/s^3	$(\omega t)^2/2$	$-3YL/\omega^2$	(C-13e)

The solution for V(t) in terms of V(0), $\dot{V}(0)$, and L can be obtained by adding these terms, multiplied by the appropriate functions.

The sine and cosine terms can be written in terms of the rotation operators $A(\omega t)$ and $A(-\omega t)$, using the formula $A(\pm\omega t) = \cos\theta I \pm \sin\theta P$. If

$$V = IV \cos\omega t + PW \sin \omega t$$

or

$$= A(\omega t)D + A(-\omega t)E$$

then,

$$V=D+E \qquad D = \frac{1}{2}(V+W)$$

$$W=D-E \qquad E = \frac{1}{2}(V-W) \tag{C-14}$$

and,

$$D = \frac{1}{2}[3XV(0) + (-2X+Y)P\dot{V}(0)/\omega + (X-2Y)L/\omega^2] \tag{C-15a}$$

$$E = \frac{1}{2}[-9XV(0) + (6X+3Y)P\dot{V}(0)/\omega + (-3X-6Y)L/\omega^2] \tag{C-15b}$$

D and E are related. Inspection shows D is 1/3 the magnitude of E; and the sign of X is changed. That is, D=(-X+Y)E/3 = QE/3. The trigonometric part of the solution is then:

$$V_E = A(\omega t)QE/3 + A(-\omega t)E \qquad \text{(C-16)}$$

or,

$$V_E = (I+Q/3)\ A(-\omega t)E. \qquad \text{(C-17)}$$

This is the "E-type" elementary solution of the text, with E defined by (C-15b) or somewhat more neatly,

$$E = \frac{3}{2}\ [-3XV(0) + (2X+Y)P\dot{V}(0)/\omega - (X+2Y)L/\omega^2]. \qquad \text{(C-18)}$$

The C-type elementary solution is obtained by setting the constant term, (C-13c), equal C. Part of the term (C-13d) linear in time is related to C.

Thus,

$$C = (4X+Y)V(0) - 2P\dot{V}(0)/\omega + (X+4Y)L/\omega^2 \qquad \text{(C-19)}$$

The operator $-\frac{3}{2}X$ will convert the V and $\dot{V}$ coefficients into those of C-13d. With the acceleration L=0, the complete solution is obtained by adding to (C-16) the C element,

$$V_C = C - \frac{3}{2}\ \omega t\ PXC \qquad \text{(C-20)}$$

$$= (I - \frac{3}{2}\ \omega t\ PX)C. \qquad \text{(C-21)}$$

With constant force, the remaining element is:

$$V_L = -\ tP\ \frac{(X+4Y)L}{2\omega} - \frac{3}{2}\ t^2\ YL \qquad \text{(C-22)}$$

In the text, only $L = \begin{pmatrix} 0 \\ d \end{pmatrix} = YL$, a drag acceleration is used. The constant lift case which permits a vehicle to move along track at a constant velocity $\ell/2\omega$ is also interesting.

The local vertical general solution is the sum of (C-17, C-20, and C-22):

$$V(t) = (I+Q/3)\ A(-\omega t)E + (I-\tfrac{3}{2}\omega t\ PX)C$$

$$- tP\ \frac{(X+4Y)L}{\omega} - \frac{3}{2}\ t^2\ YL, \qquad \text{(C-23)}$$

from which equation (16) of the text may be easily written down.

<u>Inertial Form</u>

The values for E and C in terms of Z(0) and $\dot{Z}(0)$ are obtained, starting with Equations C-18 and C-19 and using transformations (A-23) and (A-24) at time zero.

$$V(0) = Z(0) \qquad \text{(C-24)}$$

$$\dot{V}(0) = \dot{Z}(0) - \omega PZ(0) \qquad \text{(C-25)}$$

(C-18) becomes:

$$E = \frac{3}{2}\left(-3XZ(0) + (2X+Y)P\dot{Z}(0)/\omega - (2X+Y)PPZ(0) - \frac{1}{\omega^2}(X+2Y)L\right) \qquad \text{(C-26)}$$

$$\boxed{E = \frac{3}{2}\left((-X+Y)Z(0) + (2X+Y)P\dot{Z}(0)/\omega - \frac{1}{\omega^2}(X+2Y)L\right) \qquad \text{(C-27)}}$$

Similarly, (C-19) becomes

$$C = \left[(4X+Y)Z(0) - 2P\dot{Z}(0)/\omega + 2PPZ(0) + \frac{1}{\omega^2}(X+4Y)L\right] \qquad \text{(C-28)}$$

$$\boxed{C = (2X-Y)Z(0) - 2P\dot{Z}(0)/\omega + \frac{1}{\omega^2}(X+4Y)L \qquad \text{(C-29)}}$$

The forms of the elementary trajectories are obtained using the transformation $Z(t) = A(\omega t)V(t)$. Thus, (C-16) becomes:

$$Z_E = E + A(2\omega t)QE/3. \tag{C-30}$$

(C-21) becomes, trivially,

$$Z_C = A(\omega t)\left[I - \frac{3}{2}\omega t\ PX\right]C. \tag{C-31}$$

(C-22) becomes, using the case L=YL of the text,

$$Z_L = A(\omega t)\left[- \frac{2tPYL}{\omega} - \frac{3}{2}t^2 YL\right]. \tag{C-32}$$

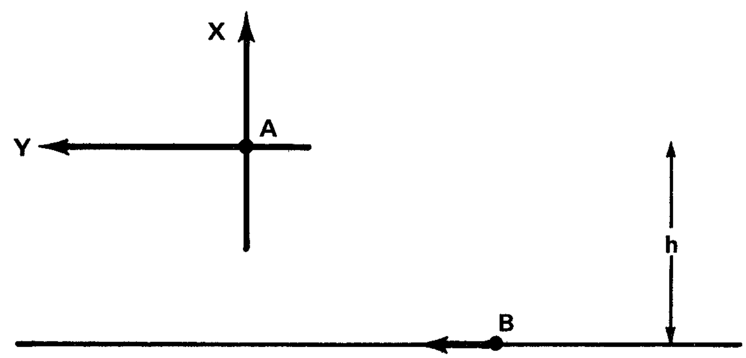
The equations of Table I are obtained by substituting for P, Q, X and Y.

APPENDIX D

Rendezvous Application

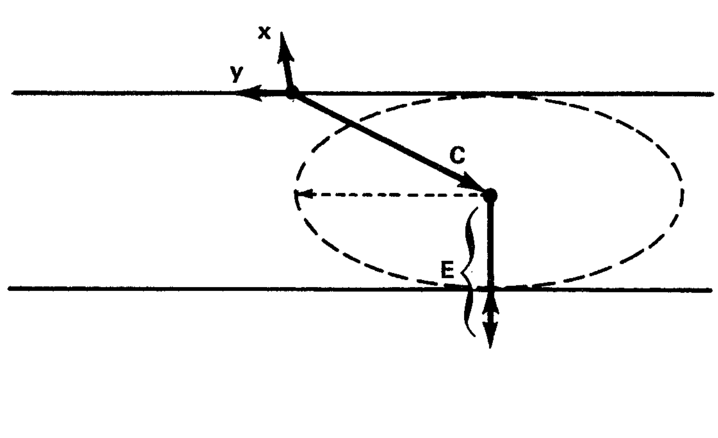
This appendix is a simple exercise to show the use of the elemental trajectories in rendezvous calculations, in particular, a Hohmann transfer from one circular orbit to another.

Local vertical coordinates are used. The target spacecraft is the origin, at A. The active spacecraft, B, lies initially on an orbit h meters below and y(0) meters aft. This is a C-trajectory, and the initial velocity (Table I) is $3\omega h/2$.



The final position will be at A, the final velocity, zero.

The transfer orbit, astronomically the ellipse tangent to both circular orbits, is a mixed C and E trajectory--a drifting ellipse. Because the burns are in track, the ellipse is tangent to the initial trajectory and to the y axis.



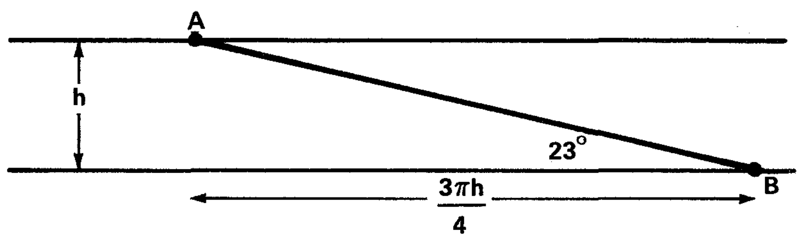
Initial condition vectors C and E are as shown. $C = (-h/2, y(0))$. E lies along the inward radius; from the figure on page 14, $E = (-\frac{3h}{4}, 0)$ for an ellipse of minor axis, h.

In the transfer orbit, the C drift velocity is $\frac{3\omega h}{4}$. The transfer is accomplished in $\omega t = \pi$, so $y(0) - 3\pi h/4$ and C for the transfer trajectory is

$$C = (-h/2, -3\pi h/4).$$

(The period of the transfer orbit is, precisely, somewhat shorter than that of the reference orbit. The error introduced using $\omega t = \pi$ is of order $|V|/|D|$, that of the expansions in general.)

The results then follow:



(1) The elevation angle at time of the first burn is

$$\tan^{-1}\left(\frac{4}{3\pi}\right) = 23°.$$

(2) The velocity prior to burn is $3\omega h/2$. After burn, it is

$$V_E + V_C = \omega 4|E|/3 + V_C = \omega h + 3\omega h/4 = 7\omega h/4.$$

$\therefore$ the burn magnitude is $\omega h/4$.

(3) At rendezvous, $\omega t = \pi$, velocity in transfer ellipse is

$$V_E + V_C = -\omega h + 3h/4 = -\omega h/4$$

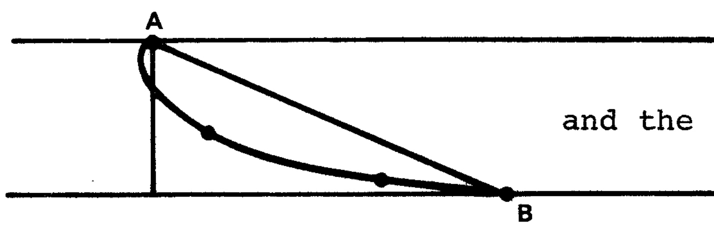
Since velocity is zero after the final burn

$\therefore$ the burn magnitude is also $\omega h/4$

(Raising an orbit h feet takes nearly the same ΔV at both Earth and Moon.)

(4) The local vertical equation can be used to obtain

$$V(t) = \begin{pmatrix} -h/2 \\ -3\pi h/4 \end{pmatrix} + \frac{3}{4}\,\omega t \begin{pmatrix} 0 \\ h \end{pmatrix} - \frac{1}{2}\begin{pmatrix} h \\ 0 \end{pmatrix} \cos\omega t + \begin{pmatrix} 0 \\ h \end{pmatrix} \sin\omega t$$

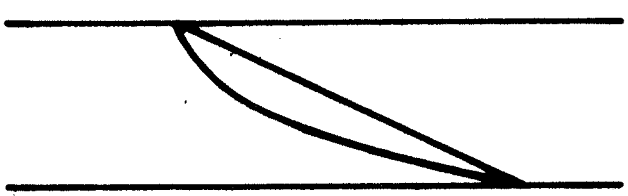


and the familiar result sketched.

Substituting $\omega t = \pi - \theta$ and expanding

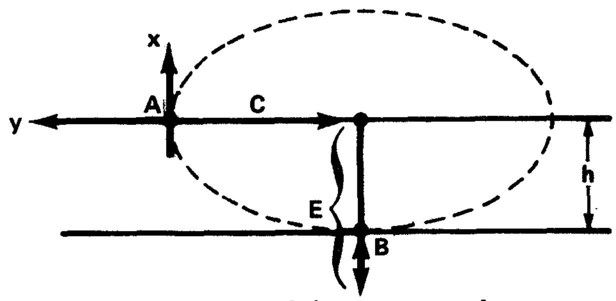
$$V(\theta) = \frac{h}{4} \begin{pmatrix} -\theta^2 \\ \theta \end{pmatrix},$$ a parabolic behavior for the last few tenths of a radian prior to rendezvous.

(5) As a conceptual alternate, the vehicle could apply constant track thrust, and follow the drag trajectory and approach directly from below. It can be shown



that the ΔV applied in track is the total Hohmann ΔV, $\omega h/2$. The radial velocity must be put in and taken out again. It is several times larger but decreases as the rendezvous is slowed down.

(6) As another alternate, the co-periodic ellipse of minor axis 2h can be used, and allows approach directly from below.



$\Delta V_1 = 2\omega h - 3\omega h/2 = \omega h/2$ (in track)

$\Delta V_2 = \omega h$ (radial only; track component zero).

In this case the penalty above the Hohmann is 2X additional.

(7) There clearly is a converse case to (6) where an approach in track is made. This is enough to demonstrate that the elemental trajectories are easy to use.

APPENDIX E

Estimates of Spacecraft Motions

This appendix briefly covers the effect of spacecraft motions on the idealized picture of particle motion in a spacecraft. The authors are indebted to several members of the Skylab Systems Analysis Department at Bellcomm for inputs, particularly, P. G. Smith and W. W. Hough.

W. W. Hough has written a good treatment of the perturbations due to predictable vehicle dynamics, and gives(6) estimates of the linear accelerations of the spacecraft structure in the crew quarters. For instance, a Skylab changing from inertial to local vertical attitude has an angular velocity comparable with the orbital rate, ω; in the crew quarters, at a radius of approximately ten meters from the center of gravity, this gives additional accelerations of order 10^{-6}g, the same as gravity gradient. Attitude control thrusters cause brief acceleration pulses, estimated as 2×10^{-3}g. The Control Moment Gyro system operates over longer times, and the effective acceleration levels in the crew quarters are smaller, somewhat under 10^{-4}g, maximum.

Clearly body-mounted objects will perceive accelerations of this magnitude, more or less modified in accord with the normal modes of vibration of the spacecraft and the particular resonances of the attaching structure. If the observer is fixed to the spacecraft, a free particle will show corresponding, apparent accelerations.

The significance of these accelerations to experimental design depends on disturbance size. Major maneuvers such as orbit change or attitude change must be lived with. No "zero-G" experiment can be performed during launch phase; the more sensitive experiments must be restricted to times when the vehicle is held stable.

With major maneuvers excluded, there is a motional environment, comprising a vibration spectrum and certain low frequency motions due to the attitude control cycle, astronaut motion, starting up of equipment, and so forth. If sensitive, experiments must be isolated from this environment. Because the motions are largely cyclic, and because total displacements are small, the requirements on vibration isolators do not appear severe.

The displacements may be estimated, and amount at most to several centimeters. Vibrational amplitudes should typically be well under a centimeter, even for low frequency modes. Gross spacecraft motions may be rotational, or translational. If rotational, the displacement at a work position is $R\theta$, R being the distance from the C.G. and θ the angular motion. Conventional thruster stabilization with, for instance, a 5-degree deadband, is a major maneuver. The displacement at a 10-meter radius is 1 meter. Half degree deadband is also available on Apollo, and is more practical, with 10 cm displacements at 10 meters and much smaller near the C.G. "Zero-G" experiments on thruster controlled spacecraft are probably best performed without attitude control, at rotational rates such that centrifugal accelerations are small. (This option is not necessarily available on multidisciplinary flights)

Control Moment Gyros as used on Skylab are specified to maintain orientation within 4 to 10 arc minutes depending on axis. Nominal performance should be measured in arc seconds. P. G. Smith calculated for us the angular displacement resulting as an astronaut jumped from one side of the Skylab to the other. This is a moderate leap, taking 9.5 seconds to cross the 22-foot ($\sim$7m) diameter. The maximum deviation of the Skylab was estimated to be 80 arc seconds. This would correspond at R = 10m to a 4mm displacement.

Translation disturbances are easy to estimate. The system C.G. is invariant. When the astronaut of mass m moves S meters, the station of mass M must move $-(m/M)S$ meters to compensate. If the astronaut weighs 100Kg and the station 10^5Kg, m/M is .001. Therefore, when the astronaut moves 22 feet (7m) the station must react, moving 7mm. Since the length of Skylab-CSM habitable area approaches 20m, maximum displacements of several centimeters are possible. It should, of course, be possible to reduce crew activities during critical experiments.

Tentative conclusions as regards observations of particle motion and zero-G experiment design are as follows (these are drawn only from the effect of spacecraft and crew disturbances):

(1) Major maneuvers must be treated separately. The subsatellite will appear to change its orbit; the experiment will not be scheduled, and the apparatus will be stowed or caged as appropriate.

(2) Thruster control will be inappropriate for a class of zero-G experiments. There will be exceptions; it is plausible that an environment involving infrequent pulses of 10^{-3}g is acceptable, at least for exploratory studies.

(3) C.M.G. attitude control is preferred. Also, the acceleration environment in a slow roll may be acceptable.

(4) Vibration isolation is required for zero-G experiments. The suspension system should be soft, that is, apply restoring accelerations weakly dependent on displacement. Given C.M.G.'s, a working distance of several centimeters should adequately isolate the experimental apparatus from most spacecraft disturbances.

(5) Observations of particle motion will contain noise due to spacecraft disturbances. With C.M.G.'s, displacements from the nominal path should rarely exceed a centimeter.