

Accumulation of Jitter: A Stochastic Model

By C. CHAMZAS*

(Manuscript received March 29, 1984)

A problem that has been considered extensively in the past is the accumulation of jitter in a chain of regenerative repeaters. For simplicity it is usually assumed that all repeaters in the chain are identical. However, this is not the case in real systems, where considerable differences among the repeaters of a chain have been observed. These random variations are due mainly to manufacturing tolerances, aging effects, temperature changes, etc. In this work, we examine the accumulation of systematic and random jitter along the chain, when the repeater transfer functions are subjected to random variations. We derive expressions for the expected value and variance of the output jitter spectrum in terms of average values of the repeater's jitter transfer function. In addition we find the expected value and the variance of the RMS jitter. Finally we examine two special cases where the timing circuit employs either a phased-locked loop or a surface acoustic wave filter, and derive some asymptotic relations for the power spectral density and root-mean-square value of the accumulated systematic jitter.

I. INTRODUCTION

In a chain of self-timed regenerative repeaters for data transmission with Pulse Amplitude Modulation (PAM), each regenerator extracts the timing information (clock) directly from the received pulse train. Ideally the output of the timing circuit should be a sine wave with frequency equal to the baud rate of the data. In practice, however, imperfections of the circuits and noise in the transmission channel disturb the timing recovery operation so that the phase of the recovered

* AT&T Bell Laboratories.

Copyright © 1985 AT&T. Photo reproduction for noncommercial use is permitted without payment of royalty provided that each reproduction is done without alteration and that the Journal reference and copyright notice are included on the first page. The title and abstract, but no other portions, of this paper may be copied or distributed royalty free by computer-based and other information-service systems without further permission. Permission to reproduce or republish any other portion of this paper must be obtained from the Editor.

clock is randomly deviated from the desired input clock phase. Such deviations, called *timing jitter*, produce a position modulation of the regenerated signal; the timing jitter tends to accumulate along the chain and degrade the system's performance. If the jitter introduced by the repeater's timing recovery circuitry is the *same* for each repeater (for example, if it depends on the data pattern), it is called *systematic jitter*. Otherwise it is called *random jitter*.

The problem of jitter accumulation in a chain of regenerative repeaters has been considered extensively in the past.^{1,2} For simplicity it is usually assumed that all repeaters in the chain are identical. However, this is not the case in real systems, where considerable differences have been observed among repeaters.³ These random variations are mainly because of manufacturing tolerances, aging effects, temperature changes, etc. In this work we examine the accumulation of systematic and random jitter along the chain, when the repeater's jitter transfer functions are subjected to random variations. Previous papers^{1,4} have treated the relationship between the jitter transfer function and the retiming circuit parameters.

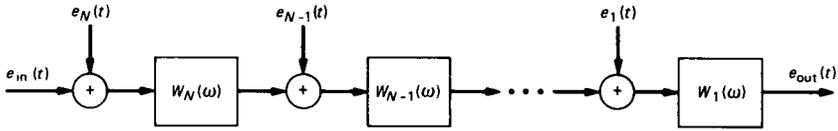
In this paper, we derive expressions for the expected value and the variance of the output jitter spectrum in terms of averages of jitter transfer functions of the ensemble of regenerators. In addition we find the expected value and the variance of the Root-Mean-Square (RMS) jitter.

We also examine two special cases where the timing circuit employs either a Phased-Locked Loop (PLL) or a Surface Acoustic Wave (SAW) filter. Finally we consider the case of a retiming circuit exhibiting random variations in its phase only and derive some asymptotic relations for the power spectral density and the RMS value of the accumulated systematic jitter. Of particular interest perhaps is relation (40), where we show that the RMS value of the accumulated systematic jitter for a nonpeaking case is approximately equal to $\sqrt{0.36}N/\alpha$, where N is the number of regenerators and α the phase slope of the average jitter transfer function.

II. THE THEORY

2.1 The basic model

A brief review of jitter accumulation is presented here for completeness. The model we use to study the jitter accumulation is similar to the linear model attributed to Chapman, as described by Byrne et al.¹ Questions regarding the validity of this linear model will not be considered here. This problem will be addressed in a forthcoming paper. An oversimplified treatment is also given in Appendix B. According to this model, the timing jitter produced in a regenerative



$W_m(\omega)$: JITTER TRANSFER FUNCTION FOR THE m th REPEATER.

Fig. 1—Chapman's model for timing jitter accumulation in a chain of repeaters.

section is the filtered sum of the jitter coming from the previous section plus an additional equivalent jitter $e_i(t)$ inserted at the local input. This is illustrated in Fig. 1, where $W_i(\omega)$ is the jitter transfer function of the i th repeater. The major differences between our model and the ones used previously are that we do not assume $W_i(\omega)$ to be the same for every repeater.

The input jitter, $e_i(t)$, can be separated into its two components, the random part $e_{ir}(t)$ and the systematic part $e_{is}(t)$, i.e.,

$$e_i(t) = e_{ir}(t) + e_{is}(t). \quad (1)$$

The random component $e_{ir}(t)$ is different for each repeater and is usually caused by random sources, e.g., line noise, thermal noise, crosstalk, etc. The systematic component $e_{is}(t)$ is the *same* for each repeater and is usually pattern dependent.

Let $\Phi(\omega)$, $\Phi_r(\omega)$, $\Phi_s(\omega)$ be the two-sided power spectra of $e_i(t)$, $e_{ir}(t)$, and $e_{is}(t)$, respectively. Assuming that $e_{ir}(t)$ and $e_{is}(t)$ are statistically independent, then

$$\Phi(\omega) = \Phi_r(\omega) + \Phi_s(\omega). \quad (2)$$

If $S(\omega)$ is the spectrum of the output jitter then we can write

$$S(\omega) = \Phi_r(\omega)T_r(\omega) + \Phi_s(\omega)T_s(\omega), \quad (3)$$

where¹

$$T_r(\omega) = |W_1(\omega)|^2 + |W_1(\omega)W_2(\omega)|^2 + \dots + |W_1(\omega) \dots W_N(\omega)|^2 \quad (4)$$

is the total transfer function for random jitter, and

$$T_s(\omega) = |W_1(\omega) + W_1(\omega)W_2(\omega) + \dots + W_1(\omega)W_2(\omega) \dots W_N(\omega)|^2 \quad (5)$$

is the total transfer function for systematic jitter.

If we assume that

$$W_j(\omega) = W(\omega) \quad j = 1, 2, \dots, N, \quad (6)$$

then we obtain the known relations¹

$$T_r(\omega) = |W(\omega)|^2 \frac{1 - |W(\omega)|^{2N}}{1 - |W(\omega)|^2} \quad (7)$$

and

$$T_s(\omega) = |W(\omega)|^2 \left| \frac{1 - W^N(\omega)}{1 - W(\omega)} \right|^2. \quad (8)$$

As we will show, relations (7) and (8) provide us with a satisfactory approximation when the fine structure of $S(\omega)$ is not important, as in the evaluation of the RMS jitter,

$$\sigma_o^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega. \quad (9)$$

However, in some applications the detailed structure of $S(\omega)$ is essential, and the random variations of $W_i(\omega)$ must be taken into consideration. Hence it is important to describe $W_i(\omega)$ with a probabilistic model instead of the deterministic approach used in the past.

Another advantage of using this approach is that we can directly examine the impact of tolerances in manufacturing the repeater components on the accumulation of jitter.

2.2 The stochastic model

In this section we shall consider $W_i(\omega)$ to be a random variable, independent and identically distributed (i.i.d.). Thus $S(\omega)$, the spectrum of the jitter at the output of N regenerative repeaters, is also a random variable and it should be described with its expected value and variance.

With the assumption that $e_i(t)$ is a stationary stochastic process we obtain from (2) and (3)

$$E\{S(\omega)\} = \Phi_s(\omega)E\{T_s(\omega)\} + \Phi_r(\omega)E\{T_r(\omega)\} \quad (10a)$$

$$\begin{aligned} \text{Var}\{S(\omega)\} &= \Phi_s^2(\omega)\text{Var}\{T_s(\omega)\} + \Phi_r^2(\omega)\text{Var}\{T_r(\omega)\} \\ &+ 2\Phi_s(\omega)\Phi_r(\omega)[E\{T_s(\omega)T_r(\omega)\} - E\{T_s(\omega)\}E\{T_r(\omega)\}], \end{aligned} \quad (10b)$$

where $E\{X\}$ and $\text{Var}\{X\}$ are, respectively, the expected value and the variance of the random variable X . The spectra $\Phi_s(\omega)$ and $\Phi_r(\omega)$ of the input systematic and random jitter will be considered known, because we can measure them experimentally. Usually $\Phi_s(\omega)$ and $\Phi_r(\omega)$ are constant for the low frequencies where $T_s(\omega)$ and $T_r(\omega)$ are significant (white noise). Typical values for Φ_s and Φ_r are between 1 and 100 deg²/MHz.

Thus for the evaluation of the $E\{S(\omega)\}$ and $\text{Var}\{S(\omega)\}$ in (10) we need to find $E\{T_s(\omega)\}$, $E\{T_r(\omega)\}$, $\text{Var}\{T_s(\omega)\}$, $\text{Var}\{T_r(\omega)\}$ and $E\{T_s(\omega)T_r(\omega)\}$. The expressions for their analytical evaluation are

given below for a chain of N repeaters. Their derivations are given in Appendix A. For simplicity hereafter we shall omit the dependence of the various functions on the parameter ω , except if it is not obviously implied. Since the derivation of $\text{Var}\{T_s\}$ and $E\{T_s T_r\}$ is complicated, we simplified our analysis by making use of the Central Limit Theorem and assuming that the two random variables X and Y defined as $X + jY = \sum_{k=1}^N W_1 W_2 \dots W_k$ are jointly normal. Hence relations (15) and (16) are not exact, because of the above approximation.

For a chain of N repeaters

$$\bar{T}_r = E\{T_r\} = B \frac{1 - B^N}{1 - B} \quad (11)$$

$$\bar{T}_s = E\{T_s\} = B \frac{1 - B^N}{1 - B} + 2 \operatorname{Re} \frac{B}{1 - B/W} \left\{ \frac{1 - W^N}{1 - W} - \frac{1 - B^N}{1 - B} \right\} \quad (12)$$

$$E\{T_r^2\} = C \frac{1 - C^N}{1 - C} + 2 \frac{C}{1 - C/B} \left\{ \frac{1 - B^N}{1 - B} - \frac{1 - C^N}{1 - C} \right\} \quad (13)$$

$$\text{Var}\{T_r\} = E\{T_r^2\} - E^2\{T_r\} \quad (14)$$

$$\text{Var}\{T_s\} \approx 2[E^2\{X^2\} + E^2\{Y^2\} + 2E^2\{XY\} - (E^2\{X\} + E^2\{Y\})^2] \quad (15)$$

$$E\{T_s T_r\} - E\{T_s\}E\{T_r\} \approx 2[E\{X\}E\{XT_r\} + E\{Y\}E\{YT_r\} - (E^2\{X\} + E^2\{Y\})E\{T_r\}], \quad (16)$$

where

$$W = E\{W_i\} \quad (17a)$$

$$Z = E\{W_i^2\} \quad (17b)$$

$$B = E\{|W_i|^2\} \quad (17c)$$

$$C = E\{|W_i|^4\} \quad (17d)$$

and

$$D = E\{|W_i|^2 W_i\}. \quad (17e)$$

With X and Y defined as

$$X + jY = Q = \sum_{k=1}^N W_1 W_2 \dots W_k \quad (17f)$$

we have

$$E\{X\} = \operatorname{Re} E\{Q\}, \quad E\{Y\} = \operatorname{Im} E\{Q\} \quad (17g)$$

$$E\{X^2\} = [\bar{T}_s + \operatorname{Re} E\{Q^2\}]/2 \quad (17h)$$

$$E\{Y^2\} = [\bar{T} - \operatorname{Re} E\{Q^2\}]/2 \quad (17i)$$

$$E\{XY\} = \text{Im } E\{Q^2\}/2 \quad (17j)$$

$$E\{XT_r\} = \text{Re } E\{QT_r\} \quad (17k)$$

$$E\{YT_r\} = \text{Im } E\{QT_r\}, \quad (17l)$$

where

$$E\{Q\} = W \frac{1 - W^N}{1 - W} \quad (17m)$$

$$E\{Q^2\} = Z \frac{1 - Z^N}{1 - Z} + 2 \frac{Z}{1 - Z/W} \left[\frac{1 - W^N}{1 - W} - \frac{1 - Z^N}{1 - Z} \right] \quad (17n)$$

$$E\{QT_r\} = D \frac{1 - D^N}{1 - D} + 2 \frac{D}{1 - D/W} \left[\frac{1 - W^N}{1 - W} - \frac{1 - D^N}{1 - D} \right]. \quad (17o)$$

With the above formulas we can evaluate the $E\{S(\omega)\}$ and $\text{Var}\{S(\omega)\}$ in (10) in terms of the averages $W(\omega)$, $Z(\omega)$, $B(\omega)$, $C(\omega)$, and $D(\omega)$ of the individual jitter transfer functions $W_i(\omega)$. These quantities can be estimated either experimentally from a sufficient number of samples of $W_i(\omega)$, or numerically, using a Monte Carlo technique, from an appropriate model of $W_i(\omega)$. The second approach will be used in our examples. If $W_i(\omega) = W(\omega)$ $i = 1, 2, \dots, N$ then $B = |W(\omega)|^2$, $W = W(\omega)$, and after some algebra we can show that relation (12) is equivalent to relation (8).

Another quantity, beyond $S(\omega)$, which is important in jitter accumulation, is σ_o , the RMS value of the output jitter. We shall also evaluate its expected value and its variance.

The expected value is obtained easily from (9) and (10a) as

$$E\{\sigma_o^2\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E\{S(\omega)\} d\omega. \quad (18)$$

The evaluation of the variance of σ_o^2 is more difficult. This is done in Appendix A. However, the result is complicated because it involves the evaluation of $R(u, v) = E\{W_i(u)W_i(v)\}$, the autocorrelation of $W_i(\omega)$. To avoid the additional computations needed for the calculation of $R(u, v)$, we obtain an upper and lower bound for $\text{Var}\{\sigma_o^2\}$ by assuming that $W_i(\omega_1)$ and $W_i(\omega_2)$ are, respectively, highly correlated or uncorrelated. Then we can derive (see Appendix A) that

$$\frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \text{Var}\{S(\omega)\} d\omega \right|^{1/2} < \text{Var}\{\sigma_o^2\} < \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Var}\{S(\omega)\} d\omega, \quad (19)$$

where $\text{Var}\{S(\omega)\}$ is obtained from (10b). The average value of the two bounds appears to be a good estimator of the true variance of the RMS. With the relations given above we can now evaluate $E\{S(\omega)\}$, $\text{Var}\{S(\omega)\}$, $E\{\sigma_o^2\}$, $\text{Var}\{\sigma_o^2\}$.

In deriving $E\{S(\omega)\}$ in (10a) and $\text{Var}\{S(\omega)\}$ in (10b), we have made the additional assumption that $\Phi(\omega)$, the spectrum of the additive jitter $e_i(t)$, is the same for all the regenerators. This appears to contradict our assumption of different $W_i(\omega)$, since it is well known that $\Phi(\omega)$ is highly correlated with the transfer function of the regenerator. This problem can be resolved if we define $e_{is}(t)$, the systematic component of $e_i(t)$, as the part that is identical, within a constant, for all the repeaters, i.e., $e_{is}(t) = \alpha_i e_s(t)$, where α_i depends on $W_i(\omega)$. The random component $e_{ir}(t)$, the remaining part of $e_i(t)$, has a spectrum $\Phi_i(\omega)$ and is statistically independent for each repeater. Then in relation (5) we have to replace $W_i(\omega)$ by $\alpha_i W_i(\omega)$, while in relation (4) we must replace $|W_i(\omega)|^2$ by $\Phi_i(\omega) |W_i(\omega)|^2$. Using these substitutions, all the remaining expressions are still valid. The evaluation of α_i and $\Phi_i(\omega)$ in terms of the transfer function $W_i(\omega)$ is a difficult problem directly related to the problem of expressing the statistical properties of $e_i(t)$ in terms of the regenerator's transfer function. This question has been considered in Refs. 4 through 6.

III. APPLICATIONS

In the following three sections we shall present some numerical applications of the above theory. In Section 3.1 we shall examine chains employing phase-locked loops in their timing circuits.^{4,6} In Section 3.2 the timing circuit employs a SAW filter.^{7,8} Finally, in Section 3.3 we shall examine a special case, where $W_i(\omega)$ is assumed to exhibit a random variation only in its phase while its amplitude is identical for all repeaters. This case is of interest when we have a maximally flat filter and we want to examine the spectrum of the accumulated jitter for low frequencies.

All the numerical results are normalized by assuming $\Phi_s(\omega) = \Phi_r(\omega) = 1 \text{ deg}^2/\text{MHz}$.

3.1 Timing circuits with PLL

Phase-locked loops have been used extensively in the timing circuits of regenerative repeaters and their jitter performance has been examined thoroughly.^{3,4,6} If the timing extractor employs a PLL, its jitter transfer function $W(\omega)$ is equal to the phase transfer function of the PLL, under the assumption that the output of the phase detector is small. This is equivalent to the assumption of having small alignment jitter. (*Alignment jitter* refers to the deviations in alignment between the clock embedded in the incoming data stream of the regenerator and the timing clock derived from the data stream by the timing circuit of the regenerator.) The above requirement is usually satisfied, validating the linear model of the PLL. The phase transfer function of a PLL used in the literature is typically described with a second-order

model. If we want to consider parasitic elements in the PLL, then, as M. W. Hall suggested,³ a fifth-order model is more realistic, especially for high frequencies. However, the spectrum of the accumulated jitter is significant only in the low frequencies. Hence the second-order model provides an adequate description of the PLL in this frequency range, because it coincides with the fifth-order model in this range of frequencies.

Thus the jitter transfer function $W_i(\omega)$ of the i th repeater can be modeled as

$$W_i(s/j) = \frac{2\zeta_i\omega_{ni}s + \omega_{ni}^2}{s^2 + 2\zeta_i\omega_{ni}s + \omega_{ni}^2} \quad s = j\omega, \quad (20)$$

where ζ_i is the damping coefficient of the PLL and ω_{ni} its natural frequency.

We shall assume that the average values of ζ_i and ω_{ni} are $\bar{\zeta}_i = 6$ and $\bar{\omega}_{ni} = 4.5$ kHz. Also, the bandwidth of the PLL is assumed to vary from 20 kHz up to 80 kHz. To simulate the above conditions we assumed that ζ_i is uniformly distributed from 4 to 8 and ω_{ni} is also uniformly distributed from 2 kHz to 7 kHz. Using the above numbers, we calculated $W(\omega)$, $Z(\omega)$, $B(\omega)$, $C(\omega)$, $D(\omega)$ (see 17a, b, c, d, e) by numerically averaging 1000 samples of $W_i(\omega)$. In Fig. 2 we show the amplitude and phase of $W(\omega)$ as well as two samples of $W_i(\omega)$ with $\zeta_i = 6$ and $\omega_{ni} = 2$ kHz and 7 kHz. The shaded area shows the permissible range of $W_i(\omega)$. Bandwidth varies between 16 kHz and 110 kHz, while phase slope varies between -0.6 deg/kHz and -3.2 deg/kHz. In Fig. 3a we plot $E\{T_s(\omega)\}$, and $E\{T_s(\omega)\} \pm \text{Var}\{T_s(\omega)\}$ for a chain of 50 repeaters. Also in the same graph we show the $T_s(\omega)$ evaluated by using relation (8), i.e., assuming that all repeaters have identical PLLs with $W(\omega) = E\{W_i(\omega)\}$. In Fig. 3b we show the same results for 200 repeaters. Figure 4 shows the same curves for the random jitter component.

To check our theoretical results, we did a complete numerical Monte-Carlo simulation of 200 chains and computed the various parameters, including $E\{\sigma_o^2\}$ and $\text{Var}\{\sigma_o^2\}$. The resulting graphs were indistinguishable from those shown in Figs. 3 and 4; the numerical values obtained for $E\{\sigma_o^2\}$ and $\text{Var}\{\sigma_o^2\}$ are shown in Table I. All the numbers agree with the results of our theoretical analysis. Also, the expected value of the RMS of the accumulated jitter, systematic and random, is shown in Fig. 5 with the $\pm 3 \text{Var}\{\sigma_o^2\}$ curves (99.7 percent confidence). Also in the same graph we plot the RMS values for the accumulated jitter for a chain having identical repeaters with jitter transfer function equal to the average value $E\{W_i(\omega)\}$.

We can draw the following conclusions from the previous example:

(1) The model of identical repeaters having as jitter transfer function

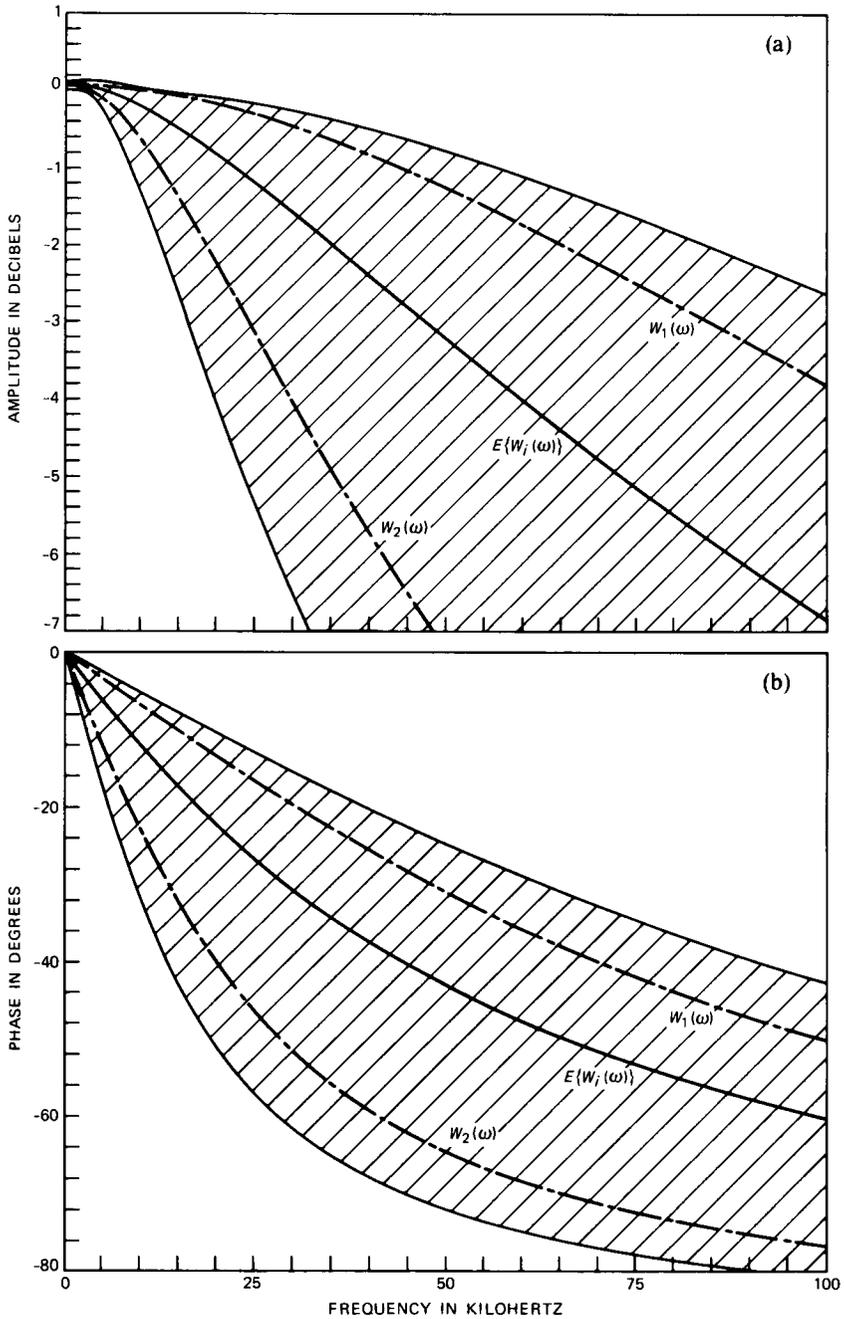


Fig. 2—(a) Amplitude and (b) phase of $W_1(\omega)$, $W_2(\omega)$, $E\{W_1(\omega)\}$ for the PLL model. $W_1(\omega)$ is obtained with $\zeta = 6$ and $\omega_{ni} = 7$ kHz. $W_2(\omega)$ is obtained with $\zeta = 6$ and $\omega_{ni} = 2$ kHz. Shaded area indicates the permissible range of $W_1(\omega)$.

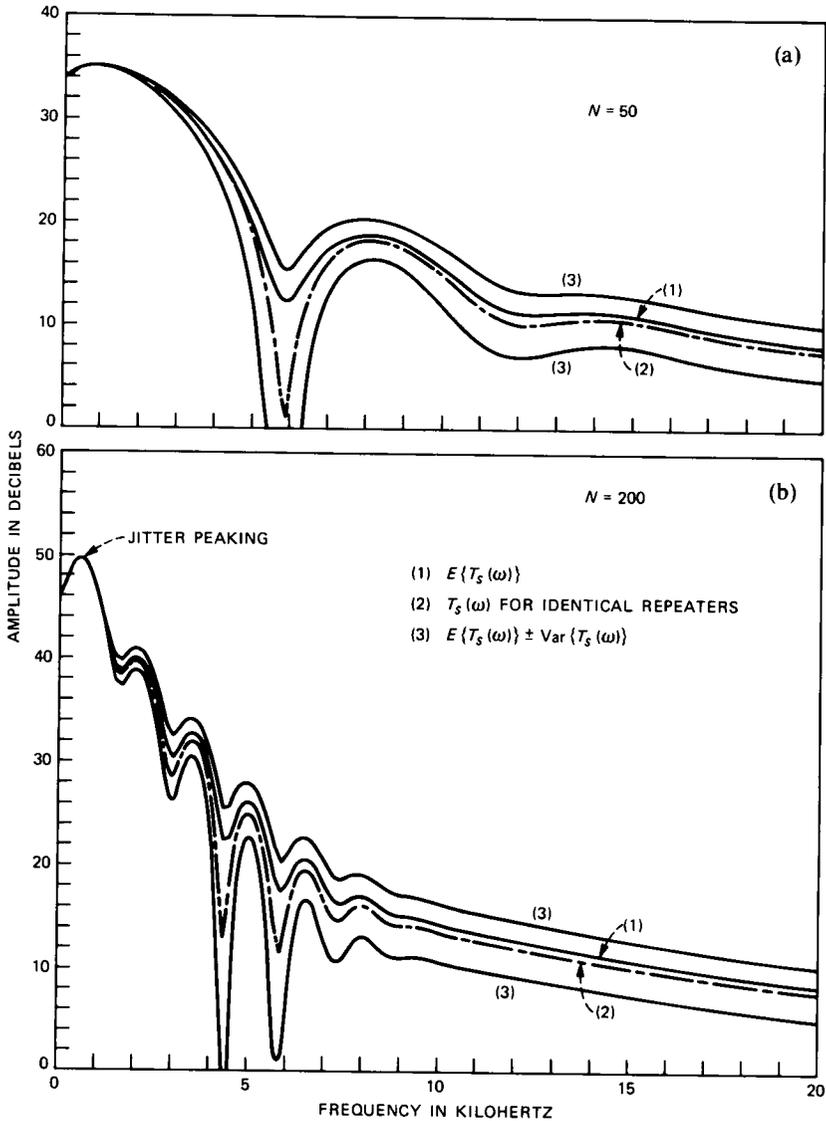


Fig. 3—Total transfer function $T_s(\omega)$ for systematic jitter for (a) 50 repeaters and (b) 200 repeaters with PLL using the stochastic model and the model with identical repeaters.

the average jitter transfer function underestimates slightly the average RMS value and the power spectrum density. (2) The variance of the RMS jitter was only 5 percent (see Table I and Fig. 5) even if the bandwidth of the PLL jitter transfer function varied from 16 kHz up to 110 kHz. This implies that the model of identical repeaters provides

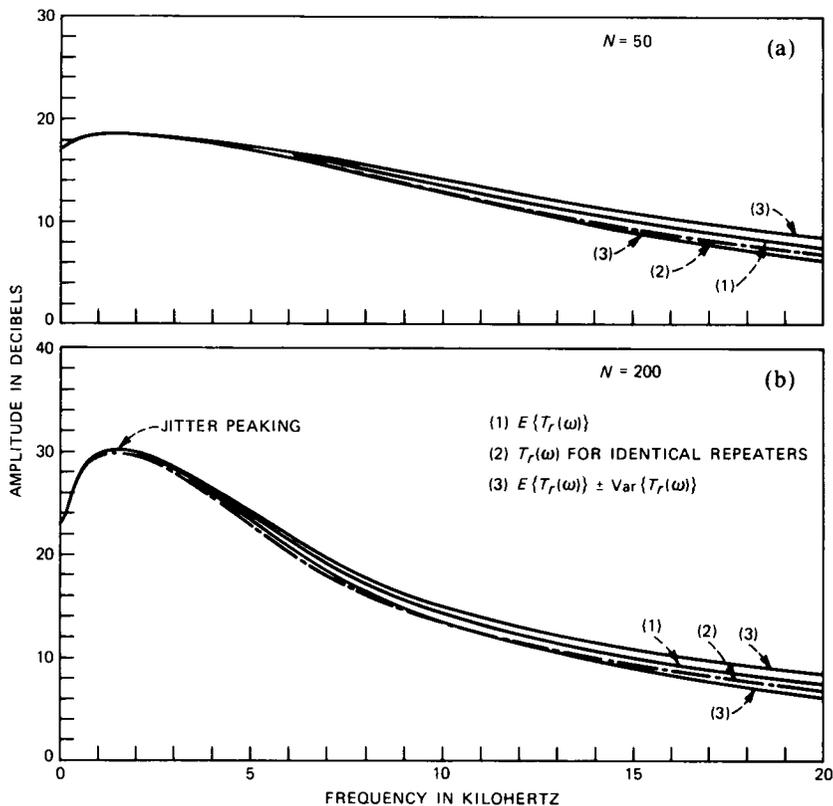


Fig. 4—Total transfer function $T_r(\omega)$ for random jitter for (a) 50 repeaters and (b) 200 repeaters with PLL using the stochastic model and the model with identical repeaters.

us with a reliable estimation of the RMS jitter. (3) The total transfer function $T_r(\omega)$ has a smaller variance than $T_s(\omega)$.

3.2 Timing circuits with SAW filters

The introduction of optical-fiber transmission systems has made possible data transmission rates of several hundred megabits per second. This introduced significant changes in the construction of the timing extracting circuits in the regenerative repeaters. The popular PLLs had to be replaced, because their implementation above 100 Mb/s has been difficult, especially in integrated circuit form. Currently, SAW filters⁷ have emerged as their replacements. This actually represents a return to passive filtering after many years of using the PLL.

In this section we analyze a system where the tuned filter in the timing circuit is a passive SAW filter. In the present analysis we will

Table 1—Theoretical and numerical values for $E\{\sigma_c^2\}$ and $\text{Var}\{\sigma_c^2\}$ for chains with 50, 100, 200, and 300 repeaters

N	$E\{\sigma_c^2\} \text{ deg}^2 *$						$\text{Var}\{\sigma_c^2\}$					
	Using Random Model		Using Identical Repeaters		Numerical Result (200 chains)		Lower Bound		Upper Bound		Numerical Result (200 chains)	
	Syst.	Ran.	Syst.	Ran.	Syst.	Ran.	Syst.	Ran.	Syst.	Ran.	Syst.	Ran.
50 PLL	18.57	1.44	18.20	1.36	18.61	1.45	0.18	0.01	1.85	0.18	1.06	0.15
	28.94	8.45	28.82	8.00			0.08	0.05	0.93	0.90		
100 PLL	51.49	2.70	50.57	2.50	51.57	2.72	0.56	0.01	4.50	0.26	2.27	0.18
	58.17	13.94	57.98	13.01			0.23	0.08	1.53	1.47		
200 PLL	188.85	7.73	184.55	7.05	189.16	7.77	2.68	0.05	17.82	0.60	8.61	0.36
	116.67	22.85	116.38	20.97			1.02	0.11	3.27	2.18		
300 PLL	540.64	22.41	522.45	20.12	550.53	22.68	10.12	0.21	62.89	1.80	32.93	1.25
	175.19	30.43	174.83	27.58			2.02	0.15	4.87	2.80		

* The input jitter is $\Phi_s(\omega) = \Phi_r(\omega) = 1 \text{ deg}^2/\text{MHz}$.

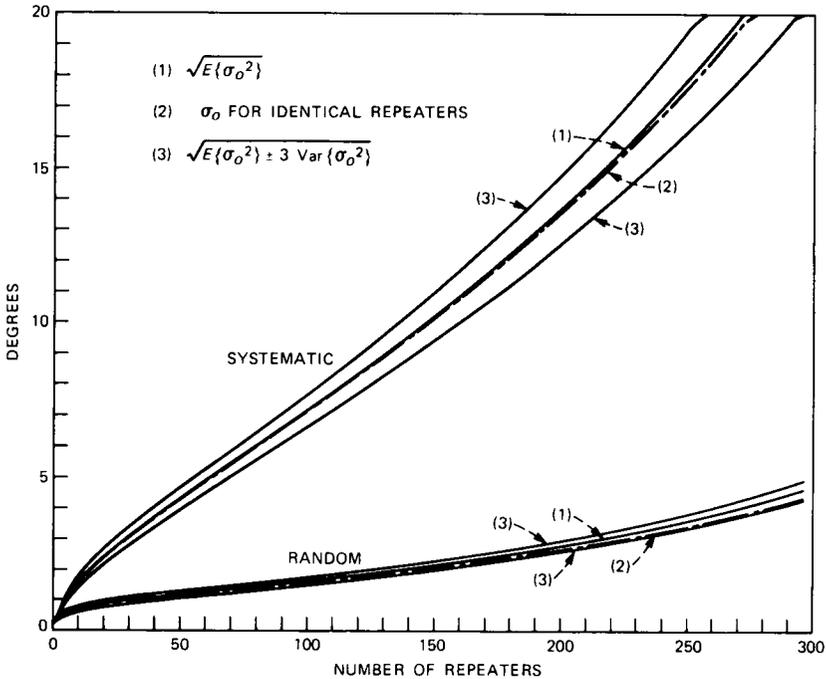


Fig. 5—RMS values of systematic and random jitter for repeaters with PLLs. Curves show: (1) stochastic model; (2) conventional model with identical repeaters; and (3) 99.7-percent confidence interval.

not consider the effects introduced by the prefilter and nonlinear device, which often precede the tuned circuit. It can be shown that this simplification is not restrictive.² A simple proof is also presented in Appendix B.

Let $H(\omega) = A(\omega)e^{j\phi(\omega)}$ be the transfer function of the SAW filter. Then, as has been shown,² the jitter transfer function of the regenerator is approximately given by

$$W(\omega) = \frac{H(\omega - \omega_0)e^{j\phi(\omega_0)} + H(\omega + \omega_0)e^{-j\phi(\omega_0)}}{2A(\omega_0)}, \quad \text{for } |\omega| < \omega_0, \quad (21)$$

where ω_0 is the baud rate of the data. Relation (21) is valid under the assumption that the accumulated jitter does not have large components at high frequencies. The exact conditions for the validity of (21) will be studied in a forthcoming work. A simple derivation of (21) is given also in Appendix B.

Let us now define the normalized low-pass equivalent of $H(\omega)$ as

$$H_L(\omega) = \frac{H(\omega + \omega_0)e^{-j\phi(\omega_0)}}{A(\omega_0)} \quad \text{for } \omega > -\omega_0. \quad (22)$$

Thus $W(\omega)$ is now given by

$$W(\omega) = \frac{[H_L(\omega) + H_L^*(-\omega)]}{2}, \quad (23)$$

where * denotes complex conjugate.

A model for transversal SAW filters related to its design electrical characteristics, i.e., number of fingers, distance of transducers, terminating impedances, etc, is given in Ref. 8. Since in the present simulation we are interested only in its transfer function, we will use a simpler representation, and model $H_L(\omega)$, the low-pass equivalent of the SAW filter, as a two-pole filter. Its transfer function is

$$H_L(s) = \frac{s_1 s_2}{(s - s_1)(s - s_2)} e^{-cs}, \quad (24)$$

where

$$s_i = -a_i + jb_i = -(1 \pm \epsilon)a + jb_i \quad i = 1, 2$$

are its two poles and c is a linear phase slope used to correct the phase of the model, since SAW filters are not minimum phase filters. For jitter studies our model provides an adequate description for SAW filters.

Denoting by BW the bandwidth of the SAW filter, ω_c its center frequency, and ω_0 the baud rate of the data, the various SAW filter parameters can be defined in terms of α , b_i , and the asymmetry factor ϵ as follows:

$$\omega_c - \omega_0 = (b_1 + b_2)/2 \quad \text{mistuning} \quad (25)$$

$$b = (b_1 - b_2)/2 \quad a = (a_1 + a_2)/2$$

$$\alpha = 2(\omega_c - \omega_0)/BW \quad \text{detuning factor}$$

$$\omega_n^2 = a^2 + b^2 \quad \text{natural frequency}$$

$$\zeta = a/\omega_n \quad \text{damping factor}$$

$$Q = \omega_c/BW \quad \text{filter's quality factor.} \quad (26)$$

The natural frequency ω_n can be determined approximately from the BW (for $\epsilon \ll 1$) by

$$\omega_n^2 = \left(\frac{BW}{2}\right)^2 \frac{1}{-(1 - 2\zeta^2) + ((1 - 2\zeta^2)^2 + 1)^{1/2}}. \quad (27)$$

For the simulation we used the following numerical values for the range of bandwidth BW , static offset, damping factor ζ , and asymmetry factor ϵ :

$$160 \text{ kHz} < \frac{BW}{2} < 240 \text{ kHz}$$

$$|f_c - f_0| < 50 \text{ kHz}$$

$$0.60 < \zeta < 0.80$$

$$-0.1 < \epsilon < 0.1$$

$$c = -0.2 \text{ deg/kHz.} \quad (28)$$

We assumed the above parameters to be uniformly distributed between their upper and lower limits. If the baud rate of the data is 300 MHz, then the above values correspond to filters having Q 's between 625 and 940. To illustrate the relation of the above parameters to $H_L(\omega)$ and $W(\omega)$, we plot in Fig. 6 the passband of the SAW filter and the corresponding jitter transfer functions for $\zeta = 0.65$ and various combinations of ϵ and α . Notice that the frequency scale has been normalized to $BW/2$, the bandwidth of $H_L(\omega)$.

In Fig. 6a we plot the passband of the SAW filters for $\zeta = 0.65$ and $\epsilon = 0, 0.1$. In Fig. 6b we plot the corresponding jitter transfer function

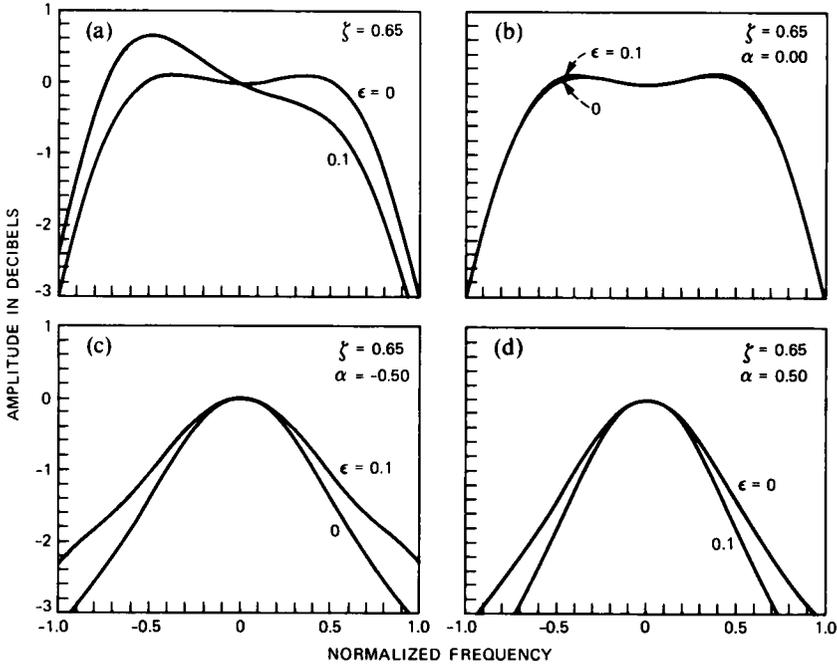


Fig. 6—(a) Underdamped SAW filter with $\zeta = 0.65$ and asymmetry factor $\epsilon = 0$ and 0.1 . Jitter transfer functions obtained with various detuning factors $\alpha = 2(\omega_c - \omega_0)/BW$ for (b) $\alpha = 0$ (0.105 dB jitter peaking), (c) $\alpha = -0.50$, and (d) $\alpha = 0.50$.

$W(\omega)$ when $f_c = f_0$ ($\alpha = 0$). For this case the asymmetry is almost canceled. This is expected because from relation (23) we can see that asymmetries in $H_L(\omega)$ that are odd with respect to f_c will be canceled when we form $W(\omega)$. In Fig. 6c we plot the corresponding jitter transfer function when the detuning factor is $\alpha = -0.5$, i.e., the baud rate is 100 kHz less than the center frequency of the filter. Figure 6d shows the jitter transfer function for $\alpha = 0.5$. In Fig. 7 we plot $E\{W_i(\omega)\}$ (1000 samples) as well as two extreme samples of $W_i(\omega)$. Most of the $W_i(\omega)$ will be between $W_1(\omega)$ and $W_2(\omega)$. The shaded area shows the permissible range of $W_i(\omega)$. In Fig. 8 we plot the average total transfer function for systematic jitter, $E\{T_s(\omega)\}$, and $E\{T_s(\omega)\} \pm \text{Var}\{T_s(\omega)\}$ for 50 and 200 repeaters. The variance of $T_s(\omega)$ in this example is much less than the variance of the example with PLL. This is due to the narrow distribution of the SAW filter phase slopes (-0.5 deg/kHz to -0.8 deg/kHz). Finally, in Fig. 9 we plot the expected value of the RMS of the accumulated jitter, systematic and random, with the $\pm 3 \text{Var}\{\sigma_0^2\}$ curves (99.7 percent confidence). Some numerical values for the accumulated RMS jitter and its variance are shown in Table I for $N = 50, 100, 200,$ and 300 . To find the true RMS value we have to multiply the numbers shown in Table I with the $\Phi_s(\omega)$ and $\Phi_r(\omega)$ measured in deg²/MHz. For example, if $N = 200$ and a PLL is used, then assuming $\Phi_s = 20$ deg²/MHz and $\Phi_r = 5$ deg²/MHz, we obtain

$$\sigma_o = [20 \cdot 189 + 5 \cdot 7.73]^{1/2} = 61.8 \text{ degrees.} \quad (29)$$

Using the same numbers for the SAW filters we obtain

$$\sigma_o = [20 \cdot 117 + 5 \cdot 22.9]^{1/2} = 49.5 \text{ degrees.} \quad (30)$$

Thus the simulated SAW filters accumulate less jitter, even with a bandwidth larger than the bandwidth of PLLs. This is due to the fact that PLLs have an inherent jitter peaking and because their phase is smaller than the SAW filters [see eq. (40)]. To obtain the corresponding peak-to-peak values we usually multiply the RMS value with a peak-to-peak/RMS factor. Typical values for this factor are between 8 and 15.

From the above example it becomes clear that, using our random model, we can tolerate larger manufacturing variances, because we can accept retiming circuits exhibiting substantial jitter peaking. For example, $W_1(\omega)$ (0.4 dB jitter peaking) in Fig. 7 can be accepted if the expected average jitter transfer function $W(\omega)$ of the manufactured SAW filters possess a moderate jitter peaking (i.e., less than 0.1 dB).

3.3 Timing circuits with random phase

In this section we will consider the dependence of the jitter power spectrum on phase variations. This case is of interest when we want

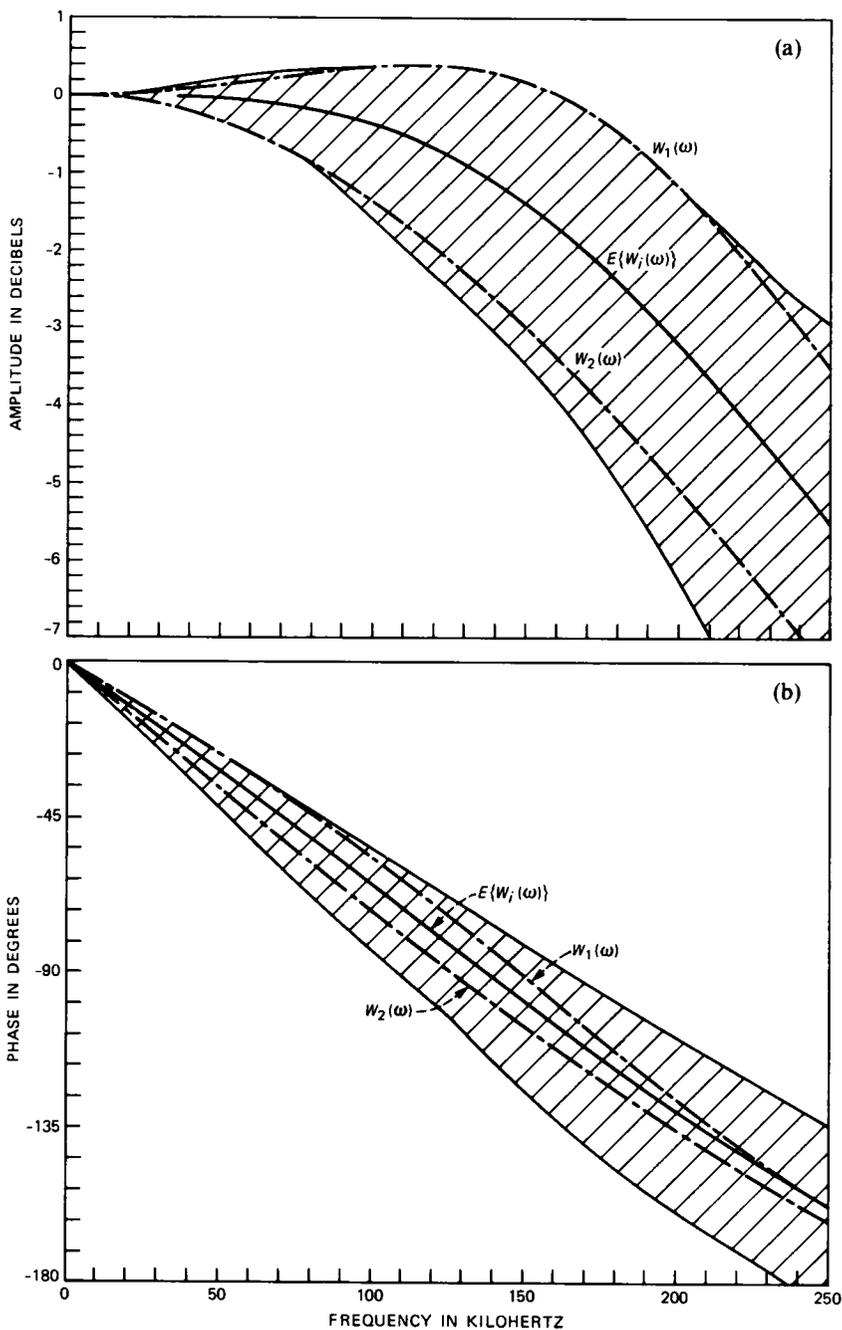


Fig. 7—(a) Amplitude and (b) phase of $E\{W_i(\omega)\}$, $W_1(\omega)$, $W_2(\omega)$ for the SAW model. $W_1(\omega)$ is obtained with $BW/2 = 240$ kHz, $\alpha = 0$, $\zeta = 0.60$, and $\epsilon = 0.1$ (0.4 dB jitter peaking). $W_2(\omega)$ is obtained with $BW/2 = 160$ kHz, $\alpha = 0.25$, $\zeta = 0.60$, and $\epsilon = 0.1$. Area with lines indicates the permissible range of $W_i(\omega)$.

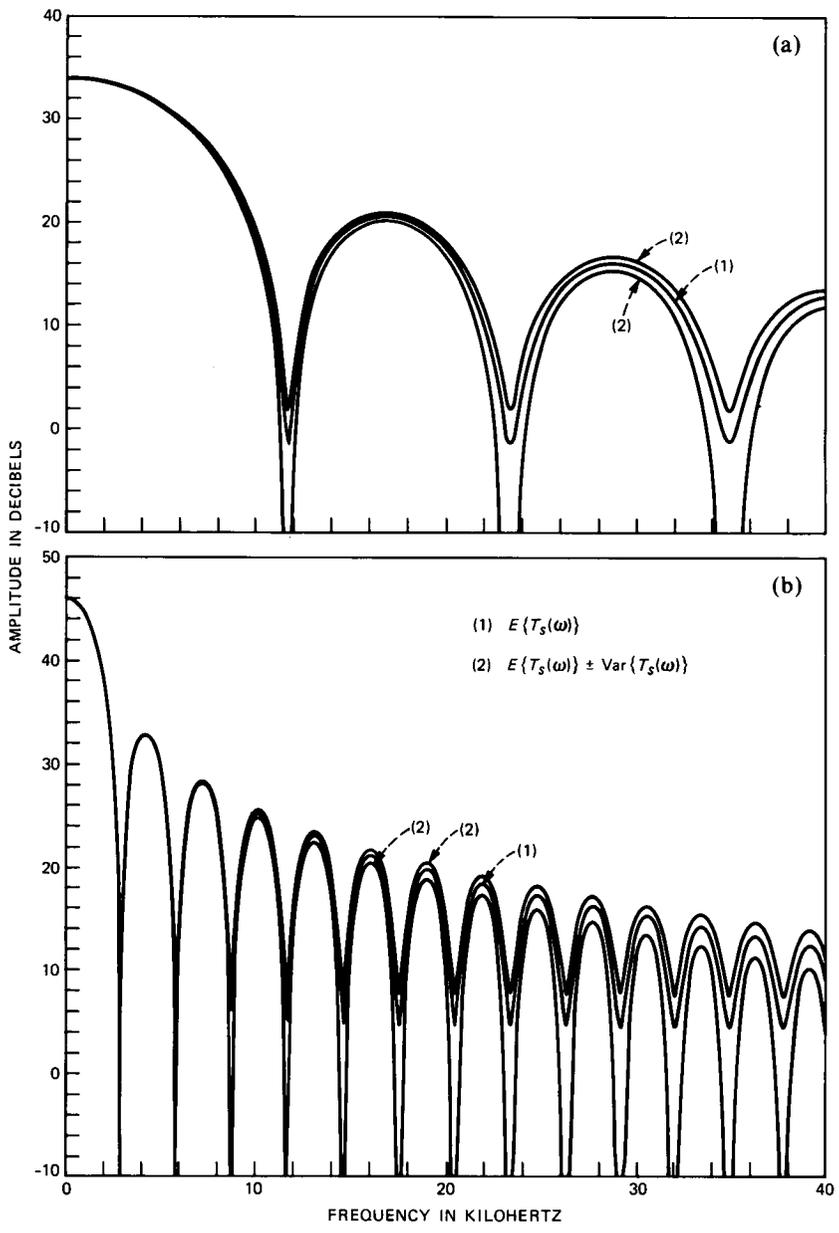


Fig. 8—Total systematic jitter transfer function $T_s(\omega)$ and its variance for 50 and 200 repeaters.

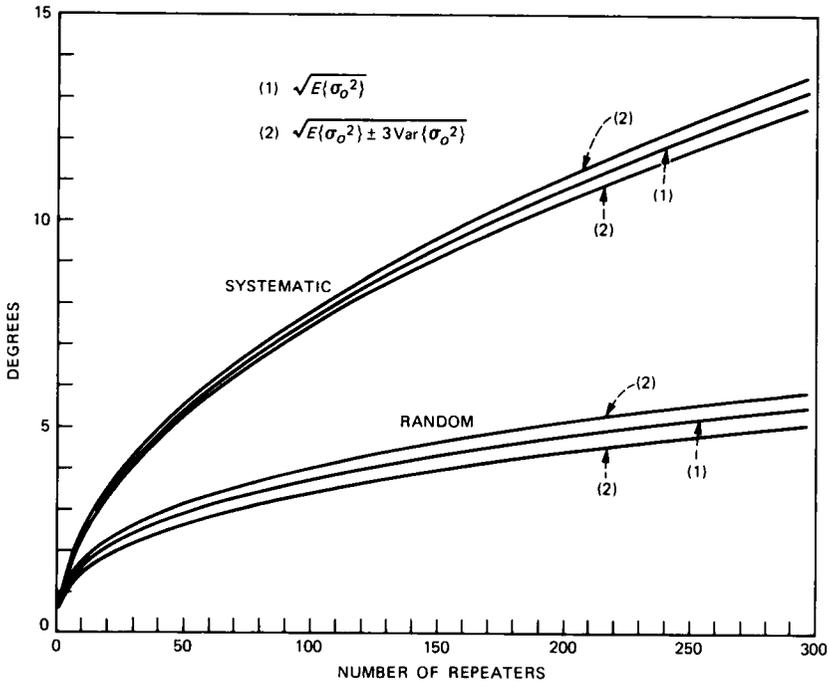


Fig. 9—RMS values of systematic and random jitter for repeaters with SAW filters.

to estimate the jitter spectral density for low frequencies.⁹ We shall assume that

$$W_i(\omega) = A(\omega)e^{j[\phi(\omega)+\eta_i(\omega)]}, \quad (31)$$

where $\eta_i(\omega)$ are random variables with zero mean and independent and identically distributed (i.i.d.). Let $\Phi_\eta(X)$ be the characteristic density function of η_i , i.e.,

$$\Phi_\eta(X, \omega) = E\{e^{j\eta_i(\omega)X}\} = \int_{-\infty}^{\infty} f_\eta(X, \omega)e^{j\eta X} d\eta. \quad (32)$$

Then [see relations (17)] all the needed statistics for $W_i(\omega)$ can be obtained analytically if $\Phi_\eta(X, \omega)$ is known, i.e.,

$$\begin{aligned} W(\omega) &= E\{W_i(\omega)\} = A(\omega)e^{j\phi(\omega)} \Phi_\eta(1, \omega) \\ Z(\omega) &= E\{W_i^2(\omega)\} = A^2(\omega)e^{j2\phi(\omega)} \Phi_\eta(2, \omega) \\ D(\omega) &= E\{|W_i(\omega)|^2 W_i(\omega)\} = A^3(\omega)e^{j\phi(\omega)} \Phi_\eta(1, \omega). \end{aligned} \quad (33)$$

If $\eta_i(\omega)$ is assumed to have zero mean and to be uniformly distributed between $-\Psi(\omega)$ and $\Psi(\omega)$, then

$$\Phi_{\eta}(X, \omega) = \frac{\sin \Psi(\omega)X}{\Psi(\omega)X}. \quad (34)$$

If $\eta_i(\omega)$ is assumed to be zero mean and Gaussian distributed with variance $\sigma(\omega)$, then

$$\Phi_{\eta}(X, \omega) = e^{-X^2\sigma^2(\omega)/2}. \quad (35)$$

Using (34), (35), (33), and (10) we can obtain $E\{T_s(\omega)\}$ and $\text{Var}\{T_s(\omega)\}$. In Fig. 10 we plot $E\{T_s(\omega)\}$ for a transversal filter with maximally flat

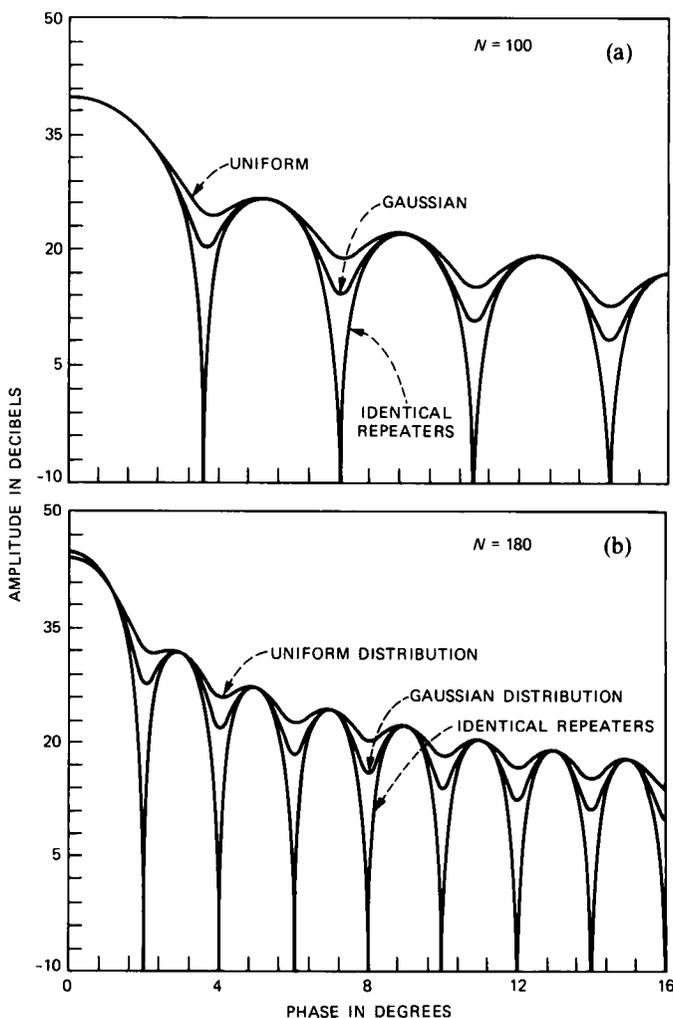


Fig. 10—Systematic jitter power density versus phase for identical repeaters, repeaters with uniform phase distribution, and repeaters with Gaussian phase distribution for (a) $N = 100$ repeaters and (b) $N = 180$ repeaters.

transfer function ($\zeta = \sqrt{2}/2$) as a function of its phase $\phi(\omega)$ for 100 and 180 regenerators for the cases $\eta_i = 0$, η_i uniformly distributed with $\Psi_i(\omega) = 8.1^\circ$ and η_i gaussian distributed with $\sigma_i(\omega) = 8.1^\circ/3$.

In Fig. 11 we plot $E\{T_s\}$ and its variance for the case of uniform distribution.

Finally, we would like to note the following approximate relations.

1. Large variance approximation—If the variance of the phase is large for $\omega = \omega_0$, then $W(\omega_0) \ll 1$ because $\Phi_\eta(1, \omega_0)$ is small [see relations (33) and (34)], and from relation (12) we obtain

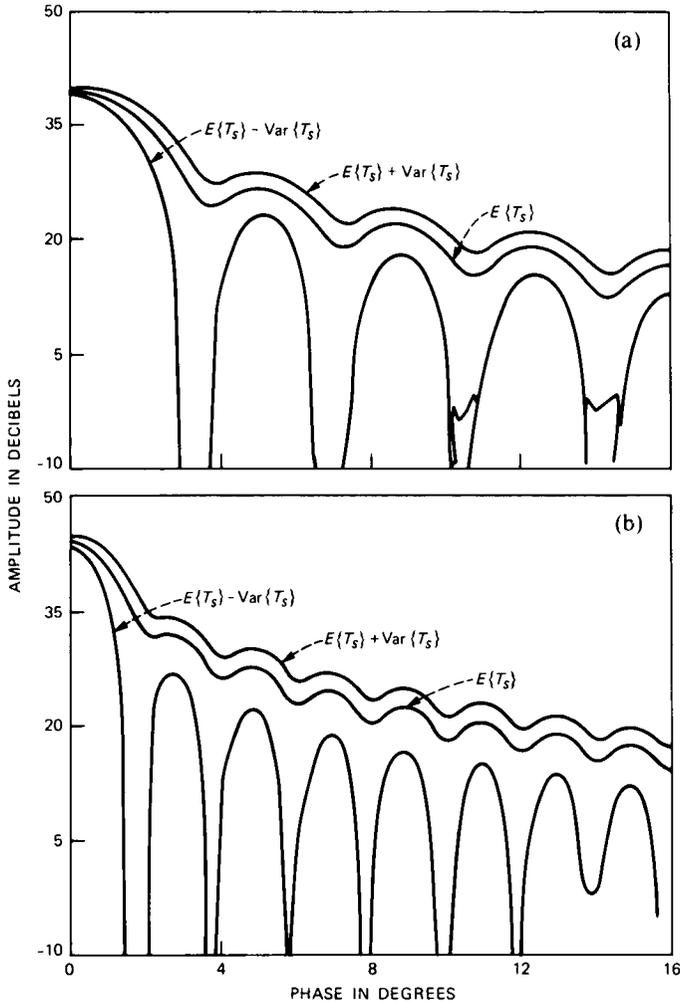


Fig. 11—Mean and variance of systematic jitter power density versus phase for repeaters with uniform phase distribution for (a) $N = 100$ repeaters and (b) $N = 180$ repeaters.

$$\lim_{\sigma(\omega_0) \rightarrow \infty} E\{T_s(\omega_0)\} = E\{T_r(\omega_0)\}. \quad (36)$$

This is expected because the randomness of the phase removes the coherent accumulation of the systematic jitter for $\omega = \omega_0$.

2. Low frequencies and long chain approximation—For large N the jitter energy is concentrated near $\omega = 0$. For this region, we can assume that $A(\omega) \approx 1$ and approximate relation (12) with

$$E\{T_s\} \approx \text{Re} \left\{ \frac{2[1 - W^N]W - N(1 - W^2)}{(1 - W)^2} \right\} \quad \omega \approx 0. \quad (37)$$

3. High frequencies and long chain—For high frequencies where $A(\omega) \ll 1$ and large N we obtain that $E\{T_s\} \approx E\{T_r\} = E\{|W_i|^2\}$.

4. Small-phase variance approximation—If we assume that the variance $\sigma(\omega)$ of the phase $\phi(\omega)$ is small, after some algebra we can obtain the following relation:

$$E\{T_s\} \approx A_N + \sigma^2(\omega) \sum_{n=1}^N A_n, \quad (38)$$

where

$$A_n(\omega) = |A(\omega)|^2 \left| \frac{1 - A^n(\omega)e^{jn\phi(\omega)}}{1 - A(\omega)e^{j\phi(\omega)}} \right|^2$$

and

$$\sigma^2(\omega) \ll \frac{1}{N}.$$

For low frequencies, $\omega \approx 0$, we can assume $A(\omega) \approx 1$ and the term $A_n(\omega)$ in relation (38) can be approximated with

$$A_n(\omega) = \left| \frac{\sin \frac{n\phi(\omega)}{2}}{\sin \frac{\phi(\omega)}{2}} \right|^2. \quad (39)$$

5. Linear phase, low frequencies, small variance approximation—Let us assume $\phi(\omega) \approx -\alpha\omega$ for $\omega \approx 0$. Then $\sigma^2(\omega) = S^2\omega$, where S is the variance of α , and relation (39) becomes

$$A_n(\omega) = \left| \frac{\sin \frac{n\bar{\alpha}\omega}{2}}{\sin \frac{\bar{\alpha}\omega}{2}} \right|^2.$$

For large N and assuming no jitter peaking and small-phase variance

we can approximate the variance σ_o^2 of the systematic jitter at the output of N regenerators with

$$E\{\sigma_o^2\} = \Phi_s(0) \left[\frac{0.36}{\bar{\alpha}} N + \frac{S^2}{3} \left(\frac{0.36}{\bar{\alpha}} \right)^3 N^2 \right] \text{deg}^2, \quad (40)$$

where

- $\Phi_s(0)$ is the spectral density in deg^2/MHz
- $\bar{\alpha}$ is the average phase slope in deg/kHz
- S^2 is the variance of α
- N is the number of regenerators.

Relation (40) shows that for long chains of regenerators exhibiting no jitter peaking the RMS value of the systematic accumulated jitter is determined mainly by the dc phase slope of the jitter transfer function and not by its shape. This is illustrated in Fig. 12, where we evaluate the RMS value of the systematic accumulated jitter for chain, consisting of identical regenerators having the jitter transfer function the $W(\omega)$ shown in Fig. 7. A curve using numerical integration is compared versus the curve predicted by the simple formula $\sigma_o = 0.6\sqrt{N}/\alpha$ ($\alpha = 0.63 \text{ deg}/\text{kHz}$). For a first-order filter with band-

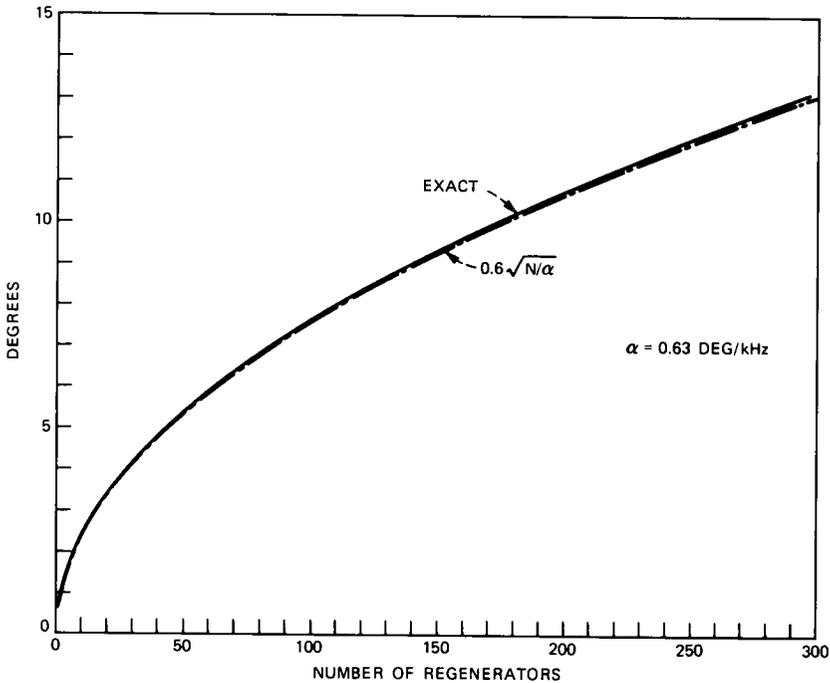


Fig. 12—RMS systematic jitter accumulation using numerical integration and relation (40).

width B , we have $B = 0.36/\alpha$ and the above relation becomes $\sigma_o = \sqrt{BN}$. This is the widely used relation (69) given in Ref. 1, which is valid only for first-order filters.

IV. CONCLUSION

We have presented a generalized model for the accumulation of jitter. This model differs from those used previously in that we do not assume all the repeaters have the same transfer functions. We have derived analytical expressions for the variance and the mean of the accumulated jitter in terms of the repeater's jitter transfer function $W_i(\omega)$.

We have also presented some numerical results for PLL and SAW filters. From our numerical simulations we found that for long chains the variance of the RMS jitter is about 5 percent. This implies that we can still reliably estimate RMS jitter by assuming that the jitter transfer function of all the repeaters is equal to the average jitter transfer function. Another result obtained from our modeling is that retiming circuits exhibiting large jitter peaking are acceptable if their average jitter transfer function does not have jitter peaking.

V. ACKNOWLEDGMENTS

I would like to thank R. L. Rosenberg for his help on modeling the SAW filters and for his comments on various aspects of this work. Also I would like to express my appreciation to P. K. Runge for his continuous encouragement.

REFERENCES

1. C. J. Byrne, B. J. Karafin, and D. B. Robinson, Jr. "Systematic Jitter in a Chain of Digital Regenerators," *B.S.T.J.*, 42 (November 1963), pp. 2679-714.
2. U. Mengali and G. Pirani, "Jitter Accumulation in PAM Systems," *IEEE Trans. Commun.*, COM-28, No. 8 (August 1980), pp. 1172-83.
3. M. W. Hall, private communication.
4. E. Rosa, "Analysis of Phase-Locked Timing Extraction Circuits for Pulse Code Transmission," *IEEE Trans. Commun.*, COM-22 (September 1974), pp. 1236-49.
5. L. E. Franks and J. P. Bubrowski, "Statistical Properties of Timing Jitter in a PAM Timing Recovery Scheme," *IEEE Trans. Commun.*, COM-22 (July 1974), pp. 913-20.
6. D. L. Duttweiler, "The Jitter Performance of Phase-Locked Loops Extracting Timing From Baseband Data Waveforms," *B.S.T.J.*, 55 (January 1976), pp. 37-58.
7. R. L. Rosenberg, C. Chamzas, and D. A. Fishman, "Timing Recovery With SAW Transversal Filters in the Regeneration of Undersea Long-Haul Fiber Transmission Systems," to be published in publication *IEEE J. Selected Areas Commun.* (Joint Issue with *IEEE/OSA Journal of Lightwave Technology*), special issue on Undersea Lightwave Communications, December 1984.
8. D. A. Fishman, R. L. Rosenberg, and C. Chamzas, unpublished work.
9. C. D. Anderson and D. L. Keller, "SL Supervisory Analysis of Jitter-Peaking Effects in Digital Long-Haul Transmission Systems Using SAW Filter Retiming," to be published in *IEEE J. Selected Areas Commun.* (Joint Issue with *IEEE/OSA Journal of Lightwave Technology*), special Issue on Undersea Lightwave Communications, December 1984.
10. A. Papoulis, *Signal Analysis*, New York: McGraw-Hill, 1977.

APPENDIX A

Stochastic Evaluation of Jitter Parameters

For simplicity the dependence on ω is omitted in most of the following formulas.

A.1 Evaluation of $E\{T_r(\omega)\}$

The transfer function $T_r(\omega)$ for the random jitter is given by [see (4)]

$$T_r = |W_1|^2 + |W_1 W_2|^2 + \dots + |W_1 W_2 \dots W_N|^2. \quad (41)$$

With the assumption that the W_i are i.i.d., we obtain the expected value of T_r as

$$E\{T_r\} = B + B^2 + B^3 + \dots + B^N = B \frac{1 - B^N}{1 - B}, \quad (42)$$

where

$$B = E\{|W_i|^2\}$$

and this is the desired relation (11).

A.2 Evaluation of $E\{T_s(\omega)\}$

The transfer function $T_s(\omega)$ for the systematic jitter is given [see (5)] by

$$T_s = \left| \sum_{k=1}^N W_1 W_2 \dots W_k \right|^2. \quad (43)$$

The expected value of T_s may be written

$$\begin{aligned} E\{T_s\} &= E \left\{ \sum_{k=1}^N W_1 W_2 \dots W_k \sum_{m=1}^N W_1^* W_2^* \dots W_m^* \right\} \\ &= E \left\{ \sum_{k=1}^N |W_1 W_2 \dots W_k|^2 \right. \\ &\quad \left. + 2 \operatorname{Re} \sum_{k=1}^N \sum_{m=1}^{k-1} |W_1 W_2 \dots W_m|^2 W_{m+1} W_{m+2} \dots W_k \right\}. \end{aligned}$$

Defining

$$W = E\{W_i\}, \quad B = E\{|W_i|^2\} \quad (44)$$

and using that W_i are i.i.d. we obtain

$$\begin{aligned} E\{T_s\} &= \sum_{k=1}^N B^k + 2 \operatorname{Re} \sum_{k=1}^N \sum_{m=1}^{k-1} B^m W^{k-m} \\ &= B \frac{1 - B^N}{1 - B} + 2 \operatorname{Re} \sum_{k=1}^N B W^{k-1} \frac{1 - (B/W)^{k-1}}{1 - (B/W)} \end{aligned}$$

or

$$E\{T_s\} = B \frac{1 - B^N}{1 - B} + 2 \operatorname{Re} \frac{B}{1 - B/W} \left\{ \frac{1 - W^N}{1 - W} - \frac{1 - B^N}{1 - B} \right\}, \quad (45)$$

which is the desired relation (12).

For large ω where $|W_i(\omega)| \ll 1$ we can obtain from (42) and (45) that

$$E\{T_s\} \approx E\{T_r\} \approx B(\omega). \quad (46)$$

A.3 Evaluation of $\operatorname{Var}\{T_s\}$

To evaluate the variance of $T_s(\omega)$, the transfer function of the random jitter, we need only to find $E\{T_r^2\}$, since

$$\operatorname{Var}\{T_r\} = (E\{T_r^2\} - E^2\{T_r\})$$

and $E\{T_r\}$ has been evaluated in (42):

$$E\{T_r^2\} = E \left\{ \sum_{k=1}^N |W_1 W_2 \dots W_k|^2 \sum_{m=1}^N |W_1 W_2 \dots W_m|^2 \right\}. \quad (47)$$

Defining $B(\omega)$ as in (44) and

$$C(\omega) = E\{|W_i(\omega)|^4\}, \quad (48)$$

we obtain

$$\begin{aligned} E\{T_r^2\} &= E \left\{ \sum_{k=1}^N |W_1 W_2 \dots W_k|^4 \right. \\ &\quad \left. + 2 \sum_{k=1}^N \sum_{m=1}^{N-1} |W_1 W_2 W_m|^4 |W_{m+1} W_{m+2} \dots W_k|^2 \right\} \\ &= \sum_{k=1}^N C^k + 2 \sum_{k=1}^N \sum_{m=1}^{N-1} C^m B^{k-m} \end{aligned}$$

or

$$E\{T_r^2\} = C \frac{1 - C^N}{1 - C} + 2 \frac{C}{1 - C/B} \left\{ \frac{1 - B^N}{1 - B} - \frac{1 - C^N}{1 - C} \right\}, \quad (49)$$

and this is relation (13).

A.4 Evaluation of $\operatorname{Var}\{T_s(\omega)\}$

A direct evaluation of the $\operatorname{Var}\{T_s\}$ is possible but the resulting formula is lengthy. To simplify our analysis we will make an additional assumption.

Let us define [see (17f)]

$$Q = \sum_{k=1}^N W_1 W_2 \dots W_k = X + jY \quad (50)$$

and

$$\bar{X} = E\{X\}, \quad \bar{Y} = E\{Y\}.$$

Then, if N is large, since W_i are independent we can assume that X and Y are jointly normal. This assumption will permit us to avoid calculations of fourth-order statistics.

Since X, Y are jointly normal, the following relations are valid (see Ref. 10, p. 374):

$$\begin{aligned} E\{X^4\} &= 3E^2\{X^2\} - 2\bar{X}^4 \\ E\{Y^4\} &= 3E^2\{Y^2\} - 2\bar{Y}^4 \\ E\{X^2Y^2\} &= 2E^2\{XY\} - 2\bar{X}^2\bar{Y}^2 + E\{X^2\}E\{Y^2\}. \end{aligned} \quad (51)$$

Then

$$\begin{aligned} \text{Vâr}\{T_s\} &= E\{T_s^2\} - E^2\{T_s\} \\ &= E\{X^2 + Y^2\}^2 - E^2\{T_s\} \\ &= E\{X^4\} + E\{Y^4\} + 2E\{X^2Y^2\} - E^2\{T_s\} \\ &= 3E^2\{X^2\} + 3E^2\{Y^2\} - 2\bar{X}^4 - 2\bar{Y}^4 \\ &\quad + 4E^2\{XY\} - 4\bar{X}^2\bar{Y}^2 + 2E\{X^2\}E\{Y^2\} \\ &\quad - E^2\{X^2\} - E^2\{Y^2\} - 2E\{X^2\}E\{Y^2\} \end{aligned}$$

or

$$\text{Vâr}\{T_s\} = 2\{E^2\{X^2\} + E^2\{Y^2\} + 2E\{XY\}\} - 2(\bar{X}^2 + \bar{Y}^2)^2, \quad (52)$$

which is relation (15).

To evaluate (52) we need $\bar{X}, \bar{Y}, E\{X^2\}, E\{Y^2\}, E\{XY\}$. We evaluate these terms below.

Since $Q = X + jY$ we have

$$\begin{aligned} E\{Q^2\} &= E\{X^2\} - E\{Y^2\} + 2jE\{XY\} \\ E\{T_s\} &= E\{|Q|^2\} = E\{X^2\} + E\{Y^2\}. \end{aligned}$$

Thus

$$\begin{aligned} \bar{X} &= \text{Re } E\{Q\} \\ \bar{Y} &= \text{Im } E\{Q\} \\ E\{X^2\} &= \{E\{T_s\} + \text{Re } E\{Q^2\}\}/2 \\ E\{Y^2\} &= \{E\{T_s\} - \text{Re } E\{Q^2\}\}/2 \\ E\{XY\} &= \text{Im } E\{Q^2\}/2. \end{aligned} \quad (53)$$

Hence, we need evaluate only $E\{Q\}, E\{Q^2\}$, since $E\{T_s\}$ has already been calculated in (45). From (50) and (44) we obtain

$$E\{Q\} = W \frac{1 - W^N}{1 - W}. \quad (54)$$

Also, following a similar method with the evaluation of $E\{T_s\}$, we can find that

$$E\{Q^2\} = Z \frac{1 - Z^N}{1 - Z} + 2 \frac{Z}{1 - Z/W} \left\{ \frac{1 - W^N}{1 - W} - \frac{1 - Z^N}{1 - Z} \right\}, \quad (55)$$

where

$$Z(\omega) = E\{W_i^2(\omega)\}$$

and this relation completes the evaluation of $\text{Var}\{T_s\}$.

A.5 Evaluation of $E\{T_s T_r\}$

A direct evaluation of $E\{T_s T_r\}$ is possible but the derivation and the resulted formula are lengthy. To facilitate our analysis, let us assume that $Q = X + jY$ and T_r are jointly normal. Then

$$E\{T_s T_r\} = E\{X^2 T_r + Y^2 T_r\}. \quad (56)$$

But since X , Y and T_r are assumed jointly normal we have

$$\begin{aligned} E\{X^2 T_r\} &= 2\bar{X}E\{X T_r\} - 2\bar{X}^2 E\{T_r\} + E\{X^2\}E\{T_r\} \\ E\{Y^2 T_r\} &= 2\bar{Y}E\{Y T_r\} - 2\bar{Y}^2 E\{T_r\} + E\{Y^2\}E\{T_r\} \end{aligned} \quad (57)$$

and

$$\begin{aligned} E\{T_s T_r\} - E\{T_s\}E\{T_r\} \\ = 2\{\bar{X}E\{X T_r\} + \bar{Y}E\{Y T_r\} - E\{T_r\}(\bar{X}^2 + \bar{Y}^2)\}. \end{aligned} \quad (58)$$

All the above terms have been evaluated in (53) except for $E\{X T_r\}$ and $E\{Y T_r\}$. Since $Q = X + jY$ and T_r is real, it is enough to find $E\{Q T_r\}$. From relations (50) and (4) we obtain

$$\begin{aligned} E\{Q T_r\} &= E \left\{ \sum_{k,m=1}^N W_1 \dots W_k / W_1 \dots W_m \right\}^2 \\ &= \sum_{k=1}^N \sum_{m=1}^K D^m W^{k-m} + \sum_{m=2}^N \sum_{k=1}^{m-1} D^k W^{m-k} \end{aligned}$$

or

$$E\{Q T_r\} = D \frac{1 - D^N}{1 - D} + 2 \frac{D}{1 - DW} \left[\frac{1 - W^N}{1 - W} - \frac{1 - D^N}{1 - D} \right], \quad (59)$$

and then

$$\begin{aligned} E\{X T_r\} &= \text{Re } E\{Q T_r\} \\ E\{Y T_r\} &= \text{Im } E\{Q T_r\}. \end{aligned} \quad (60)$$

A.6 Evaluation of $\text{Var}\{\sigma_o^2\}$

We have defined

$$\sigma_o^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega. \quad (61)$$

Hence

$$\begin{aligned} \text{Var}^2\{\sigma_o^2\} &= E\{\sigma_o^4\} - E^2\{\sigma_o^2\} \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [E\{S(u)S(v)\} - E\{S(u)\}E\{S(v)\}] dudv \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(u, v) dudv. \end{aligned} \quad (62)$$

To facilitate our calculations we will assume that the random term is small compared to the systematic term and we shall approximate the spectrum $S(\omega)$ as

$$S(\omega) \approx \Phi_s(\omega)T_s(\omega). \quad (63)$$

A direct evaluation of $\text{Var}\{\sigma_o^2\}$ is possible but requires knowledge of

$$\begin{aligned} R_1(u, v) &= E\{W_i(u)W_j(v)\} \\ R_2(u, v) &= E\{W_i(u)W_j^*(v)\}. \end{aligned} \quad (64)$$

A simpler approach is to evaluate an upper and lower bound for $\text{Var}\{\sigma_o^2\}$. We can obtain an upper bound if we assume that $S(u)$ and $S(v)$ are highly correlated, i.e.,

$$E\{(S(u) - \bar{S}(u))(S(v) - \bar{S}(v))\} = \text{Var}\{S(u)\}\text{Var}\{S(v)\}, \quad (65)$$

and a lower bound if we assume that $S(u)$ and $S(v)$ are uncorrelated, i.e.,

$$E\{(S(u) - \bar{S}(u))(S(v) - \bar{S}(v))\} = \text{Var}^2\{S(u)\}\delta(u - v). \quad (66)$$

Then

$$\frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \text{Var}^2\{S(\omega)\} d\omega \right|^{1/2} < \text{Var}\{\sigma_o^2\} < \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Var}\{S(\omega)\} d\omega. \quad (67)$$

$\text{Var}\{S(\omega)\}$ has already been evaluated in (10b). It is expected that for long chains (large N) the true value will be closer to the lower bound, while for short chains (small N), it will be closer to the upper bound. From our numerical simulations we found the average of the two bounds to be a good estimator for $\text{Var}\{\sigma_o^2\}$.

APPENDIX B

On the Jitter Transfer Function of a Tuned Circuit

In most applications the jitter transfer function of a timing recovery circuit is approximated with the phase transfer function of the timing passive bandpass filter. In this appendix we derive in a simple way the phase transfer function of an arbitrary filter. The limits of the applicability of the derived formula are discussed. Finally, the results are extended for the case when a prefilter followed by a squarer is used.

B.1 Phase transfer function

In this part the phase transfer function of a narrow passive bandpass filter is shown to be

$$W(\omega) = \frac{H(\omega - \omega_0)e^{j\phi(\omega_0)} + H(\omega + \omega_0)e^{-j\phi(\omega_0)}}{2A(\omega_0)}, \text{ for } |\omega| < \omega_0, \quad (68)$$

where $H(\omega) = A(\omega)e^{j\phi(\omega)}$ is the transfer function of the bandpass filter and ω_0 is the baud rate of the received data. In Fig. 13 we illustrate the above relation.

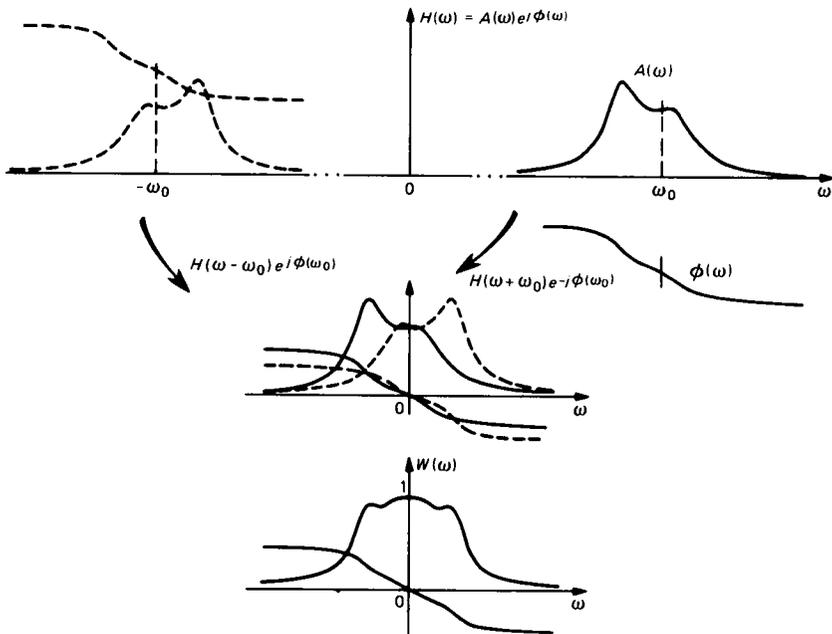


Fig. 13—Construction of the jitter transfer function $W(\omega)$ from $H(\omega)$, the transfer function of the retiming circuit filter.

Proof:

Let

$$f(t) = e^{j\omega_0 t + e(t)} \quad (69)$$

be the input of the bandpass filter and $e(t)$ the input jitter. We decompose $e(t)$ into its low-frequency and high-frequency parts, $e_L(t)$ and $e_H(t)$, respectively. We define $e_L(t)$ as the component of $e(t)$ with frequencies much less than the bandwidth of the bandpass filter. Thus

$$e(t) = e_L(t) + e_H(t). \quad (70)$$

Then if $|e_H(t)| \ll 1$ we can write $f(t)$ as

$$f(t) = e^{j\omega_0 t + e_L(t)} [1 + je_H(t) - \dots], \quad (71)$$

which is a narrowband process centered at ω_0 . Let $H_L(\omega)$ be the normalized low-pass equivalent of $H(\omega)$, where

$$H_L(\omega) = \frac{H(\omega + \omega_0)}{A(\omega_0)} e^{-j\phi(\omega_0)} \quad \omega > -\omega_0. \quad (72)$$

Then the output of the filter is⁴

$$g(t) = f(t) * h(t) = e^{j\omega_0 t} A(\omega_0) e^{j\phi(\omega_0)} \{h_L(t) * e^{je(t)}\}, \quad (73)$$

where $*$ indicates convolution and $h(t)$, $h_L(t)$ are the impulse responses corresponding to $H(\omega)$ and $H_L(\omega)$. The output phase is the phase of the term $h_L(t) * \exp(je(t))$, plus the static phase shift of $e^{j\phi(\omega_0)}$.

Using that [see (72)]

$$\int_{-\infty}^{\infty} h_L(t) dt = H_L(0) = 1$$

and because of the definition of $e_L(t)$, we can assume $e_L(t)$ to be constant compared with $h_L(t)$, i.e.,

$$h_L(t) * e^{je_L(t)} \approx H_L(0) e^{je_L(t)} = e^{je_L(t)}.$$

We therefore obtain

$$\begin{aligned} h_L(t) * e^{je(t)} &\approx e^{je_L(t)} \{1 + je_H(t) * h_L(t)\} \\ &\approx e^{j[e_L(t) + e_H(t)]} * h_L(t) \\ &= e^{j[e(t) * h_L(t)]}. \end{aligned} \quad (74)$$

Thus, since $e(t)$ is real,

$$\begin{aligned} g(t) &= A(\omega_0) e^{j[\omega_0 t + \phi(\omega_0)]} e^{j[e(t) * h_L(t)]} \\ &= A(\omega_0) e^{-[e(t) * \text{Im}h_L(t)]} e^{j[\omega_0 t + \phi(\omega_0) + e(t) * \text{Re}h_L(t)]}, \end{aligned} \quad (75)$$

where

$A(\omega_0)e^{-e(t)*\text{Im}h_L(t)}$ is an amplitude modulation term. Thus the phase transfer function is the Fourier transform of $w(t) = \text{Re}\{h_L(t)\}$, that is,

$$W(\omega) = \frac{H_L(\omega) + H_L^*(-\omega)}{2}, \quad (76)$$

and using (72) we obtain, finally, (68).

To derive (9) we have made two assumptions:

$$(a) |e_H(t)| \ll 1 \quad \text{and} \quad |e_H(t)*h_L(t)| \ll 1 \quad (77)$$

$$(b) |e_L(t)| \approx \text{constant}. \quad (78)$$

When $W(\omega)$ is going to be used to examine the accumulation of jitter in a chain of repeaters, the validity of (10) and (11) must be questioned for every repeater in the chain. For the N th repeater, $e_L(t)$ represents the accumulated jitter appearing in the clock of the $(N-1)$ -th repeater, while $e_H(t)$ represents the additional jitter generated by the N th repeater section, and it is always very small. Thus, assumptions (a) are valid in general, while assumption (b) is true only if jitter peaking does not occur. As is known, jitter peaking occurs if $\max |W(\omega)| > 1$, and in such a case the accumulated jitter grows exponentially. Thus relation (68) can be used to evaluate jitter accumulation when we have no jitter peaking. If jitter peaking is present, then

$$e_L(t) \approx A_0 \cos \omega_p t,$$

where ω_p is the peaking frequency, i.e.,

$$|W(\omega_p)| = \max |W(\omega)|$$

and (68) can be used only if $|e(t)| \ll 1$. This limits the applicability of the formula to short chains of regenerators. It is our feeling that in the case of jitter peaking, the linear model in (68) will overestimate jitter accumulation for long chains, because the nonlinear model will shift energy from the peaking frequency band to other bands. Preliminary simulations appear to agree with the above statement.

Relation (68) also suggests that

1. Filters with symmetric ripples in their passband are undesirable because they will always create jitter peaking. This is due to the normalization factor $2A(\omega_0)$ in (68).

2. Even with a monotonic filter, jitter peaking can occur if the data frequency, f_0 , is placed away from the filter's center frequency f_c .⁷ Define the detuning parameter as follows:

$$\alpha = \frac{\omega_0 - \omega_c}{B},$$

where B is the bandwidth of the low-pass filter $H_L(\omega)$. Then jitter peaking usually occurs if $\alpha > 1$.

These results are illustrated in Fig. 14, where the phase transfer function of a second-order Butterworth is plotted for various α .

B.2 Timing circuit with prefilter and squarer

In case a prefilter and a squarer are used we can modify the above analysis and also take the above circuits into consideration.

With ω_0 denoting the baud rate of the received data, the component that is going to generate the clock is located at $\omega_0/2$. Therefore, if $e(t)$ is the jitter present in the input data, the component of the input located at $\omega_0/2$ can be represented as

$$f(t) = e^{j[\omega_0 t + e(t)]/2}. \tag{79}$$

Notice that $e(t)$ also contains the jitter generated by the repeater.

Let us also define as $p(t)$ the output of the prefilter when the symbol 1 is transmitted, and let $P(\omega) = B(\omega)e^{j\psi(\omega)}$ be the Fourier Transform of $p(t)$, i.e.,

$$p(t) \xleftrightarrow{FT} P(\omega) = B(\omega)e^{j\psi(\omega)}. \tag{80}$$

Then, using the same assumptions we used in deriving relations (74) and (75) and defining

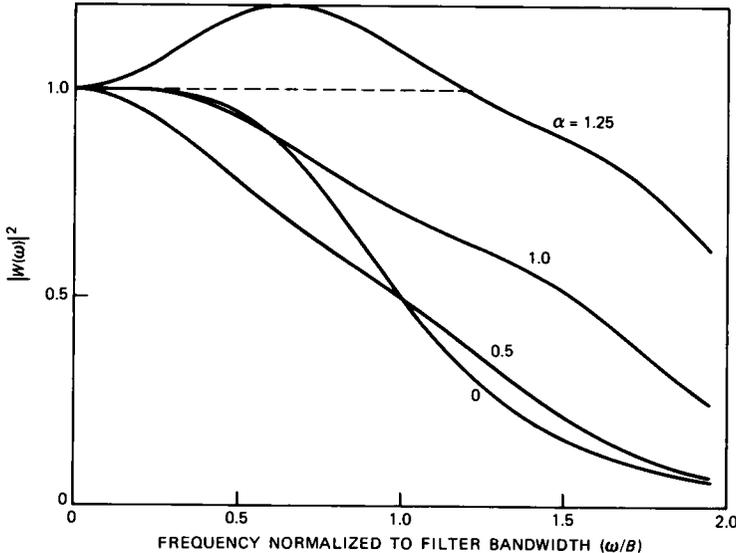


Fig. 14—Jitter transfer function of a second-order Butterworth filter for various values of the detuning parameter α .

$$P_L(\omega) = \frac{P(\omega + \omega_0/2)}{B(\omega_0/2)} e^{-j\Psi(\omega_0/2)} \quad (81)$$

as the low-pass equivalent of $P(\omega)$, we obtain

$$\begin{aligned} g(t) &= [f(t)*p(t)]^2*h(t) \\ &\approx B(\omega_0/2)e^{j[\omega_0 t + 2\Psi(\omega_0/2)]} e^{j[e(t)*p_L(t)]}*h(t) \\ &\approx A(\omega_0)B(\omega_0/2)e^{j[\omega_0 t + 2\Psi(\omega_0/2) + \phi(\omega_0)]} e^{j[e(t)*p_L(t)*h_L(t)]}. \end{aligned} \quad (82)$$

Reasoning as in (76) we obtain

$$W(\omega) = \frac{P_L(\omega)H_L(\omega) + P_L^*(-\omega)H_L^*(-\omega)}{W(0)}. \quad (83)$$

Since $P(\omega)$ is much wider than $H_L(\omega)$, we can assume that

$$P_L(\omega)H_L(\omega) \approx H_L(\omega), \quad (84)$$

which implies that the presence of a prefilter will not change significantly the jitter transfer function of the repeater. However, the jitter generated within the repeater may significantly depend on the prefilter and squarer.

AUTHOR

Christodoulos Chamzas, Diploma degree (Electrical and Mechanical Engineering), the National Technical University of Greece, Athens, Greece, 1974; M.S., 1975, Ph.D., 1979 (Electrical Engineering), Polytechnic Institute of New York, Farmingdale, N.Y.; AT&T Bell Laboratories, 1982—. From 1979 to 1982 Mr. Chamzas was an Assistant Professor with the Department of Electrical Engineering at Polytechnic Institute of New York, where he is currently a part-time Visiting Professor for the Imaging Institute. At AT&T Bell Laboratories he has been working on problems in high-speed pseudorandom noise generators, the Submarine Lightwave System and adaptive echo cancellers. His primary interests are in signal processing and communication systems. Member, Technical Chamber of Greece, Sigma Xi.