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Waiting Time Convexity in the M/G/1 Queue

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Strong bounds are obtained on the complementary waiting time distribution for the M/G/1 queue using the α-convexity structural characteristic of the distribution. This notion is discussed and a sufficient condition is obtained.

I. INTRODUCTION

This paper investigates the α-convexity[1] of the complementary waiting time distribution in the M/G/1 queue and shows how a sufficient condition for α-convexity is obtained, in terms of the service time distribution. This structure characteristic will permit strong bounds to be obtained on the complementary waiting time distribution. For convenience, the definition of α-convexity and some of its properties are given below.

II. α-CONVEXITY

A function $f(x)$ is said to be α-convex on an interval I if $e^{\alpha x}f(x)$ is convex on I. Of course, ordinary convexity corresponds to $\alpha = 0$. A sufficient condition for α-convexity is

$$e^{-\alpha x}\frac{d^2}{dx^2}(e^{\alpha x}f(x)) \geqslant 0, \qquad x \in I. \tag{1}$$

This is the same as

$$f''(x) + 2\alpha f'(x) + \alpha^2 f(x) \geqslant 0, \qquad x \in I. \tag{2}$$

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A function may be α-convex without being convex; for example, consider $f(x) = x^3$, which is α-convex for $x \geqslant 0$ and $\alpha \geqslant 0$. For $\alpha = 1$, however, x^3 is α-convex for $-3-\sqrt{3} \leqslant x \leqslant -3+\sqrt{3}$ as seen by use of (2).

The α-convexity of a function may permit stronger bounds than convexity to be obtained on the function or on integrals of the function. For example, let $p(x) \geqslant 0$ and let $f(x)$ be convex on I; then Jensen's inequality[2] states

$$\int_I f(x)p(x)dx \geqslant f(\mu) \int_I p(x)dx,$$

$$\mu = \int_I xp(x)dx \Big/ \int_I p(x)dx. \tag{3}$$

If $f(x)$ is α-convex on I, then, since

$$\int_I f(x)p(x)dx = \int_I e^{\alpha x}f(x)e^{-\alpha x}p(x)dx, \tag{4}$$

one has

$$\int_I f(x)p(x)dx \geqslant e^{\alpha\mu}f(\mu) \int_I e^{-\alpha x}p(x)dx,$$

$$\mu = \int_I xe^{-\alpha x}p(x)dx \Big/ \int_I e^{-\alpha x}p(x)dx. \tag{5}$$

This result can be stronger than (3).

An example is provided by

$$K = \int_0^\infty \frac{e^{-x}}{1+x}dx, \tag{6}$$

whose value is $K = 0.5963$. From (3), one has $K \geqslant 0.5$. Since, for $x \geqslant 0$, $1/(1+x)$ is α-convex for all α, one may apply (5) to obtain

$$K \geqslant \frac{e^{\frac{\alpha}{\alpha+1}}}{\alpha+2}, \tag{7}$$

which, for $\alpha = (\sqrt{5}-1)/2$, yields $K \geqslant 0.5596$.

Let $\tilde{f}(s)$ be the Laplace transform of a function $f(x)$, that is,

$$\tilde{f}(s) = \int_0^\infty e^{-sx}f(x)dx, \tag{8}$$

and let the transform be absolutely convergent for $s > 0$. Then the approximation sequence, $f_n(x)$ $(n = 0, 1, 2, \cdots)$, introduced in the Laplace inversion theory, is given by[1]

$$f_n(x) = \frac{(-1)^n}{n!} s^{n+1}\tilde{f}^{(n)}(s) \Bigg|_{s=\frac{n+1}{x}}, \tag{9}$$

from which, in particular,

$$f_0(x) = \frac{1}{x}\tilde{f}\left(\frac{1}{x}\right),$$
$$f_1(x) = -\frac{4}{x^2}\tilde{f}'\left(\frac{2}{x}\right). \tag{10}$$

The approximation sequence may be used to obtain bounds on $f(x)$; thus, if $f(x)$ is convex for $x \geqslant 0$, then

$$f(x) \leqslant f_{n+1}(x) \leqslant f_n(x), \qquad x \geqslant 0, \qquad n \geqslant 0. \tag{11}$$

If $f(x)$ is α-convex on $x \geqslant 0$, then the bound of (11) may be strengthened. Let $\tilde{f}(s - \alpha)$ be absolutely convergent for $s > 0$; then it is the transform of a function $g(x)$ for which

$$f(x) = e^{-\alpha x}g(x). \tag{12}$$

Application of (11) now provides the inequality

$$f(x) \leqslant e^{-\alpha x}g_{n+1}(x) \leqslant e^{-\alpha x}g_n(x), \qquad x \geqslant 0, \qquad n \geqslant 0. \tag{13}$$

This is a much tighter inequality than (11), especially for the tail of $f(x)$, and constitutes the main tool for bounding the M/G/1 waiting time distribution.

The Bernstein theorem,[3] which states that $f(x) \geqslant 0$ if $\tilde{f}(s)$ is completely monotone and, conversely, may be used to translate condition (1) in terms of $\tilde{f}(s)$. Thus let $f''(x)$ be continuous on $(0, \infty)$; then $f(x)$ is α-convex on $(0, \infty)$ if and only if

$$(s + \alpha)^2\tilde{f}(s) - (s + 2\alpha)f(0+) - f'(0+) \tag{14}$$

is completely monotone in s on $(0, \infty)$ and is absolutely convergent for $s > 0$.

A function, $f(x) > 0$, is said to be log-convex if $\ln f(x)$ is convex on some interval I. The condition for log-convexity is

$$f''(x)f(x) - f'(x)^2 \geqslant 0, \qquad x \in I. \tag{15}$$

In particular, log-convexity implies convexity; hence $e^{\alpha x}f(x)$ is convex, and a log-convex function is α-convex for all α. The converse is also true. This follows from (2) on observing that the discriminant of the quadratic in α is $f'(x)^2 - f''(x)f(x)$; hence α-convexity for all α implies (15) and the log-convexity of $f(x)$. An interesting corollary of this is that the sum of log-convex functions is log-convex since, clearly, the sum of functions convex for the same α is again convex for this α. This theorem and eq. (14) permit ascertaining the log-convexity of $f(x)$ from its Laplace transform.

III. α-CONVEXITY IN M/G/1

The starting point for this investigation of α-convexity in M/G/1 is the Pollaczek-Khintchine formula.[4] Let $B(x)$ be the service time distribution and $F(x)$ the complementary waiting time distribution; also let $\hat{B}(s)$, $\hat{F}(s)$ be the corresponding Laplace-Stieltjes transforms. Then

$$\hat{F}(s) = \frac{\rho s - \lambda[1 - \hat{B}(s)]}{s - \lambda[1 - \hat{B}(s)]}, \qquad \rho < 1, \tag{16}$$

in which λ is the arrival rate, μ is the service rate, and $\rho = \lambda/\mu$ is the offered load. It is convenient to use the forward recurrence time distribution, $\theta(x)$, corresponding to $B(x)$. Since the Laplace-Stieltjes transform, $\hat{\theta}(s)$, of $\theta(x)$ is

$$\hat{\theta}(s) = \mu \frac{1 - \hat{B}(s)}{s}, \tag{17}$$

one has

$$\hat{F}(s) = \rho \frac{1 - \hat{\theta}(s)}{1 - \rho\hat{\theta}(s)},$$

$$\tilde{F}(s) = \frac{\rho}{s} \frac{1 - \hat{\theta}(s)}{1 - \rho\hat{\theta}(s)}, \tag{18}$$

in which $\tilde{F}(s)$ is the corresponding Laplace transform. Clearly,

$$\theta(s) \sim \frac{\mu}{s}, \qquad s \to \infty; \tag{19}$$

hence

$$\hat{F}(s) \sim \rho - \frac{\mu\rho(1 - \rho)}{s}, \qquad s \to \infty. \tag{20}$$

Thus,

$$F(0+) = \rho, \qquad F'(0+) = -\mu\rho(1 - \rho). \tag{21}$$

The information is now available to apply condition (14). That expression now takes the form

$$\frac{\tilde{N}(s)}{1 - \rho\hat{\theta}(s)}, \tag{22}$$

$$\tilde{N}(s) = \mu\rho(1 - \rho) + \frac{\alpha^2\rho}{s} - \left\{\rho(1 - \rho)(s + \lambda + 2\alpha) + \frac{\alpha^2\rho}{s}\right\}\hat{\theta}(s).$$

The function $1/[1 - \rho\hat{\theta}(s)]$ is the Laplace-Stieltjes transform of a monotone increasing function on $(0, \infty)$. If we write $\tilde{N}(s)$ in the form

$$\tilde{N}(s) = \frac{\alpha^2\rho}{s} + \lambda(1 - \rho)\hat{B}(s) - \frac{\lambda(1 - \rho)(\lambda + 2\alpha)}{s}[1 - \hat{B}(s)] - \frac{\lambda\alpha^2}{s^2}[1 - \hat{B}(s)]. \tag{23}$$

we see that $\tilde{N}(s)$ is a Laplace transform. For the function of (22) to be completely monotone, it is therefore sufficient that $\tilde{N}(s)$ be the transform of a nonnegative function. If we let $b(x)$ be the service-time density function, and $r(x)$ the corresponding rate function, that is,

$$r(x) = \frac{b(x)}{1 - B(x)}, \tag{24}$$

this condition may be written in the following two forms:

$$\frac{\alpha^2\rho}{\lambda(1 - \rho)} + b(x) \geqslant (\lambda + 2\alpha)[1 - B(x)] + \frac{\alpha^2}{1 - \rho}\int_0^x [1 - B(u)]du,$$

$$r(x) + \frac{\alpha^2}{\mu(1 - \rho)}\frac{1 - \theta(x)}{1 - B(x)} \geqslant \lambda + 2\alpha \tag{25}$$

to assure the α-convexity of $F(x)$.

The case $\alpha = 0$ of (25) yields the interesting result that convexity of $F(x)$ is guaranteed by

$$r(x) \geqslant \lambda, \qquad x \geqslant 0, \tag{26}$$

which also implies

$$1 - B(x) \leq e^{-\lambda x}, \qquad x \geq 0. \tag{27}$$

Application of (25) will now be made to the following class of distribution functions $B(x)$:

$$B(x) = \int_0^\infty (1 - e^{-xu})dG(u) \tag{28}$$

in which $G(x)$ is a distribution function on $(0, \infty)$. Condition (25) now becomes

$$\int_0^\infty e^{-xu}\left(u - \lambda - 2\alpha + \frac{\alpha^2}{1 - \rho}\frac{1}{u}\right)dG(u) \geq 0. \tag{29}$$

Now the integrand is nonnegative if

$$2u \geq \lambda + 2\alpha + \sqrt{(\lambda + 2\alpha)^2 - \frac{4\alpha^2}{1 - \rho}}, \tag{30}$$

which suggests the introduction of the quantity c defined by

$$c = \inf_x [x; G(x) > 0]. \tag{31}$$

Thus condition (30) need be satisfied only for $u \geq c$, and hence

$$2c \geq \lambda + 2\alpha + \sqrt{(\lambda + 2\alpha)^2 - \frac{4\alpha^2}{1 - \rho}} \tag{32}$$

assures (29) and the α-convexity of $F(x)$. One implication of (32) is the following constraint on α:

$$\alpha \leq c(1 - \rho) - \sqrt{c(1 - \rho)(\lambda - c\rho)}. \tag{33}$$

Application of (33) to the exponential case $B(x) = e^{-\mu x}$ yields

$$\alpha \leq \mu(1 - \rho), \tag{34}$$

which, of course, is consistent with the known result

$$F(x) = \rho e^{-\mu(1-\rho)x}. \tag{35}$$

As another illustration, consider

$$B(x) = 1 - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-2x} \tag{36}$$

for which $c = 1$, $\mu = 4/3$. One has

$$\alpha \leqslant 1 - \frac{3}{4}\lambda - \frac{1}{2}\sqrt{\lambda\left(1 - \frac{3}{4}\lambda\right)}. \tag{37}$$

Since

$$\hat{B}(s) = \frac{1}{2}\frac{1}{s+1} + \frac{1}{s+2}, \tag{38}$$

use of (18) yields

$$\tilde{F}(s) = \frac{\lambda}{4}\frac{3s+5}{s^2 + (3-\lambda)s + 2 - \frac{3}{2}\lambda}. \tag{39}$$

If we let

$$\gamma = \frac{-3 + \lambda + \sqrt{1+\lambda^2}}{2},$$

$$\delta = \frac{-3 + \lambda - \sqrt{1+\lambda^2}}{2},$$

$$A = \frac{3\sqrt{1+\lambda^2} + 1 + 3\lambda}{2},$$

$$B = \frac{3\sqrt{1+\lambda^2} - 1 - 3\lambda}{2}, \tag{40}$$

then we have

$$F(x) = \frac{\lambda}{4\sqrt{1+\lambda^2}}(Ae^{\gamma x} + Be^{\delta x}). \tag{41}$$

This distribution is, in fact, log-convex; thus (37) is overly restrictive. This might have been expected since (25) is only a sufficient condition for α-convexity.

IV. BOUNDS

When values of α have been determined by use of the complete monotonicity of $\tilde{N}(s)$ in (23), or by use of (25), then (13) may be applied to $\tilde{F}(s)$ in (18). It is, of course, advantageous to use as large a value of α as possible consistent with the requirement that $\tilde{F}(s - \alpha)$ be absolutely convergent for $s > 0$. Applying (10) and (13) to (18)

Table I—Numerical inversion for different α

	$F(x)$			
X	Exact Results	$\alpha = 0$	$\alpha =$ 0.34549	$\alpha =$ 0.69098
1	0.1718	0.1815	0.1753	0.1727
2	0.0834	0.0997	0.0883	0.0841
3	0.0414	0.0590	0.0460	0.0416
4	0.0207	0.0368	0.0245	0.0208
5	0.0103	0.0239	0.0133	0.0104
6	0.0052	0.0161	0.0073	0.0052
7	0.0026	0.0112	0.0041	0.0026
8	0.0013	0.0080	0.0023	0.0013
9	0.0007	0.0058	0.0013	0.0007

provides the following explicit bounds:

$$F(x) \leq \frac{e^{-\alpha x}}{1 - \alpha x}\left[1 - \frac{1 - \rho}{1 - \rho\hat{\theta}\left(\frac{1}{x} - \alpha\right)}\right],$$

$$F(x) \leq \frac{4e^{-\alpha x}}{(2 - \alpha x)^2}\left[1 - \frac{1 - \rho}{1 - \rho\hat{\theta}\left(\frac{2}{x} - \alpha\right)}\right]$$

$$+ \frac{4e^{-\alpha x}}{2x - \alpha x^2}\frac{\rho(1 - \rho)\hat{\theta}'\left(\frac{2}{x} - \alpha\right)}{\left[1 - \rho\hat{\theta}\left(\frac{2}{x} - \alpha\right)\right]^2}. \tag{42}$$

As a numerical example, the approximation $F_4(x)$ was calculated for $\tilde{F}(s)$ of (39) with $\lambda = 0.5$. Table I shows the exact results obtained from (41). It also shows the inversion with $\alpha = 0$ (that is, no α-enhancement used), the results with the value of α obtained from (37) (0.34549), and, finally, the calculation using the optimum choice, $\gamma = -\alpha$, i.e., $\alpha = 0.69098$.

As expected, the table shows improved accuracy as α is increased and, in particular, in the tracking of the tail behavior. This, of course, is the primary goal of α-enhancement. Observe that the approximate values are larger than the exact values, as implied by the α-convexity of $F(x)$ and (13).

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