TM-71-1033-1

TECHNICAL MEMORANDUM

TRACKING ANALYSIS OF A FIRST ORDER PHASE-LOCKED LOOP WITH TWO SINEWAVES MODULATION

Bellcomm

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W., WASHINGTON, D.C. 20024

COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Tracking Analysis of a First Order Phase-Locked Loop with Two Sinewaves Modulation

TM- 71-1033-1

DATE- January 21, 1971

FILING CASE NO(S)- 103-6

AUTHOR(S)- S. Y. Lee
L. D. Nelson

FILING SUBJECT(S)- (ASSIGNED BY AUTHOR(S))- Communication Systems
Phase Locked-Loop Tracking

ABSTRACT

This paper gives a tracking analysis of a first order phase-locked loop with two sinewaves modulation. By assuming the fluctuation about the d-c value of the phase error (not necessarily the total error) to be small, a linearized model is obtained. A periodic solution of this model is then derived for the output phase of the first order loop. It is shown that this periodic solution is unique. Using this periodic solution, analytical tracking behaviors of the output phase of this model are obtained. Computer plots of the phase error for some typical parameters used in Apollo communication systems are included.

BA-145A (8-68)

SEE REVERSE SIDE FOR DISTRIBUTION LIST

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

SUBJECT: Tracking Analysis of a First Order Phase-Locked Loop with Two Sinewaves Modulation - Case 103-6

DATE: January 21, 1971

FROM: S. Y. Lee
L. D. Nelson

TM-71-1033-1

TECHNICAL MEMORANDUM

I. INTRODUCTION

Basically, a phase-locked loop is an electronic servomechanism that operates as a coherent detector by continuously correcting the frequency of its local oscillator according to a measurement of the error between the phase of the incoming signal and that of its local oscillator. The simplest form of loop is shown in Figure 1. The precise relationship between the input and error signals is a nonlinear integro-differential equation from which very little information concerning the tracking behavior is analytically available. With the assumption of the total loop phase error being small, Viterbi[2] and others obtained a mathematically equivalent linearized model for the analysis of a single sinewave modulation. The purpose of this memorandum is to develop a method of analysis for two sinewaves modulation, and to derive analytically its tracking behavior using a linearized model.

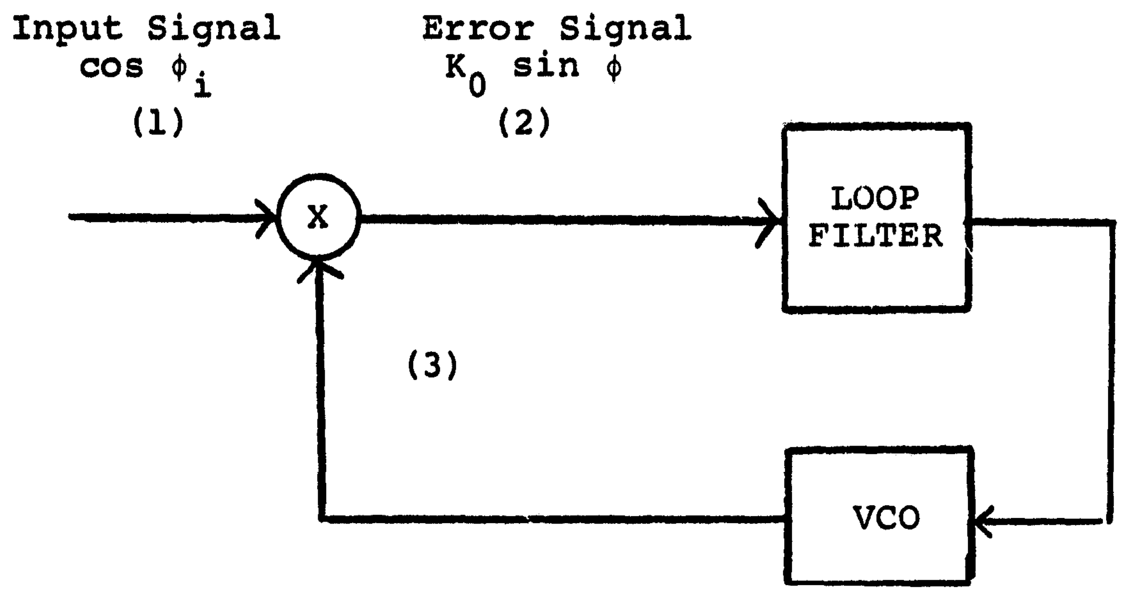


Figure 1 Basic Configuration of a General Phase-Locked Loop

II. BACKGROUND

General Equation for the Phase Error of the Phase-Locked Loop

The basic operation of the phase-locked loop given in Figure 1 is described in References [1] and [2]. Furthermore, in these same references, a general nonlinear differential equation which gives the precise relationship between the input phase signal ϕ_i at point (1) and the phase error ϕ at point (2) of the phase-locked loop is derived. In an operational notation this equation is

$$s\phi + KF(s)\sin\phi = s\phi_i - s\omega_0 t \tag{1}$$

or equivalently

$$\dot{\phi} + KF(s)\sin\phi = \dot{\phi}_i - \omega_0 \tag{2}$$

where $s = \frac{d}{dt}$, $K = K_0K_1K_2$ is defined as loop gain, ϕ denotes the phase $\phi(t)$ is the multiplier output (i.e., the error signal), $K_1F(s)$ is the linear transfer function of the loop filter, ω_0 is the center frequency of the voltage controlled oscillator (VCO), K_0 is the phase-detector gain factor, and K_2 is the gain of the VCO. By using the result of (1) we shall proceed to derive the effects of two sinewaves modulation on the tracking behaviors of the phase-locked loop.

III. THE PHASE-LOCKED LOOP WITH TWO SINEWAVES MODULATION

The differential equation for the phase-locked loop preceded by a limiter for the case of two sinewaves modulation can be obtained as follows: Let the input signal at point 1 be represented by $\cos\omega_c t + a\cos(\omega_c + \omega_d)t$, where $a < 1$, and ω_c and ω_d are the singular frequencies. Then the resultant input signal can be derived from the vectorial addition of the unmodulated carriers as shown in Figure 2,

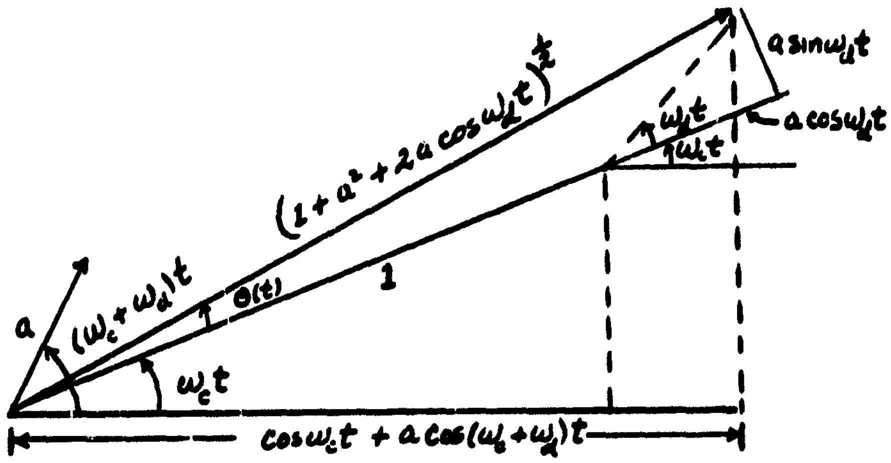


Figure 2 - Vector Addition of the Unmodulated Carriers

$$\cos\omega_c t + a\cos(\omega_c+\omega_d)t = \sqrt{1+a^2+2a\cos\omega_d t}\ \cos[\omega_c t+\tan^{-1}\frac{a\sin\omega_d t}{1+a\cos\omega_d t}] \qquad (3)$$

The unit gain limiter removes the amplitude variations. Therefore, the input signal becomes

$$\cos\phi_i = \cos[\omega_c t + \tan^{-1}\frac{a\sin\omega_d t}{1+a\cos\omega_d t}] \qquad (4)$$

Hence by substituting for ϕ_i from (4) and defining

$$\theta(t) = \theta = \tan^{-1} \frac{a\sin \omega_d t}{1+a\cos\omega_d t} \tag{5}$$

the general equation for the phase error of the phase-locked loop with two sinewaves modulation becomes

$$\dot{\phi} + KF(s)\sin\phi = \omega_c - \omega_0 + \dot{\theta} \tag{6}$$

Instead of solving (6) as is, we first obtain $\dot{\theta}$ as an infinite series. Expanding (5),

$$\theta(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n}{n} \sin n\, \omega_d t \tag{7}$$

and differentiating term by term, the series obtained is uniformly convergent and, thus,

$$\dot{\theta}(t) = \frac{\omega_d a(a+\cos\omega_d t)}{1+a^2+2a\cos\omega_d t} = \omega_d \sum_{n=1}^{\infty} (-1)^{n+1} a^n \cos n\omega_d t \tag{8}$$

Substituting (8) into (6) yields

$$\dot{\phi} + KF(s)\sin\phi = \omega_c - \omega_0 + \omega_d \sum_{n=1}^{\infty} (-1)^{n+1} a^n \cos n\omega_d t \qquad (9)$$

Assume the solution of (9), ϕ, has the form of a constant term plus a fluctuating term such that

$$\phi = \phi_0 + \psi(t) \ . \qquad (10)$$

Then (9) and (10) yield

$$\dot{\psi} + KF(s)[\sin\phi_0\cos\psi + \cos\phi_0\sin\psi] =$$

$$\omega_d \sum_{n=1}^{\infty} (-1)^{n+1} a^n \cos n\omega_d t + \omega_c - \omega_0 \qquad (11)$$

Assuming the fluctuations about the dc value of the phase error are small (this is generally true; see Section V) we have

$$\dot{\psi}_1 + KF(s)[\sin\phi_0 + \psi_1\cos\phi_0] = \omega_d \sum_{n=1}^{\infty} (-1)^{n+1} a^n \cos n\omega_d t + \omega_c - \omega_0 \qquad (12)$$

For the case of the first-order loop F(s) = 1 and equation (12) reduces to

$$\dot{\psi}_1 + K[\sin\phi_0 + \psi_1\cos\phi_0] = \omega_c - \omega_0 + \omega_d \sum_{n=1}^{\infty} (-1)^{n+1} a^n \cos n\omega_d t \qquad (13)$$

By noting that (13) is a linear differential equation in $\psi_1(t)$ and $\dot{\theta}(t)$ is of period $\frac{2\pi}{\omega_d}$, the steady state solution of (13) must also have period $\frac{2\pi}{\omega_d}$. Thus, assume a steady state series solution for $\psi_1(t)$ to be of the form of

$$\psi_1(t) = \psi_1 = \sum_{n=1}^{\infty} [A_n \cos n\omega_d t + B_n \sin n\omega_d t] \quad . \qquad (14)$$

Then

$$\dot{\psi}_1(t) = \dot{\psi}_1 = \omega_d \sum_{n=1}^{\infty} n[B_n \cos n\omega_d t - A_n \sin n\omega_d t] \ . \qquad (15)$$

Substituting (14) and (15) into (13) yields

$$\omega_d \sum_{n=1}^{\infty} n[B_n \cos n\omega_d t - A_n \sin n\omega_d t] + K\sin\phi_0 + \omega_0 - \omega_c$$
$$+ K\cos\phi_0 \sum_{n=1}^{\infty} [A_n \cos n\omega_d t + B_n \sin n\omega_d t] = \omega_d \sum_{n=1}^{\infty} (-1)^{n+1} a^n \cos n\omega_d t \qquad (16)$$

By taking the integral of (16) from 0 to $\frac{2\pi}{\omega_d}$ it is seen that

$$K\sin\phi_0 + \omega_0 - \omega_c = 0 . \tag{17}$$

Equating coefficients of cos $n\omega_d t$ and sin $n\omega_d t$, for each n there are two equations in two unknowns,

$$-n\omega_d A_n + K\cos\phi_0 B_n = 0$$

$$K\cos\phi_0 A_n + n\omega_d B_n = (-1)^{n+1} a^n \omega_d$$

Solving for A_n and B_n,

$$A_n = \frac{(-1)^{n+1} a^n b}{b^2 + n^2} \qquad B_n = \frac{(-1)^{n+1} a^n n}{b^2 + n^2} \tag{18}$$

where

$$b = \frac{K\cos\phi_0}{\omega_d} \tag{19}$$

Thus substituting into (14) and the analog of (10) yields the complete solution for the linearized model of the phase error of the first order phase-locked loop with the effect of two sinewaves modulation,

$$\phi_1(t) = \phi_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n (b\cos n\omega_d t + n\sin n\omega_d t)}{b^2 + n^2} \tag{20}$$

where b is given in (19) and from (17),

$$\phi_0 = \sin^{-1} \frac{\omega_c - \omega_0}{K} \tag{21}$$

Let $\alpha = \tan^{-1} (\frac{-n}{b}) = \tan^{-1} \frac{-n\omega_d}{K\cos\phi_0}$ and (20) can be written as

$$\phi_1(t) = \phi_0 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n \cos(n\omega_d t + \alpha)}{[b^2 + n^2]^{1/2}} \tag{22}$$

or

$$\phi_1(t) = \phi_0 + \omega_d \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n \cos\left(n\omega_d t + \tan^{-1} \frac{-n\omega_d}{K\cos\phi_0}\right)}{[(K\cos\phi_0)^2 + (n\omega_d)^2]^{1/2}} \tag{23}$$

The fundamental solution $\Phi(t,t_0)$ of the homogeneous equation

$$\dot{\psi}_1 = -\psi_1 K\cos\phi_0$$

is

$$\Phi(t,t_0) = e^{-(t-t_0)K\cos\phi_0}$$

The ϕ_0 is "small" (less than $\frac{\pi}{2}$ in absolute value) in practice; hence, $\cos\phi_0 \neq 0$. The period of $\dot{\theta}$ being $\frac{2\pi}{\omega_d}$ then forces

$$\Phi(\frac{2\pi}{\omega_d}, 0) - 1 \neq 0 .$$

Equation (12) thus satisfies the conditions of a theorem in [7] and on page 363 in [6] which implies that the solution (20) is unique.

The preceding series (20) (or (22)) is uniformly convergent for $0 \leq a < 1$ and, hence, is the Fourier series of the function $\phi_1(t)$. Thus, the output phase error for the first order loop with two sinewaves modulation is a periodic fluctuation of zero average value about the phase error that would be present if there were only one sinewave modulation. It should be noted that if the fluctuation due to two sinewaves modulation were not assumed small, then the average effect would not be expected to be zero since the nonlinear sine function would weigh positive and negative fluctuation unevenly.

IV. LIMITING BEHAVIOR OF THE OUTPUT PHASE EQUATION

By examining (23), it can be seen that there are three interesting special cases. These are:

Case A - When the input signal is one sinewave then the phase error is ϕ_0, which is derived in prior literature.

Case B - As expected, when K is infinite which corresponds to the ideal phase-locked loop (i.e., tracking the $\phi_i = \omega_c t + \theta(t)$ exactly) then the phase errors ϕ_1, ϕ_0, and $\psi_1(t)$ are zero.

Case C - When ω_d becomes large relative to $K\cos\phi_0$ such that

$$\frac{1}{(n\omega_d)^2 + K^2\cos^2\phi_0} \simeq \frac{1}{(n\omega_d)^2} \qquad (24)$$

then the solution of (23) yields

$$\phi_1 \cong \phi_0 + \theta - K\cos\phi_0 \int \theta dt \quad , \; (\omega_d{}^2 >> K^2\cos^2\phi_0) \qquad (25)$$

by noting that

$$\theta = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n}{n} \sin n\omega_d t$$

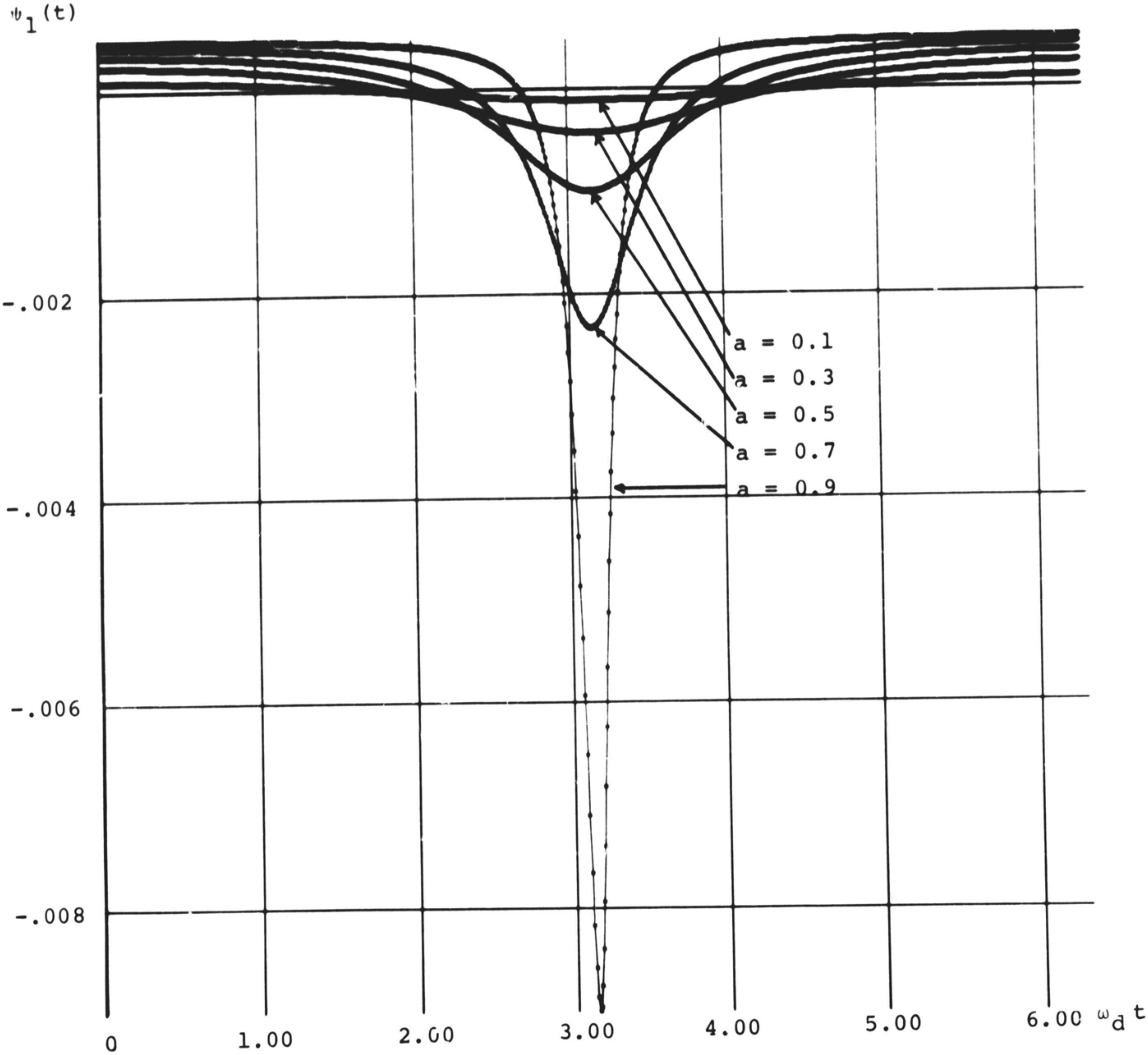
and

$$\int \theta dt = - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n}{n^2\omega_d} \cos n\omega_d t$$

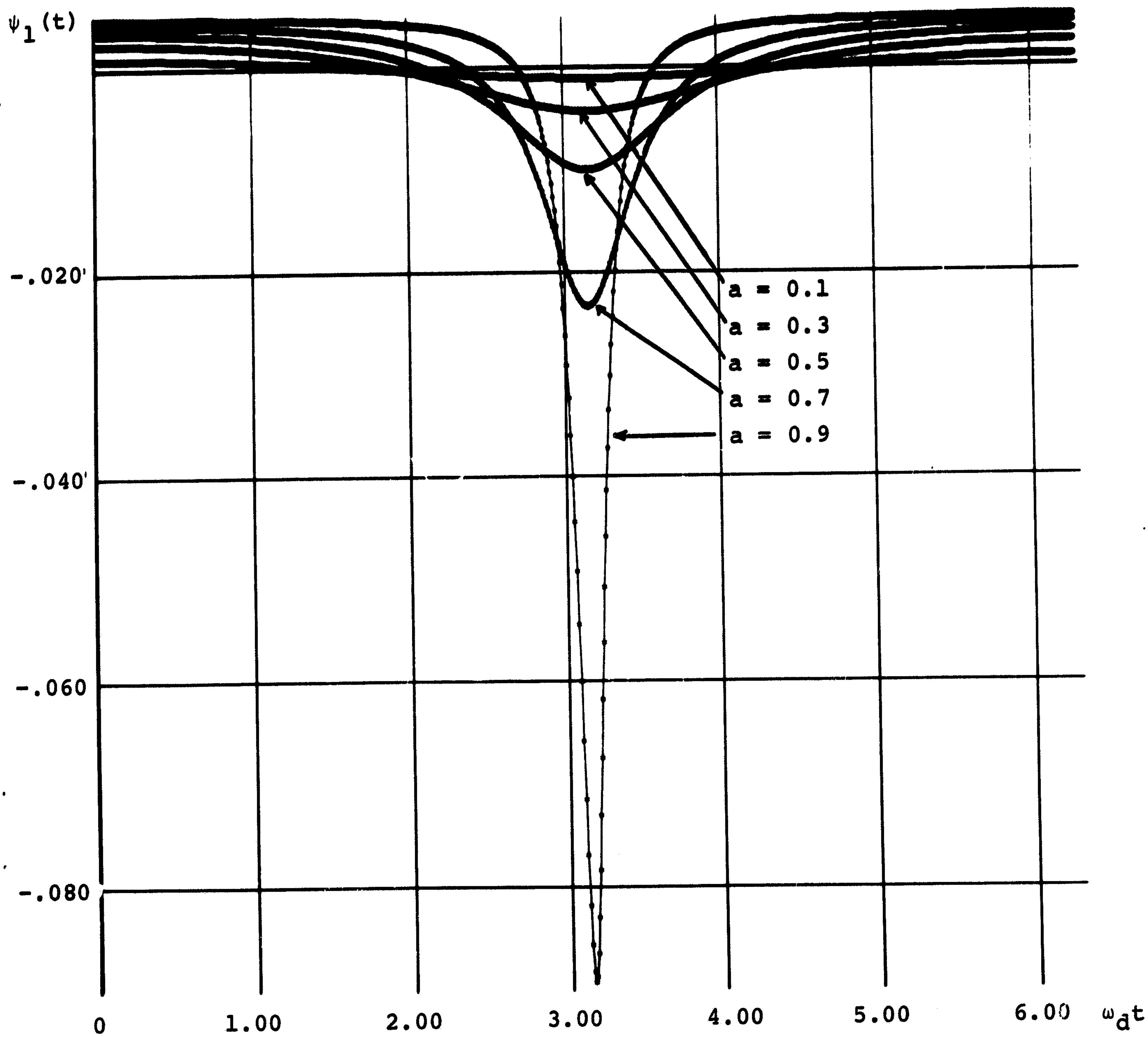
V. COMPUTER PLOTS OF THE PHASE ERROR FOR SOME TYPICAL PARAMETERS USED IN APOLLO COMMUNICATION SYSTEM

The tracking behaviors of the first order loop can be seen by considering (23) when the system parameters are known. For Apollo communication systems, we know K is approximately equal to 10^6 and ω_d is approximately equal to 10^3. Therefore from (19), $b \approx 10^3$. Substituting this value into (23) and noting that the contribution of ϕ_0 only corresponds to a d-c shift, we can determine the behavior of the phase error $\phi_1(t)$ by plotting $\psi_1(t)$ versus $\omega_d t$ using "a" as a parameter. Examples of these plots and others for different values of b follow. As a "goodness" criterion for this linear model we must examine $|\sin\psi_1 - \psi_1| < \varepsilon$ for all values of t, where ε is an arbitrarily assigned small number. For the examples, suppose we let $\varepsilon = .01$, then only two cases do not satisfy the goodness criterion. These cases are where (b=5, a=.9) and (b=10, a=.9); $|\sin\psi_1 - \psi_1|$ attains the values .16 and .05 respectively. It should be noted that for the linear model derived in this memorandum the assumption of $\sin\psi \cong \psi$ is less restrictive than if the assumption is $\sin\phi \cong \phi$.

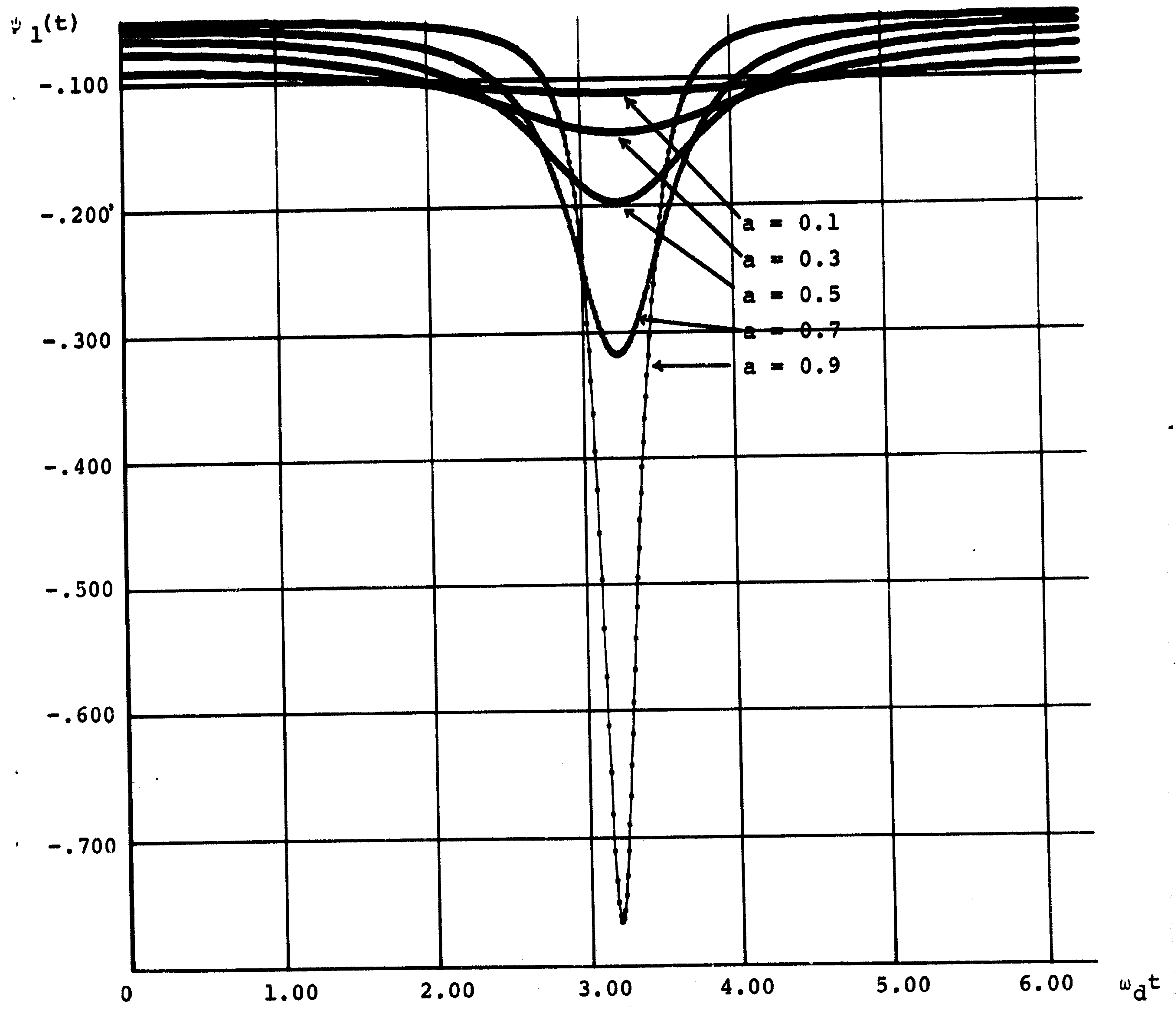
(1) b = 1000


$\psi_1(t)$
-.002
-.004
-.006
-.008
a = 0.1
a = 0.3
a = 0.5
a = 0.7
a = 0.9
0
1.00
2.00
3.00
4.00
5.00
6.00
$\omega_d t$

(2) b = 100

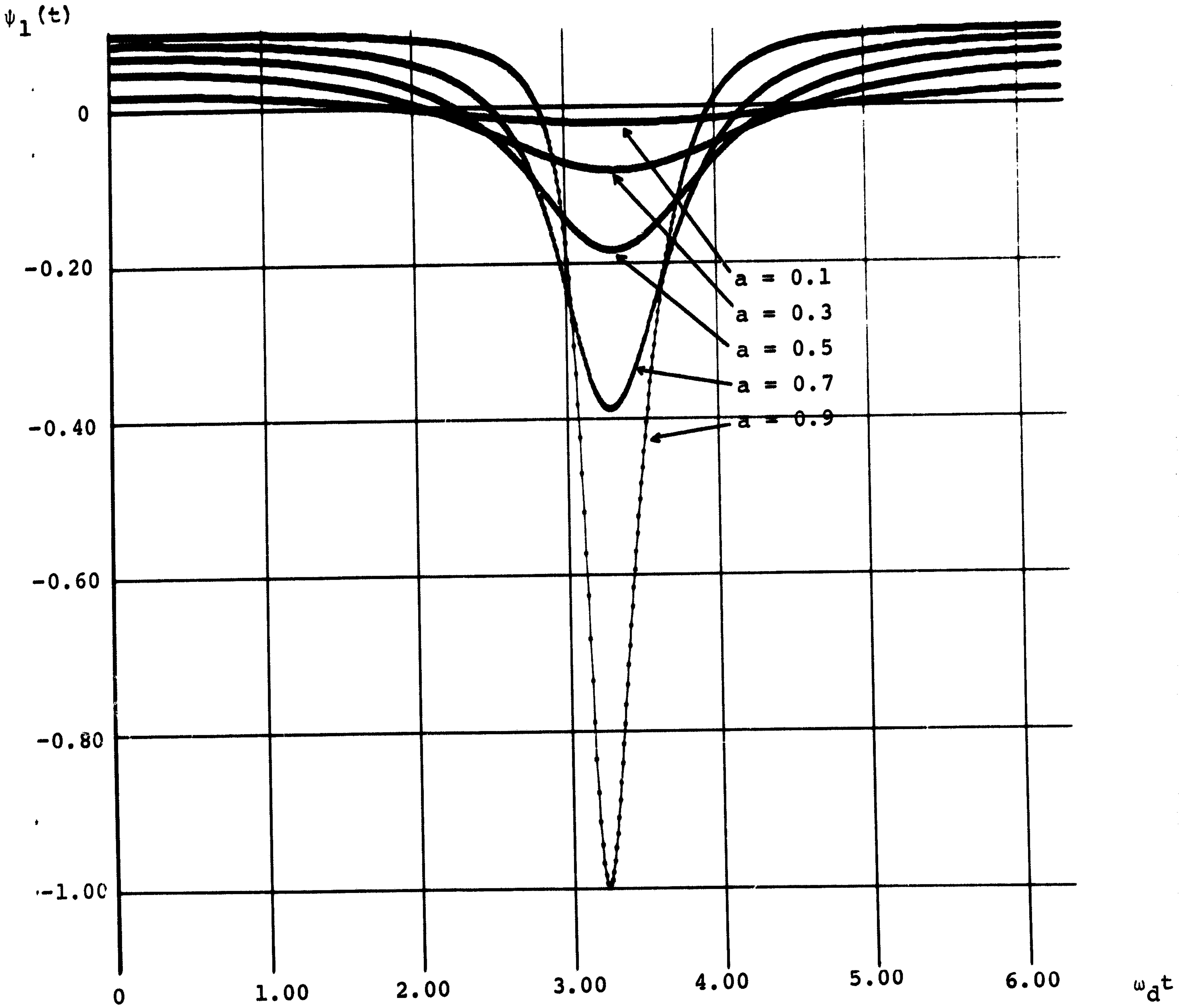

$\psi_1(t)$
-.020
-.040
-.060
-.080
a = 0.1
a = 0.3
a = 0.5
a = 0.7
a = 0.9
0
1.00
2.00
3.00
4.00
5.00
6.00
$\omega_d t$

(3) b = 10



$\psi_1(t)$
-.100
-.200
-.300
-.400
-.500
-.600
-.700
a = 0.1
a = 0.3
a = 0.5
a = 0.7
a = 0.9
0
1.00
2.00
3.00
4.00
5.00
6.00
$\omega_d t$

(4) b = 5


$\psi_1(t)$
0
-0.20
-0.40
-0.60
-0.80
-1.00
a = 0.1
a = 0.3
a = 0.5
a = 0.7
a = 0.9
0
1.00
2.00
3.00
4.00
5.00
6.00
$\omega_d t$

ACKNOWLEDGMENT

The authors would like to thank Mr. S. C. Chu for his helpful suggestions. The able programming assistance of Mr. R. R. Singers is gratefully acknowledged. Thanks are due to Department 2034 for their comments of this paper.

S. Y. Lee

S. Y. Lee

Larry D. Nelson

1033-$\frac{\text{SYL}}{\text{LDN}}$-jf L. D. Nelson

Attachment
References

REFERENCES

1. Gardner, F. M., Phase-Lock Techniques, John Wiley & Sons, Inc., New York, 1966.

2. Viterbi, A. J., "Acquisition and Tracking Behavior of Phase-Locked Loops", Symposium on Active Networks and Feedback Systems, Polytechnic Institute of Brooklyn, April 19-21, 1960, pp. 583-618.

3. Schuchman, L., "Time to Cycle Slip in First and Second Order Phase-Lock Loops", Bellcomm Technical Memorandum, TM-69-2034-8, December 19, 1969.

4. Van Trees, H. L., "Functional Techniques for Analysis of the Nonlinear Behavior of Phase-Locked Loops", presented at WESCON, San Francisco, California, August 20-23, 1963.

5. Tausworthe, R. C., "Theory and Practical Design of Phase-Locked Receivers", NASA Technical Report No. 32-819, Vol. 1, February 15, 1966.

6. Hahn, W., Stability of Motion, Springer-Verlag Book Company, New York, 1967.

7. Fearnsides, J. J. and Chu, S. C., "A Minimum-Fuel Mode for Steady-State Attitude Control of an Earth-Orbiting Spacecraft", IEEE Trans. on Auto. Control, Vol. AC-15, June 1970, pp. 362-365.

8. Panter, P. F., Modulation, Noise and Spectral Analysis, McGraw-Hill Book Company, New York, 1965.