Bellcomm, Inc.

BELLCOMM, INC. TR-68-340-7

VARIANCE OF LUNAR SLOPES

November 27, 1968

A. H. Marcus

Work performed for Office of Manned Space Flight, National Aeronautics and Space Administration, under Contract NASW-417.

TABLE OF CONTENTS

ABSTRACT

The root-mean-square (r.m.s.) slope over a finite span on a cratered planetary surface is derived theoretically. The r.m.s. slopes from Ranger VII observations of Mare Cognitum are in very good agreement with predictions.

VARIANCE OF LUNAR SLOPES

1.0 THEORY

A geometrical parameter often useful in characterizing surface roughness is the slope over a finite span of length r, denoted $S(r)$. Let $Z(\underset{\sim}{R})$ be the elevation at a point $\underset{\sim}{R}$, and $Z(\underset{\sim}{R} + \underset{\sim}{r})$ the elevation at another point $\underset{\sim}{R} + \underset{\sim}{r}$, whose distance from $\underset{\sim}{R}$ is

$$r = (\text{length of } \underset{\sim}{r})$$

We assume that the elevations are statistically homogeneous, i.e., the distribution of $Z(\underset{\sim}{R})$ is the same for each point $\underset{\sim}{R}$. The surface is also statistically isotropic, thus, the distribution of slopes over a finite span of length r is the same whatever the direction in which the slope is measured. We may, thus, define $S(r)$ by

$$S(r) = [Z(\underset{\sim}{R} + \underset{\sim}{r}) - Z(\underset{\sim}{R})] \;/\; r \qquad (1)$$

with the understanding that $S(r)$ has the same distribution for each point $\underset{\sim}{R}$ and direction $\underset{\sim}{r}/r$.

Let $E\{Q\}$ denote the expectation (average value) of any random variable Q for which this quantity is finite (this can be justified in applications). Since $Z(\underset{\sim}{R}+\underset{\sim}{r})$ and $Z(\underset{\sim}{R})$ are assumed to have the same distribution,

$$E\{S(r)\} = 0 \qquad (2)$$

The mean-square slope $E\{S^2(r)\}$ is, by virtue of Eq. (1), equal to the <u>slope variance</u> $\sigma^2(r)$

$$\sigma^2(r) = E\{S^2(r)\}-E^2\{S(r)\} = E\{S^2(r)\} \quad (3)$$

It is obvious that S is distributed symmetrically about S = 0, and that if the surface is not too rough S will have a roughly bell-shaped probability density function. Thus, the dispersion of S is well described by the mean-square slope, whatever the actual distribution of S. The variance of Z (mean square deviation of surface elevations) is denoted by

$$\mathrm{Var}\{Z\} = c(0) \quad (4)$$

The covariance c(r) between $Z(\underset{\sim}{R})$ and $Z(\underset{\sim}{R} + \underset{\sim}{r})$ is defined by

$$c(r) = E\left\{Z(\underset{\sim}{R} + \underset{\sim}{r})Z(\underset{\sim}{R})\right\} - E^2\left\{Z(\underset{\sim}{R})\right\} \quad (5)$$

and correlation coefficient $\rho(r)$ by

$$\rho(r) = c(r)/c(0) \quad (6)$$

Thus, it may be shown that

$$E\,\{S^2(r)\} = 2c(0)\left[\frac{1-\rho(r)}{r^2}\right] \quad (7)$$

The root-mean-square (r.m.s.) slope over a span of length r, $\sigma(r)$, is

$$\sigma(r) = \sqrt{c(0)}\,\sqrt{2[1-\rho(r)]/r^2} \tag{8}$$

We are interested in the way in which $\sigma(r)$ depends on r.

In the first place, we usually have $\rho(r)\to 0$ as $r\to\infty$, thus for sufficiently large r (r much larger than the maximum diameter of the largest object affecting roughness of the surface)

$$\sigma(r) \sim 1/r \tag{9}$$

In the limit $r\to 0$ the r.m.s. slope $\sigma(r)$ may be finite, whence

$$\sigma(0) = \sqrt{-c(0)\rho''(0)} \tag{10}$$

where

$$\rho''(0) = \frac{d^2}{dr^2}\,\rho(r)\bigg|_{r=0}$$

On sufficiently rough surfaces, this limit may be infinite; such roughness is observed, however, only on an infinitesimally small scale which is of little practical importance.

The correlation function for a surface covered with primary impact craters has been calculated theoretically by Marcus (1968 a,b) improving earlier results of Chernov (1967). The complete model is described in (Marcus, 1968 a,b,c); the most important parameters are:

s, the crater population index ($2\leq s\leq 4$)

δ, the exponent relating crater depth to crater diameter (if $\delta = 1$, depth is proportional to diameter)

x_m, the maximum diameter of any crater in the region of interest

C, the density of craters with diameters larger than some small diameter x_0.

The correlation functions $\rho(r)$ are taken from Figure 3 in (Marcus, 1968b). In Fig. 1 of this paper we exhibit as a function of log r the logarithm of the <u>normalized</u> <u>r.m.s.</u> <u>slope</u>

$$\log[\sigma(r)/\sqrt{c(0)}] = 1/2 \log [2(1-\rho(r))/r^2] \tag{11}$$

for $\delta=1$, $x_m=100$, $x_0=1$, s = 2,3,4, and other parameters as shown.

We see from Figure 1 that as a rough approximation

$$\sigma(r) \sim 1/r^\eta \tag{12}$$

where

$$\eta=1 \text{ for } r>x_m \tag{13}$$

$$\eta\leq 1 \text{ for } x_0<r<x_m$$

$$(\eta \sim 1 \text{ for } s=4,\ \eta \sim 1/2 \text{ for } s=3) \tag{14}$$

This is in accord with theory. (13) agrees with (9), of course. Marcus showed that for an intermediate range of values of r

$$1-\rho(r) \sim r^{\mu-1} \tag{15}$$

where

$$\mu = 3 + 2\delta - s, \ 1<\mu<3 \quad (16)$$

thus (12) applies with

$$\eta = \frac{s}{2} - \delta \quad (17)$$

and in particular, $\eta = 1$ for ($s=4$, $\delta=1$), and $\eta = 1/2$ for ($s=3,\delta=1$). For $s=2$ and $\delta=1$, (15) fails; in this case, $[1-\rho(r)]r^2$ is a very slowly decreasing function of r which is not a power function.

In Figure 2 we exhibit some hypothetical r.m.s. slopes $\sigma(r)$ for x_m = 100 meters and for four types of possible lunar terrain:

a. s = 2, C = 0.2 - a very rough terrain.

b. s = 2, C = 0.08 - rough mare, like Rangers VII and VIII (Shoemaker et. al., 1967).

c. s = 3, C = 0.2 - smooth mare.

d. s = 4, C = 0.2 - extremely smooth mare.

C is, in this example, the number of craters per square meter larger than x_o = 1 meter diameter.

Note that the ordinate in Fig. 2 is arctan $[\sigma(r)]$, even though in general

$$(E\{[\arctan S(r)]^2\})^{1/2} \neq \arctan [(E\{S^2(r)\})^{1/2}]$$

But if the slope distribution is sufficiently concentrated near S = 0, i.e., not too dispersed, these quantities are almost equal.

If s is nearly equal to 2, the results shown in Figure 2 are easily extended to somewhat different values of x_m. It can be shown (Marcus, 1968 ab) that for $r>x_o$, $\rho(r)$ depends on r, x_o and x_m through r/x_m and r/x_o only; and if x_o is much smaller than x_m, then $\rho(r)$ depends on r, x_o, and x_m principally through the value of r/x_m. Since $c(0)$ is proportional to

$$x_m^{2+2\delta-s} - x_o^{2+2\delta-s}$$

it depends principally on the value of x_m for x_m much larger than x_o, if s is not too large. Thus, the chosen value of x_o does not much affect the value of $\rho(r)$ or $\sigma(r)$, for moderate and large values of r, when $s = 2$ and x_m is much larger than x_o. If we have a graph of log $\sigma(r)$ against log(r) for fixed C, $s = 2$, and a large value of x_m, then we can approximate log $\sigma(r)$ for the same C, for $s = 2$, and a different value of x_m, by "sliding" the curve horizontally from the original endpoint value at x_m to the same endpoint value at the new x_m. We illustrate this method in the next section. The method fails in practice if x_m is too large (greater than 200-300 meters) since larger craters have a much larger population index s.

2.0 APPLICATION TO MARE COGNITUM

Schloss (1966) has calculated r.m.s. slopes from a Ranger VII map of Mare Cognitum. His estimated $\sigma(r)$ values are also shown in Fig. 2. The largest crater affecting roughness in the region he studied seems to have diameter x=22 meters. Choosing this value as x_m, we construct a hypothetical $\sigma(r)$ function, as described above, for $s=2$, and $C=0.08$. There is a surprisingly good fit between the data and the theoretical curve.

3.0 SLOPE DISTRIBUTIONS

Our interest in the r.m.s. slope $\sigma(r)$ grew from the hope that if the actual slope distribution were nearly Gaussian, then it could also be nearly completely described by $\sigma(r)$. The actual slope distributions obtained by Schloss (1967) (see Figure 3) are roughly symmetric about zero, but are not quite Gaussian. Attempts to derive a theoretical slope distribution along lines developed earlier (Marcus, 1968abc) have not been successful. It may, however be instructive to consider the formulation and its mathematical difficulties.

The basic function is the <u>joint characteristic function</u> (ch.f.) $\phi(u_1,u_2;r)$ of elevations $Z(\underset{\sim}{R})$ at a point $\underset{\sim}{R}$ and $Z(\underset{\sim}{R} + \underset{\sim}{r})$ at a point $\underset{\sim}{R} + \underset{\sim}{r}$ whose distance from $\underset{\sim}{R}$ is r = (length of $\underset{\sim}{r}$). The ch.f. $\phi(u_1,u_2;r)$ is the bivariate Fourier transform of the joint probability density function of $Z(\underset{\sim}{R})$ and $Z(\underset{\sim}{R} + \underset{\sim}{r})$, say $p(z_1,z_2;r)$:

$$\phi(u_1,u_2;r) = \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \; p(z_1,z_2;r) \exp(iu_1z_1+iu_2z_2) \qquad (18)$$

Assuming the same model as in (Marcus, 1968bc) we obtain

$$\phi(u_1,u_2;r) = \exp\left(-\int\xi(x)dx\iint[\exp(iu_1\zeta(x,r_1) + iu_2\,\zeta(x,r_2))-1]\,\bar{\nu}_2(r_1,r_2;r)\,dr_1dr_2\right) \qquad (19)$$

where $\xi(x)dx$ is the expected number of craters per unit area of diameters x to $x + dx$, $\zeta(x,r)$ is the elevation change caused at some point by the formation of a crater of diameter x at another point at distance r from the first point, and

$$\bar{\nu}_2(r_1,r_2;r) = 4r_1r_2/[4r_1^2r_2^2 - (r_1^2 + r_2^2 - r^2)^2]^{1/2}$$

$$\text{if } r_1>0,\ r_2>0,\ |r_1-r_2| <r,\ r_1+r_2>r$$

$$\bar{\nu}\,(r_1,r_2;r) = 0 \text{ otherwise} \qquad (20)$$

If $\phi(u_1,u_2;r)$ were available, we could readily compute the <u>ch.f. of S(r)</u>, $\phi_S(w;r)$:

$$\phi_S(w;r) = \phi(-w/r,w/r;r) \tag{21}$$

the joint ch.f. of $Z(\underset{\sim}{R})$ and S(r), $\phi_{Z,S}(u_1,w;r)$

$$\phi_{Z,S}(u_1,w;r) = \phi(u_1-w/r,w/r;r) \tag{22}$$

and so on. At present, these calculations do not seem to lead to any explicit ch.f. Even the higher moments and mixed moments of these random variables involve complicated integrals which must be integrated numerically (Marcus, 1968a,b). There is thus little promise of obtaining additional useful results about the lunar slope distribution from the above theory.

2015-AHM-kse

A. H. Marcus

A. H. Marcus

REFERENCES

1. Chernov, Yu. A., (1967), The Meteoroid Mass Distribution and Autocorrelation Function for a Planetary Surface Eroded by a Meteor Stream, Sov. Astron. - A.J., 10, 853-859.

2. Marcus, A. H., (1968a), Covariance Function of Elevations on a Cratered Planetary Surface. Part I, Bellcomm Technical Report, TR-68-340-3.

3. Marcus, A. H., (1968b), Covariance Function of Elevations on a Cratered Planetary Surface. Part II, Bellcomm Technical Report, TR-68-340-5.

4. Marcus, A. H., (1968c), Distribution of Elevations on a Cratered Planetary Surface, Bellcomm Technical Report, TR-68-340-4.

5. Schloss, M., (1966), Quantifying Terrain Roughness on Lunar and Planetary Surfaces, J. Spacecraft and Rockets, 3, 283-285.

6. Shoemaker, E. M., R. M. Batson, H. E. Holt, E. C. Morris, J. J. Rennilson, and E. A. Whitaker, (1967), Television Observations from Surveyor V, In Surveyor V Mission Report, Part II: Science Results, J.P.L. TR-32-1246, Pasadena, November 1, 1967, pp. 14-19.

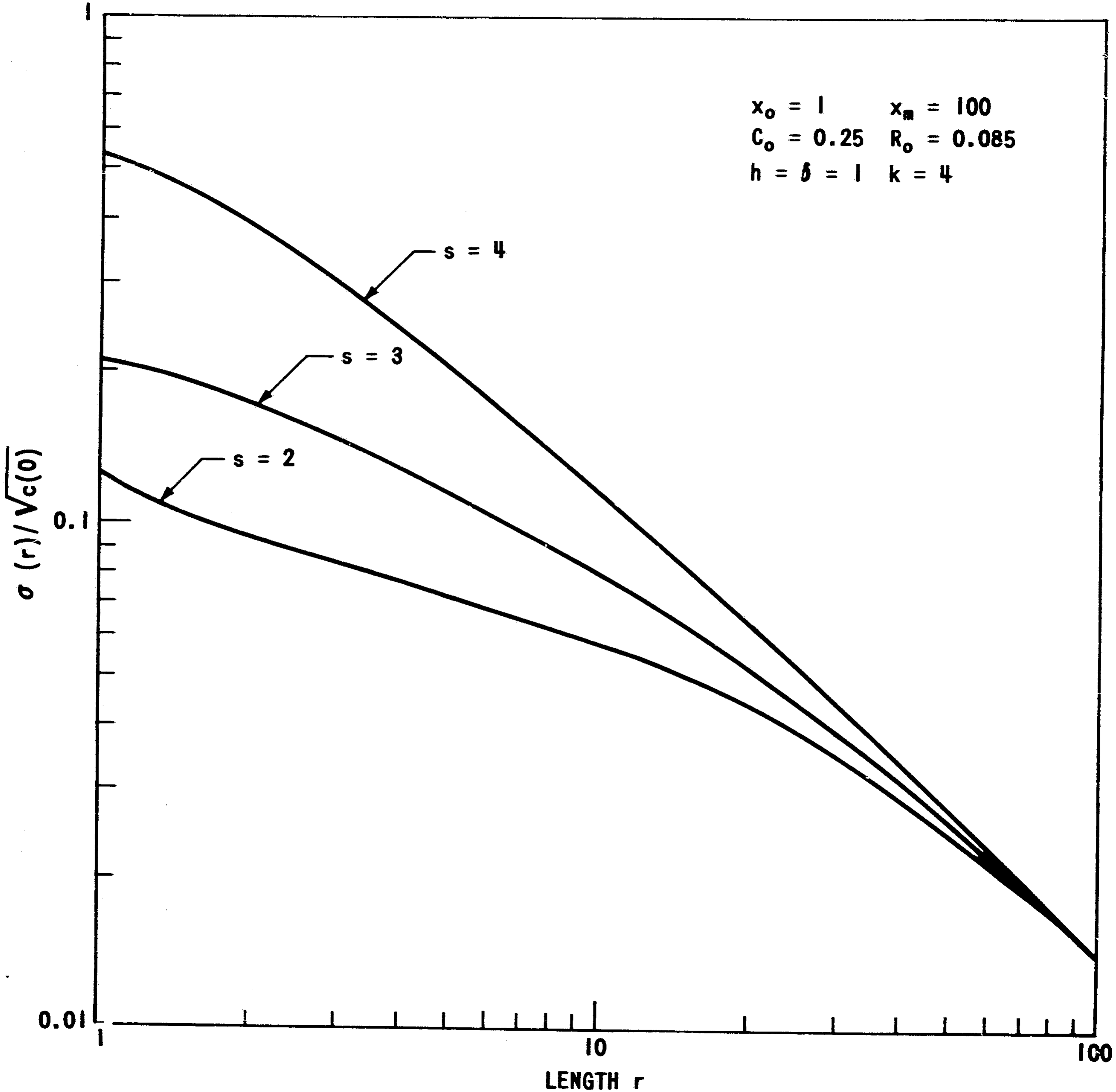

FIGURE 1 - THEORETICAL NORMALIZED R.M.S. SLOPE vs. SLOPE LENGTH

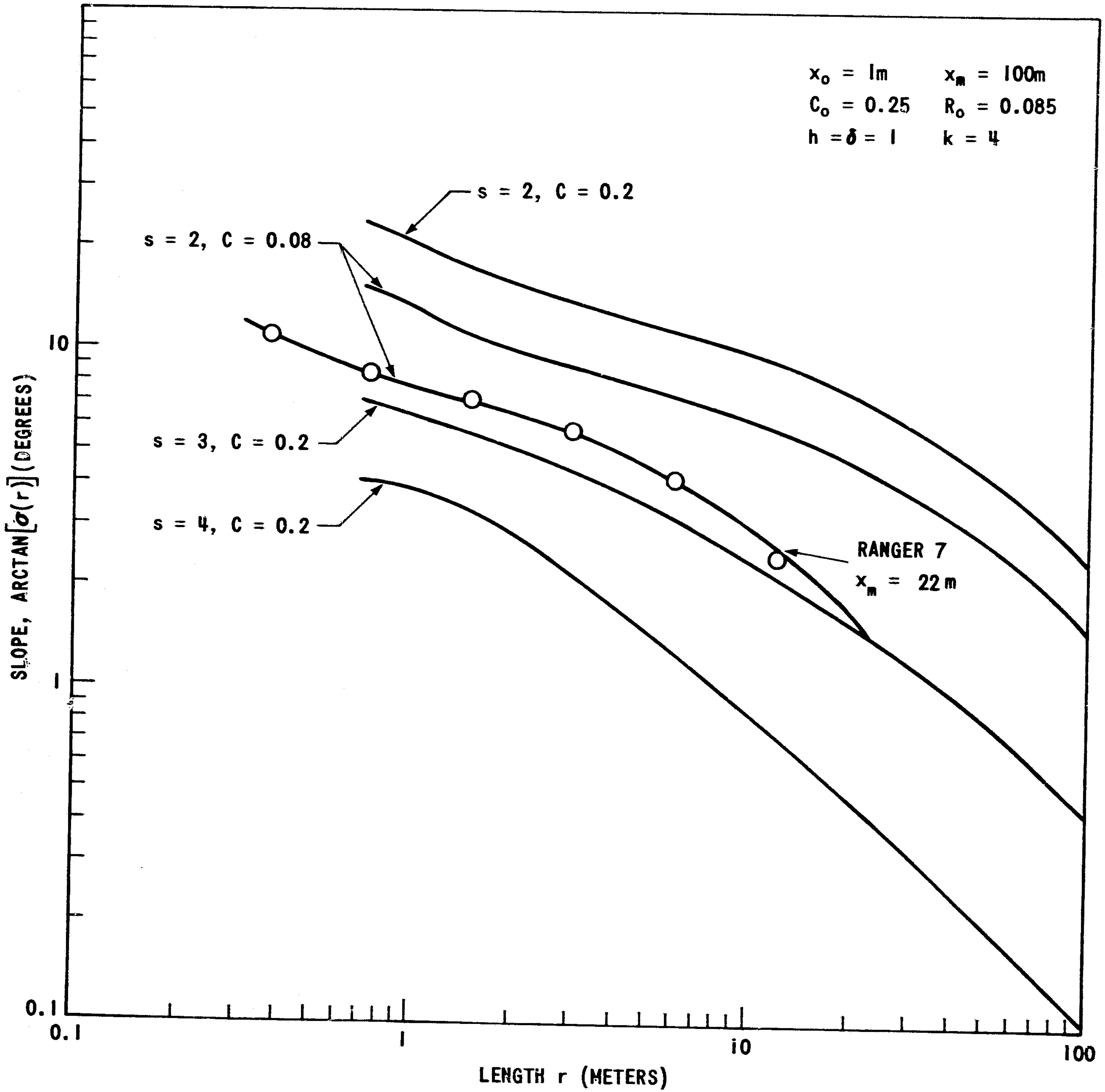

FIGURE 2 - R.M.S. SLOPE vs. SLOPE LENGTH

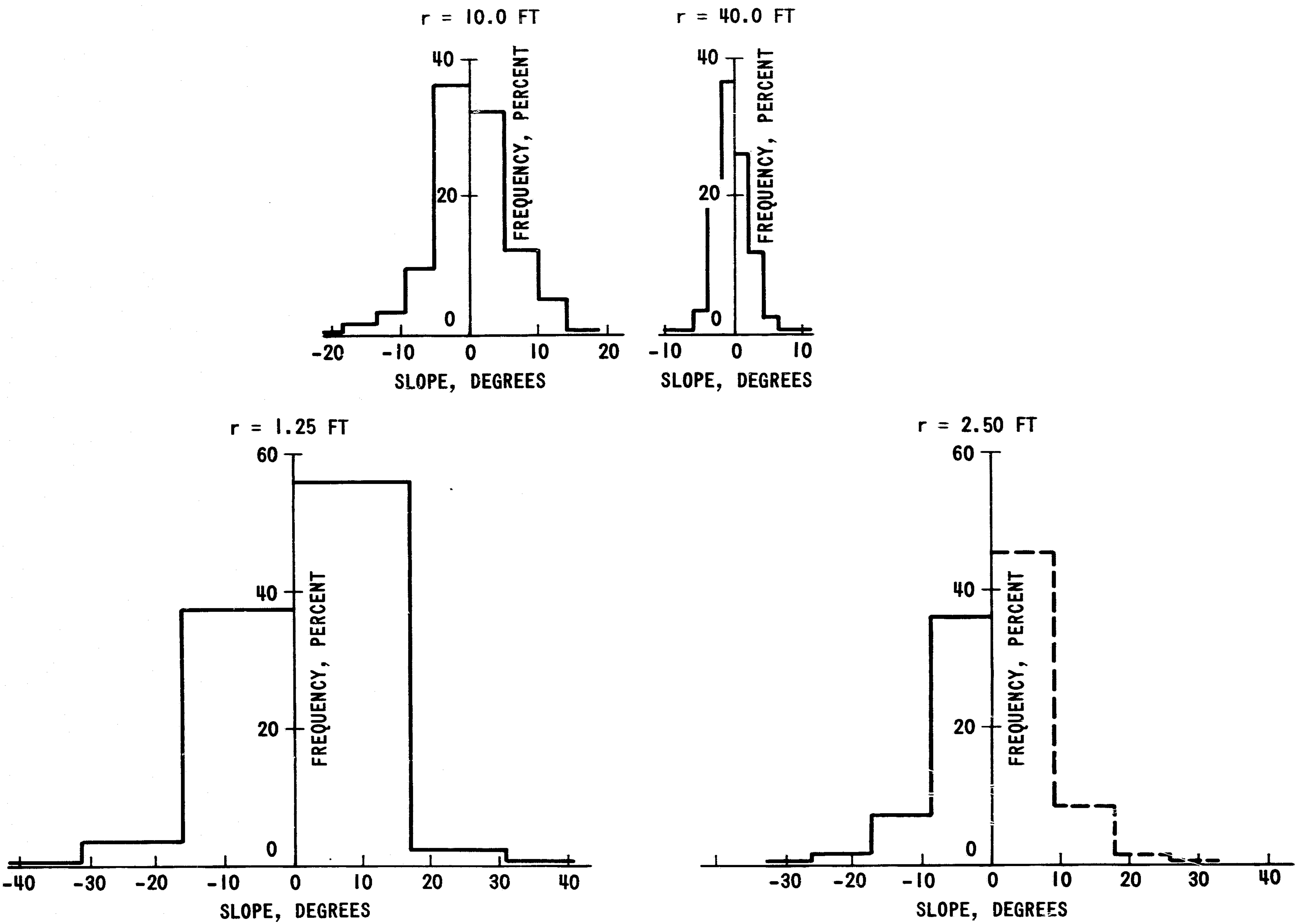

FIGURE 3 - SLOPE HISTOGRAMS IN COMBINED DIRECTIONS (SCHLOSS, 1966)