AT&T Technical Journal
Vol. 64, No. 1, January 1985
Printed in U.S.A.

Nonlocal Input-Output Expansions

By I. W. SANDBERG*

(Manuscript received July 16, 1984)

In a recent series of papers by this writer on the existence, determination, and properties of power-series-like expansions for expressing a nonlinear system's outputs in terms of its inputs, the emphasis is primarily on *locally* convergent expansions. Here we report on related general results concerning nonlocal expansions, including in particular material concerning the *size* of the region of convergence. One of the results given provides useful *necessary and sufficient* conditions under which f^{-1} has a generalized power-series expansion, where f is a certain important general type of invertible map (that, for example, might take one set of complex-valued signals defined on $[0, \infty)$ into another).

I. INTRODUCTION

In a recent series of papers including Refs. 1 through 3 on the existence, determination, and properties of power-series-like expansions for expressing a nonlinear system's outputs in terms of its inputs, the emphasis is primarily on *locally* convergent expansions. Here, in Section II, we report on related general results concerning nonlocal expansions, including in particular material concerning the *size* of the region of convergence.

More specifically, Theorem 1 in Section II gives necessary and sufficient conditions under which f^{-1} has a generalized power series expansion (see Section II for the details) when f is an invertible locally Lipschitz map between certain general subsets of two complex Banach

* AT&T Bell Laboratories.

spaces. Theorem 2 provides an algorithm for obtaining the expansion whenever it exists.

In Section 2.4 we use Theorems 1 and 2 to prove results concerning a system model considered in Ref. 3 and in earlier papers (e.g., Ref. 2). This model is characterized by five operators: a nonlinear operator N and linear operators, A, B, C, and D. The system input v and corresponding output w are related by the equations

$$y = Nx \tag{1}$$

$$x = Av + Cy \tag{2}$$

$$w = Dv + By, \tag{3}$$

where x and y, respectively, can be interpreted as the input and output of the nonlinear portion of the system.* In Ref. 3 it is assumed that y, x, v, and w are n-vector valued and defined for $t \geq 0$, where as usual n is an arbitrary positive integer; here these quantities are allowed to belong to a general complex Banach space $\mathscr{B}$, but we shall be interested mainly in the case where $\mathscr{B}$ is the space L_∞ of bounded functions considered in Ref. 3. Theorem 4 in Section 2.4 shows that, in an interesting and important setting, w can be expressed as a certain power series in v that converges for $Av \in V$, where V is *any* open ball in $\mathscr{B}$ centered at the origin such that there is an open subset V_0 of $\mathscr{B}$ for which $(I - CN)$ (I is the identity operator on $\mathscr{B}$) is a homeomorphism of V_0 onto V, with $(I - CN)^{-1}$ locally Lipschitz on V in the sense of Section 2.1. Theorem 4 is actually somewhat more general than is indicated above, and it together with its relation to earlier work is discussed in Section 2.4.2. A pertinent example is given in Appendix B.

II. HOMOGENEOUS POLYNOMIALS AND EXPANSIONS FOR MAPS BETWEEN SUBSETS OF COMPLEX BANACH SPACES

2.1 Preliminaries

Throughout the paper, $\mathscr{B}_0$ and $\mathscr{B}$ are Banach spaces such that there is a linear homeomorphism q of $\mathscr{B}$ onto $\mathscr{B}_0$. (Thus, $\mathscr{B}_0$ and $\mathscr{B}$ are taken to be isomorphic. We shall be interested mainly in the case in which in fact $\mathscr{B}_0 = \mathscr{B}$.) The same symbols $\|\cdot\|$ and θ, respectively, are used to denote the norm and zero element in $\mathscr{B}_0$ as well as in $\mathscr{B}$. Unless stated otherwise, $\mathscr{B}_0$ and $\mathscr{B}$ are assumed to be over the field of *complex* scalars.

* The equations of a very large class of systems can be written in the form (1) through (3), with N memoryless. It is not necessary that the equations of the system to be studied be given *at the outset* in the form (1) through (3) (see Ref. 3, Appendices I and II). In Ref. 3, y, x, and w are taken to belong to an *extended* space $L_{\infty e}$. For the purposes of this paper extended space concepts are not needed.

A map g from a nonempty open subset U of $\mathscr{B}$ or of $\mathscr{B}_0$ into $\mathscr{B}_0$ or $\mathscr{B}$, respectively, is *locally Lipschitz* on U if for each $a \in U$ there are a positive number c_a and an open ball $\beta_a \subset U$ centered at a such that $\|g(u_1) - g(u_2)\| \leqslant c_a\|u_1 - u_2\|$ for u_1 and u_2 in β_a. We use $d^m g(a)$ to denote the mth-order Fréchet derivative (Ref. 4, pp. 149, 181) of g at a point $a \in U$, assuming it exists. A sufficient condition for g to be locally Lipschitz on U is that it be continuously Fréchet differentiable in U.

For V_0 and V any nonempty open subsets of $\mathscr{B}_0$ and $\mathscr{B}$, respectively, $H(V_0, V)$ denotes the set of all homeomorphisms h of V_0 onto V such that h is locally Lipschitz on V_0. Our results are concerned with this class of maps with V assumed to be a *c-star* about some $p \in \mathscr{B}$, by which is meant that $V = \{x \in \mathscr{B}: x = p + W\}$, where W is a subset of $\mathscr{B}$ with the property that $zw \in W$ for $w \in W$ and any complex scalar z with $|z| \leqslant 1$. The concept of a c-star is of importance in studies of the region of convergence of abstract power series expansions (see Ref. 5, Theorems 26.5.9 and 26.6.1). In this paper we could have restricted attention to the case where V is an open ball centered at p, which clearly is a c-star about p, but this was not done because no significant simplification results.

Given any positive integer m, by an *m-linear* map Q from $\mathscr{B}^m$ into $\mathscr{B}_0$, we mean that $Q(h_1, \cdots, h_m)$ is linear (i.e., additive and homogeneous) separately in each h_j. Such a map is *symmetric* if $Q(h_1, \cdots, h_m)$ is symmetric in the variables $h_1, \cdots, h_m$. A map $M(\cdot)$ from $\mathscr{B}$ into $\mathscr{B}_0$ is called a *homogeneous polynomial* of degree m if there exists an m-linear map Q from $\mathscr{B}^m$ into $\mathscr{B}_0$ such that $M(h) = Q(h, \cdots, h)$ for all h*. We now come to a definition of central importance in our results.

For $p \in \mathscr{B}$ and $V \subset \mathscr{B}$ an open c-star about p, let $\mathscr{P}(p, V)$ denote the set of all maps g from V into $\mathscr{B}_0$ such that there are homogeneous polynomials $g_m(p, \cdot)$ of degree m ($m = 1, 2, \cdots$), from $\mathscr{B}$ into $\mathscr{B}_0$, with the properties that

$$\sum_{m=1}^{\infty} g_m(p, w)$$

converges in $\mathscr{B}_0$ for each $(p + w) \in V$, and

$$g(p + w) = g(p) + \sum_{m=1}^{\infty} g_m(p, w), \qquad (p + w) \in V. \tag{4}$$

---

* This definition of a homogeneous polynomial $M(\cdot)$ is not the same as, but is equivalent to, the one given in Ref. 5, that $M(zh) = z^m M(h)$ and $M(p + zh) = \sum_{j=1}^{m} M_j(p, h)z^j$ for p and h in $\mathscr{B}$ and any complex scalar z, where the M_j do not depend on z. Also, the first definition given above describes the same class of maps $M(\cdot)$ if "m-linear" is replaced with "symmetric m-linear".

The set $\mathscr{P}(p, V)$ is of course a set of maps g that admit a generalized power series expansion in the sense indicated. The expansion (4) for any $g \in \mathscr{P}(p, V)$ is *unique* in the sense that if

$$g(p + w) = g(p) + \sum_{m=1}^{\infty} h_m(p, w), \qquad (p + w) \in V \tag{5}$$

[by which is meant, in particular that the sum in (5) converges] with each $h_m(p, \cdot)$ a homogeneous polynomial of degree m, then $g_m(p, \cdot) = h_m(p, \cdot)$ for all m. This follows from a simple argument due to Graves (Ref. 6, p. 174). (See also Ref. 1, Section 2.7.)

Finally, we say that g belongs to $\mathscr{P}_F(p, V)$ if g belongs to $\mathscr{P}(p, V)$ and for each m there is a *continuous* symmetric m-linear Q_m from $\mathscr{B}^m$ into $\mathscr{B}_0$ that depends on p such that $g_m(p, h) = Q_m(h, \cdots, h)$ for all h. In particular, then each $g_m(p, \cdot)$ is bounded in the sense that there is a positive constant ρ_m such that $\|g_m(p, h)\| \leq \rho_m \|h\|^m$ for all m and h, with p fixed, and every $g_m(p, \cdot)$ is Fréchet differentiable on $\mathscr{B}$.

2.2 Inverses of maps in $H(V_0, V)$ and generalized power series expansions

Our first result, Theorem 1 below, provides a complete characterization of those f's in $H(V_0, V)$, with V a c-star, for which f^{-1} has a generalized power series expansion of the type described in the preceding section.

Theorem 1: Let V_0 and V be nonempty open subsets of $\mathscr{B}_0$ and $\mathscr{B}$, respectively, with V a c-star about some point p in $\mathscr{B}$. Let $f \in H(V_0, V)$. Then, $f^{-1} \in \mathscr{P}(p, V)$ if and only if f is Fréchet differentiable on V_0 and f^{-1} is locally Lipschitz on V. In addition, we have $f^{-1} \in \mathscr{P}_F(p, V)$ whenever $f^{-1} \in \mathscr{P}(p, V)$.

2.2.1 Proof of Theorem 1

We first prove the following lemma.

Lemma 1: Let V_0 and V be nonempty open subsets of $\mathscr{B}_0$ and $\mathscr{B}$, respectively, with $\mathscr{B}_0$ and $\mathscr{B}$ over the same field, either real or complex. Let h be an invertible Fréchet continuously differentiable map of V_0 onto V, with h^{-1} locally Lipschitz on V. Then $d(h^{-1})(\cdot)$ (the Fréchet derivative of h^{-1}) exists and is continuous throughout V.

Proof of Lemma 1: With q the homeomorphism mentioned in Section 2.1, define s on V_0 by $s(x) = qh(x)$ for $x \in V_0$. It is not difficult to verify that s is a continuously Fréchet differentiable invertible map of $V_0 \subset \mathscr{B}_0$ onto the open subset* $q(V)$ of $\mathscr{B}_0$, and that s^{-1} is locally Lipschitz on $q(V)$. Since the inverses h^{-1} and s^{-1} of h and s, respec-

* The set $q(V)$ is *open* by the open mapping theorem, or because it is simply the inverse image of the open set V under the continuous map q^{-1}.

tively, are related by $h^{-1} = s^{-1}q$, it follows that the lemma holds in general if it holds for $\mathscr{B} = \mathscr{B}_0$. We therefore assume throughout the remainder of the proof that $\mathscr{B}$ and $\mathscr{B}_0$ are the same spaces.

Let $a \in V_0$ be arbitrary, and let $c_{h(a)}$ and $\beta_{h(a)}$, respectively, be a positive constant and an open ball in V centered at $h(a)$ such that $\|h^{-1}(u_1) - h^{-1}(u_2)\| \leqslant c_{h(a)}\|u_1 - u_2\|$ for u_1 and u_2 in $\beta_{h(a)}$. Let $S = \{x \in \mathscr{B}: (x + a) \in V_0\}$. Let β denote the open ball in $\mathscr{B}$ centered at the origin, with the same radius as $\beta_{h(a)}$.

Define $H_a: S \to \mathscr{B}$ by $H_a(x) = h(x + a) - h(a)$, $x \in S$. The set S is open in $\mathscr{B}$. Thus, by the continuity of H_a, the inverse image $H_a^{-1}(\beta)$ is open in S and hence in $\mathscr{B}$. [The fact that $H_a^{-1}(\beta)$ is an open subset of $\mathscr{B}$ is used in connection with (6) and (7) below, and at the end of the proof.]

Let $h_a: H_a^{-1}(\beta) \to \beta$ be given by $h_a(x) = H_a(x)$, $x \in H_a^{-1}(\beta)$. By the invertibility of h, h_a is an invertible map of $H_a^{-1}(\beta)$ onto β. Let h_a^{-1} denote its inverse, and define $g: \beta \to \mathscr{B}$ by

$$g(u) = u - dh(a)h_a^{-1}u + w, \qquad u \in \beta$$

for $w \in \beta$, where $dh(a)$ is the Fréchet derivative of h at a.

By the mean value theorem in Ref. 4, p. 160, and the continuity of $dh(\cdot)$,

$$\frac{\|h_a(z_1) - h_a(z_2) - dh(a)(z_1 - z_2)\|}{\|z_1 - z_2\|} \to 0 \tag{6}$$

as $\max(\|z_1\|, \|z_2\|) \to 0$ with $z_1 \neq z_2$.* Therefore, with $\sigma > 0$ such that $\sigma c_{h(a)} < 1$, there is a $\delta > 0$ for which $\{z \in \mathscr{B}: \|z\| \leqslant \delta\} \subset H_a^{-1}(\beta)$ and

$$\frac{\|h_a(z_1) - h_a(z_2) - dh(a)(z_1 - z_2)\|}{\|z_1 - z_2\|} \leqslant \sigma \tag{7}$$

for $z_1 \neq z_2$, $\|z_1\| \leqslant \sigma$, and $\|z_2\| \leqslant \delta$. Since $\|z\| \leqslant c_{h(a)}\|u\|$ for $u \in \beta$ and $z = h_a^{-1}u$ [because then $h(a + z) = u + h(a)$, which gives $z = h^{-1}(u + h(a)) - h^{-1}(h(a))$], we have

$$\frac{\|u_1 - u_2 - dh(a)[h_a^{-1}u_1 - h_a^{-1}u_2]\|}{\|h_a^{-1}u_1 - h_a^{-1}u_2\|} \leqslant \sigma \tag{8}$$

for $u_1 \neq u_2$, $\|u_1\| \leqslant \rho$, and $\|u_2\| \leqslant \rho$, where $\rho = \min(\rho_\beta/\alpha, \delta(c_{h(a)})^{-1})$, ρ_β is the radius of β, and α is any number in $(1, \infty)$.

Now let u_1 and u_2 belong to β, and define x_1 and x_2 by $x_1 = h_a^{-1}u_1$ and $x_2 = h_a^{-1}u_2$. Clearly, $h(x_1 + a) = h(a) + u_1$ and $h(x_2 + a) = h(a) + u_2$, which gives $\|x_1 - x_2\| \leqslant c_{h(a)}\|u_1 - u_2\|$. This, together with (8) and

---

* With regard to Ref. 4, p. 160, $h_a(z_1) - h_a(z_2) - dh(a)(z_1 - z_2) = [h(a + z_1) - dh(a)z_1] - [h(a + z_2) - dh(a)z_2]$.

$\sigma c_{h(a)} < 1$, shows that the map g defined above is a contraction on the set $S_0 = \{u \in \mathscr{B}: \|u\| \leq \rho\}$. Since $h_a^{-1}(\theta) = \theta$, it is easy to see that for all $w \in \mathscr{B}$ with $\|w\| \leq \rho_0$ and $\rho_0 > 0$ sufficiently small, g maps S_0 into itself. By the contraction-mapping fixed point theorem, g has a unique fixed point* in S_0, and thus there is a unique solution $u \in S_0$ of $dh(a)h_a^{-1}u = w$, for each such w.

By the linearity of $dh(a)$, we see that for each $w \in \mathscr{B}$ there is a unique $x \in \mathscr{B}$ such that $dh(a)x = w$, the uniqueness following from the fact that h_a maps an open ball in $\mathscr{B}$ centered at θ into S_0 [recall that $h_a(\theta) = \theta$, that h_a is continuous, and that $H_a^{-1}(\beta)$ is open in $\mathscr{B}$].† This shows that $dh(a)$ is an invertible map of $\mathscr{B}$ onto $\mathscr{B}$. By the boundedness of $dh(a)$, $dh(a)^{-1}$ is bounded (Ref. 9, p. 119). Since $a \in V_0$ is arbitrary, and $dh(\cdot)$ is assumed to be continuous, it follows (Ref. 4, p. 273) that h^{-1} is continuously Fréchet differentiable on V, which proves the lemma.‡

Returning now to the proof of the theorem, suppose initially that $f^{-1} \in \mathscr{P}(p, V)$. Since the series associated with f^{-1} is a (G)-power series in the sense of Ref. 5, p. 773, this series is (G)-differentiable (i.e., Gâteaux differentiable) in V (Ref. 5, p. 773). By the (G)-differentiability and continuity of f^{-1} (which implies that f^{-1} is analytic), it follows Ref. 5, Theorem 3.17.1 that f^{-1} is Fréchet differentiable throughout V. Using the following lemma,§ which is proved in Appendix A and which is used also in Section 2.3, $d(f^{-1})(\cdot)$ is continuous.

Lemma 2: If g is a Fréchet differentiable map of a nonempty open subset V of $\mathscr{B}$ into a complex Banach space $\mathscr{B}_1$, then g is twice Fréchet differentiable on V.

---

* This part of the proof is along the lines of proofs of the classical implicit function theorem (see, for example, Ref. 7, pp. 194–5). Lemma 1 is related to a theorem stated in Ref. 8, p. 165, but the argument given there does not prove the theorem. (In the terminology of Ref. 8, it is not shown that the definition of a linearization leads to the limit given. However, the proof above shows that the theorem in Ref. 8, p. 165 is true under the additional hypotheses that L_x exists and is continuous in a neighborhood of $x = x_0$.)

† A modification of an argument given in Ref. 8, p. 165 could also have been used to obtain the uniqueness.

‡ With regard to Lemma 1 and the condition in Section 2.1 that there is a linear homeomorphism q that maps $\mathscr{B}$ onto $\mathscr{B}_0$, we note that, in the absence of that condition, the existence of $dh(\cdot)$ and $dh^{-1}(\cdot)$ as indicated in the lemma *implies* (see Proposition 1 in Section 2.3.1) that the condition holds. This shows that there is no loss of generality in assuming that the condition is met.

§ This lemma, which is an analog of a standard proposition in the classical theory of functions of a complex variable, is probably a known result, but we have not encountered it in the literature. However, for related material concerning *Gâteaux* variations, see Ref. 5. The lemma provides additional understanding concerning some results proved in Ref. 1 and other recent papers. While it does not strengthen these results, it shows that some hypotheses actually follow from others that were introduced (see, e.g., Ref. 1, Theorem 2, which is an early result concerning nonlocal expansions).

Thus f^{-1} is locally Lipschitz on V, and, by Lemma 1, $df(\cdot)$ exists on V_0. Using the analyticity of f^{-1}, by Ref. 5, Theorems 3.16.2, 26.3.4 and 26.3.6, one has $f^{-1} \in \mathscr{P}_F(p, V)$.

Now assume that f is (F)-differentiable, and thus continuously (F)-differentiable, on V_0 and that f^{-1} is locally Lipschitz on V. Using Lemma 1, $d(f^{-1})(\cdot)$ exists in V. In particular, f^{-1} is (G)-differentiable in V, and by Ref. 5, Theorems 3.16.2 and 26.3.5, we have $f^{-1} \in \mathscr{P}(p, V)$, which completes the proof of the theorem.

2.2.2 Comments

Proposition 1 in Section 2.3.1 and the proof of Theorem 1 show that under the hypotheses of the theorem, $f^{-1} \in \mathscr{P}(p, V)$ implies also that $df(a)$ is a homeomorphism of $\mathscr{B}_0$ onto $\mathscr{B}$ for every $a \in V_0$.*

For the extreme case in which both $\mathscr{B}_0$ and $\mathscr{B}$ are just the space of complex numbers with the absolute value norm, the condition that V is an open c-star about p reduces to the requirement that V is an open disk in the complex plane centered at p. In that case, by standard results in the classical theory of functions of a complex variable, $f^{-1} \in \mathscr{P}(p, V)$ implies that the ordinary derivative $(f^{-1})'(z)$ exists for $z \in V$. In addition, since f^{-1} is one-to-one on V, it follows from a known result (Ref. 10, Theorem 16-23) that $(f^{-1})'(z) \neq 0$ for $z \in V$, which shows that $f'(\cdot)$ exists on V_0. Similarly, using Ref. 10, Theorem 16-23, it follows from the hypothesis that $f'(z)$ exists for $z \in V_0$ that $f^{-1} \in \mathscr{P}(p, V)$.† Theorem 1 can be viewed as a Banach space relative of these propositions.

2.3 Construction of the expansion of f^{-1}

Here we give an algorithm, along the lines of Theorem 2 of Ref. 1, for expanding f^{-1}.

Theorem 2: Let the hypotheses of Theorem 1 be met, and assume that $f^{-1} \in \mathscr{P}(p, V)$. Then for each $l = 1, 2, \cdots$, the lth-order Fréchet derivative $d^l f(x)$ exists for $x \in V_0$, and

$$f^{-1}(p + h) = f^{-1}(p) + \sum_{m=1}^{\infty} g_m(p, h), \qquad (p + h) \in V \tag{9}$$

where

$$g_1(p, h) = (df[f^{-1}(p)])^{-1}h$$

---

* And this adds to material in Ref. 1, Section 2.6 concerning the necessity of a certain invertibility condition.

† In these observations, the hypothesis that f is locally Lipschitz is not needed. On the other hand, the Lipschitz condition is obviously a consequence of $f^{-1} \in \mathscr{P}(p, V)$.

(*the inverse exists by Proposition 1, below*), *and*

$$g_m(p, h) = -(df[f^{-1}(p)])^{-1} \sum_{l=2}^{m} (l!)^{-1} \cdot \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} d^l f[f^{-1}(p)] g_{k_1}(p, h) g_{k_2}(p, h) \cdots g_{k_l}(p, h),$$

$$m \geq 2.^* \quad (10)$$

2.3.1 Proof of Theorem 2

Proof: Since by Theorem 1 and Proposition 1 (which appears below), $f^{-1} \in \mathscr{P}(p, V)$ implies that $df(a)$ and $[df(a)]^{-1}$ exist for each $a \in V_0$, Lemma 2, the proof of Theorem 2 of Ref. 1, and Theorem 3.16.2 of Ref. 5, show that Theorem 2 holds.

Proposition 1: If V_0 and V are nonempty open subsets of $\mathscr{B}_0$ and $\mathscr{B}$, respectively, and f is a homeomorphism of V_0 onto V such that f and f^{-1} are Fréchet differentiable on their respective domains, then $df(a)$ is a homeomorphism of $\mathscr{B}_0$ onto $\mathscr{B}$ for any $a \in V_0$.

The proposition is a well-known result (see, for example, Ref. 11, p. 175, Problem 6) provable using the relations $f^{-1}[f(x)] = x$ and $f[f^{-1}(y)] = y$ for x and y in V_0 and V, respectively, and the chain rule (see Ref. 11, pp. 171–2) for differentiating a composite function.

2.4 Theorems concerning the system model

In this section attention is focused on the system model described in Section I. We use I to denote the identity map on $\mathscr{B}$.

Let A, B, C, and D be linear maps of $\mathscr{B}$ into itself, with B and C bounded. Let N be a map from a subset S of $\mathscr{B}$ into $\mathscr{B}$, for which there are nonempty open subsets V_0 and V of $\mathscr{B}$ such that $V_0 \subset S$, V is a c-star about some point $p \in \mathscr{B}$, and $(I - CN)$ is a homeomorphism of V_0 onto V, with $(I - CN)^{-1}$ locally Lipschitz on V. (It is not difficult to give important examples in which these hypotheses on N are met. This is illustrated in Appendix B.)

Let $\mathscr{I}$ (for "set of inputs") denote the collection of all $v \in \mathscr{B}$ such that $Av \in V$, and assume that there is a $v_0 \in \mathscr{B}$ for which $Av_0 = p$. We see that for each $v \in \mathscr{I}$, there are unique x, y, and w in V_0, $\mathscr{B}$, and $\mathscr{B}$, respectively, such that

$$x = Av + Cy$$

$$w = Dv + By$$

$$y = Nx.$$

* In (10), $\sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}}$ denotes a sum over all positive integers $k_1, \cdots, kl$ that add to m.

Let W be the map from $\mathscr{I}$ into $\mathscr{B}$ defined by the condition that $W(v) = w$ when $v \in \mathscr{I}$. In other words, let W be given by

$$W(v) = Dv + B(I - CN)^{-1}Av, \qquad v \in \mathscr{I} \tag{11}$$

Since N differentiable on V_0 implies (by Lemma 2) that it is continuously differentiable on V_0 and thus locally Lipschitz there, by Theorem 1 and the boundedness of C we have the following:

Theorem 3: Let N be Fréchet differentiable on V_0. Then $(I - CN)^{-1} \in \mathscr{P}_F(p, V)$.

Theorem 3 shows that, under merely the condition indicated, W has an expansion about v_0 in terms of homogeneous polynomials that is valid* for all $v \in \mathscr{I}$. With regard to actually determining the expansion for W, we have the following.

Theorem 4: Let N be Fréchet differentiable on V_0. Then for each $l = 2, 3, \cdots$ the Fréchet derivative $d^l N(\cdot)$ exists in some open neighborhood $\mathscr{N}_l$ of the point x_0 in V_0 that satisfies $x_0 - CN(x_0) = Av_0$, and for $v \in \mathscr{I}$, we have

$$W(v) = BN(x_0) + Dv + \sum_{m=1}^{\infty} BY_{(m)}(v - v_0) \tag{12}$$

where $Y_{(1)}, Y_{(2)}, \cdots$ are the homogeneous polynomials defined by the relations†

$$X_{(1)}(v - v_0) = [I - CdN(x_0)]^{-1}A(v - v_0),$$

$$X_{(m)}(v - v_0) = [I - CdN(x_0)]^{-1}C \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} d^lN(x_0)X_{(k_1)}(v - v_0) \cdots X_{(k_l)}(v - v_0) \tag{13}$$

for $m \geqslant 2$, and

$$Y_{(m)}(v - v_0) = \sum_{l=1}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} d^lN(x_0)X_{(k_1)}(v - v_0) \cdots X_{(k_l)}(v - v_0) \tag{14}$$

for $m \geqslant 1$.

2.4.1 Proof of Theorem 4

Proof: Let $\mathscr{B}^2$ denote the Banach space $\mathscr{B} \times \mathscr{B}$ with norm max $(\|\cdot\|, \|\cdot\|)$. Define G: $V_0 \times \mathscr{B} \to \mathscr{B}^2$ by

---

* Here the boundedness of B is used to ensure that $\sum_{m=0}^{\infty} Bg_m[p, A(v - v_0)]$ converges to $B\sum_{m=0}^{\infty} g_m[p, A(v - v_0)$, where $\sum_{m=0}^{\infty} g_m[p, A(v - v_0)]$ is the series for $(I - CN)^{-1} Av$.

† The inverse of $I - CdN(x_0)$ exists; see Section 2.2.2.

$$G_1(p_1, p_2) = p_1 - CNp_1$$

$$G_2(p_1, p_2) = Np_1 - p_2$$

for $(p_1, p_2) \in V_0 \times \mathscr{B}$. With β a nonempty open ball in $\mathscr{B}$ centered at θ, let

$$\mathscr{D} = \{(p_1, p_2) \in V_0 \times \mathscr{B}: G(p_1, p_2) = (q_1, q_2), (q_1, q_2) \in V \times \beta\}$$

[i.e., let $\mathscr{D} = G^{-1}(V \times \beta)$]. We see that $\mathscr{D}$ is *open* in $\mathscr{B}^2$ (because V and β are open, and G is continuous and defined on an open subset of $\mathscr{B}^2$), and that $V \times \beta$ is a *c-star* in $\mathscr{B}^2$ centered at (p, θ). The restriction F of G to $\mathscr{D}$ is a differentiable homeomorphism of $\mathscr{D}$ onto $V \times \beta$. Using this fact, it is not difficult to show that a proof of Theorem 4 can be obtained by proceeding as in Ref. 2, proof of part (ii) of Theorem 1, but with our Theorem 2 with $\mathscr{B} = \mathscr{B}_0$ employed instead of Lemma 1 in Ref. 2.* (It is easy to verify that for F as described above, F^{-1} is locally Lipschitz.)

2.4.2 Discussion

Under the conditions on N of Theorem 4, N has the representation

$$N(x) = N(x_0) + \sum_{l=1}^{\infty} (l!)^{-1} d^l N(x_0)(x - x_0)^l \tag{15}$$

in which the series converges (for example) uniformly for x in some sufficiently small ball in S centered at x_0,† and of course (15) provides an important interpretation of the maps $d^l N(x_0)$ that appear in (13) and (14).

It is often useful to observe that the operator $(I - CN)^{-1}$ in (11) can naturally be identified with the "feedback part" of the system represented by (1), (2), and (3) (see Ref. 3, Figs. 1 and 2). Aside from considerations concerning the differentiability hypothesis on N, and the existence of a v_0 as described, Theorem 4 shows that the series representation given by (12) holds for $Av \in V$, whenever V is an open c-star about some p such that the equation $x - CNx = u$ of the feedback portion is, so to speak, uniquely locally-Lipschitz-solvable in some open subset of S for every u in V. Implicit in this is the assumption that $\mathscr{B}$ is a *complex* Banach space. Since the inputs and outputs of most nonlinear systems of direct interest are *real* valued functions, the main point of Theorem 4 with regard to applications is that an

---

* See also Ref. 3, Theorem 1, with regard to the expression for the Y_m.

† See Ref. 5, Theorem 3.17.1, and Ref. 11, p. 198. Also, notice that the convergence to $N(x)$ of the right side of (15) for $x_0 = p$ and $x \in V$ is implicit in Theorem 4, since C and D can be taken to be the zero map and A and B can be assumed to be the identity map.

expansion exists, and can be constructed as specified, when the original system equations can be "complexified" so that the hypotheses concerning V_0, V, and $(I - CN)$, and of the theorem, are met. It is not difficult to give specific examples. One example is provided by the material in Appendix B.

Theorem 4 with $N(\theta) = p = v_0 = x_0 = \theta$ bears directly on the main result in Ref. 3, which concerns expansions involving iterated integrals, for the case in which $\mathscr{B}$ is the set L_∞ of bounded complex n-vector valued functions on the interval $[0, \infty)$.[3]* Our theorem shows that the locally convergent expansions described there converge in fact for $Av \in V$ for any V and V_0 such that our solvability hypotheses with $p = \theta$ are met, assuming merely that $S = \Gamma$, where Γ is the domain of definition of N in Hypothesis B.2 of Ref. 3.

In this connection, if $\mathscr{B} = L_\infty$ and N is memoryless in the sense that $(Ns)(t) = \eta[s(t), t]$, $t \geqslant 0$ for $s \in S$, with $S = \Gamma$ and η and Γ, respectively, a function and a domain of the kind described in Ref. 3, Hypothesis B.2, and if, for simplicity, $n = 1$, then $[d^l N(x_0) X_{(k_1)} (v - v_0) \cdots X_{(k_l)}(v - v_0)](t)$ is just the product $Ml(t)[X_{(k_1)}(v - v_0)](t) \cdots [X_{(k_l)}(v - v_0)(t)]$ for $t \geqslant 0$, where $Ml(t) = \partial^l \eta(z, t)/\partial z^l \big|_{z=x_0(t)}$ (see Ref. 2, Lemma 3).

REFERENCES

1. I. W. Sandberg, "Expansions for Nonlinear Systems," B.S.T.J., *61*, No. 2 (February 1982), pp. 159–99.
2. ——, "Volterra Expansions for Time-Varying Nonlinear Systems," B.S.T.J., *61*, No. 2 (February 1982), pp. 201–25.
3. ——, "On Volterra Expansions for Time-Varying Nonlinear Systems," IEEE Trans. Circuits Syst., *CAS-30* (February 1983), pp. 61–7.
4. J. Dieudonné, *Foundations of Modern Analysis*, New York: Academic Press, 1969.
5. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Providence: Amer. Math. Soc. Coll. Publ. XXXI, 1957.
6. L. M. Graves, "Riemann Integration and Taylor's Theorem in General Analysis," Trans. Amer. Mathematical Soc., *29* (January 1927), pp. 163–77.
7. L. A. Liusternik and V. J. Sobolev, *Elements of Functional Analysis*, New York: Frederick Unger Publishing Co., 1961.
8. J. C. Willems, *The Analysis of Feedback Systems*, Cambridge: M.I.T. Press, 1971.
9. I. J. Maddox, *Elements of Functional Analysis*, London: Cambridge University Press, 1970.
10. T. M. Apostol, *Mathematical Analysis*, Reading: Addison-Wesley, 1957.
11. T. M. Flett, *Differential Analysis*, London: Cambridge University Press, 1980.
12. A. A. M. Saleh, "Matrix Analysis of Mildly Nonlinear Multiple-Input, Multiple-Output Systems with Memory," B.S.T.J. *61*, No. 9 (November 1982), pp. 2221–43.
13. R. J. P. de Figueiredo, "A Generalized Fock Space Framework for Nonlinear System and Signal Analysis," IEEE Trans. Circuits and Syst., *CAS-30* (September 1983), pp. 637–47.
14. M. Fliess, M. Lamnabhi, and F. Lamnabhi-Lagarrique, "An Algebraic Approach to Nonlinear Functional Expansions," IEEE Trans. Circuits and Syst., *CAS-30* (August 1983), pp. 554–70.

* For recent work related in a general sense, see, for example, Refs. 12 to 15. The complexification needed to use the results in Ref. 3 is ordinarily trivial, because of the nature of the hypotheses involved.

15. S. Boyd, L. O. Chua, and C. A. Desoer, unpublished work.
16. R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Reading, MA: Addison-Wesley, 1983.

APPENDIX A

Proof of Lemma 2

Proof: Choose $a \in V$ and, using the continuity of g, let r and M be positive numbers such that $\mathscr{D} \subset V$ and $\|g(x)\| \leq M$ for $x \in \mathscr{D}$, where $\mathscr{D} = \{x \in \mathscr{B}: \|x - a\| < r\}$. Since g is analytic on $\mathscr{D}$ in the sense of Ref. 5, Definition 3.17.2, there are continuous symmetric m-linear maps Q_m from $\mathscr{B}^m$ into $\mathscr{B}_1$, which depend on a, such that

$$g(a) + \sum_{m=1}^{\infty} (m!)^{-1} Q_m(k, \cdots, k)$$

converges absolutely to $g(a + k)$ for $\|k\| < r$ (see Ref. 5, Theorems 3.16.2, 26.3.4, 26.3.6, and 26.6.6).

With $h \in \mathscr{B}$, consider the formal series

$$\sum_{m=1}^{\infty} (m!)^{-1} m Q_m(h, \cdots, h, \cdot) \tag{16}$$

whose terms are bounded linear maps from $\mathscr{B}$ to $\mathscr{B}_1$. Choose any $\beta_1 > 1$ and let $\beta = (\beta_1/r)$. Using an analog (Ref. 5, Theorem 3.16.3) of the Cauchy bounds and the m-linearity of Q_m,

$$\|Q_m(k, \cdots, k)\| \leq M(m!)(\beta\|k\|)^m$$

for $k \in \mathscr{B}$. Since (see Ref. 16, Proposition 2.2.11)

$$\sup\{\|Q_m(k_1, \cdots, k_m)\| : \|k_1\| \leq 1, \cdots, \|k_m\| \leq 1\}$$
$$\leq m^m (m!)^{-1} \sup\{\|Q_m(k, \cdots, k)\| : \|k\| \leq 1\} \tag{17}$$

we have

$$\|Q_m(k_1, \cdots, k_m)\| \leq m^m M \beta^m \|k_1\| \cdots \|k_m\| \tag{18}$$

for any $k_1, \cdots, k_m$ in $\mathscr{B}$. By Sterling's formula for $m!$,

$$m! > (2\pi)^{1/2} m^{1/2} m^m e^{-m}. \tag{19}$$

Thus, by (18) and (19) one has for the mth term in (16),

$$\|(m!)^{-1} m Q_m(h, \cdots, h, \cdot)\|$$
$$= \sup\{\|(m!)^{-1} m Q_m(h, \cdots, h, k)\| : \|k\| \leq 1\}$$
$$\leq (2\pi)^{-1/2} m^{1/2} M(e\beta) \ \|e\beta h\|^{(m-1)}.$$

It easily follows that there is a positive $r_1 < r$ such that (16) converges absolutely to a bounded linear map $L(a, h)$ for $\|h\| < r_1$. Since $L(a, h)$

is an (F)-power series in h in the sense of Ref. 5, Theorem 26.64, the map $L(a, \cdot)$ is analytic and hence Fréchet differentiable (Ref. 5, Theorem 3.17.1) on $\{x \in \mathscr{B}: \|x\| < r_1\}$. In particular, $L(a, \cdot)$ is Fréchet differentiable at θ.

In addition, for $\|h\| < \alpha r_1$ and $\|\delta\| < \alpha r_1$ where $\alpha \in (0, 1/2)$,

$$\begin{aligned}
&\|g(a + h + \delta) - g(a + h) - L(a, h)\delta\| \\
&\quad \leq \sum_{m=2}^{\infty} \|(m!)^{-1}[Q_m(h + \delta, \cdots, h + \delta) - Q_m(h, \cdots, h) \\
&\qquad - mQ_m(h, \cdots, h, \delta)]\| \\
&\quad \leq \sum_{m=2}^{\infty} (m!)^{-1}2^m \|Q_m\| (\alpha r_1)^{(m-2)} \|\delta\|^2, \qquad (20)
\end{aligned}$$

where $\|Q_m\|$ is the left side of (17). Using (18) and (19), it is a simple matter to verify that for α sufficiently small the extreme right side of (20) is $o(\|\delta\|)$ as $\|\delta\| \to 0$. This shows that $dg(a + h) = L(a, h)$ for all h in some neighborhood of θ, and thus that $dg(\cdot)$ is Fréchet differentiable at the arbitrary point $a \in V$.

APPENDIX B

An Example

Here we give a simple example of a map $(I - CN)$ and associated sets V_0 and V that meet the conditions at the beginning of Section 2.4. The construction of the example involves a contraction-mapping technique that is well known and often useful. We shall use L_∞ to denote the set of bounded complex-valued Lebesgue measurable functions defined on $[0, \infty)$ with the usual sup norm.*

Let $S = \mathscr{B} = L_\infty$, and let us take C to be an operator, such as a convolution operator, with induced norm ρ. We assume that $\rho > 0$. Let N be defined by $(Nx)(t) = x(t)^3$ for $t \geq 0$ and $x \in \mathscr{B}$, and take F: $\mathscr{B} \to \mathscr{B}$ to be given by $Fx = u + CNx$ for every $x \in \mathscr{B}$, where u is an element of $\mathscr{B}$.

We note that for x_0 in the closed ball $\beta(r)$ of radius r in $\mathscr{B}$ centered at θ, $dN(x_0)h(t) = 3x_0(t)^2h(t)$, $t \geq 0$, for arbitrary $h \in \mathscr{B}$, and therefore $\|CdN(x_0)\| \leq 3\rho r^2$. Thus, F is a contraction mapping on $\beta(r)$ for $0 < r < r_0$, where $r_0 = (3\rho)^{-1/2}$. Also, F takes $\beta(r)$ into itself if

$$\|u\| + \sup\{\|CNx\| : x \in \beta(r)\} \leq r,$$

* In other words, here L_∞ is the set L_∞ in Ref. 3 with $n = 1$. (The definition of L_∞ ordinarily found in the system-theoretic literature involves instead the essential sup norm. That our L_∞ in this paper, or in Ref. 3, is *complete* follows from the Cauchy criterion for uniform convergence and the standard proposition that pointwise limits of measurable functions are measurable.)

which is met when $\|u\| \leq q(r)$, where $q(r) = r - \rho r^3$. These observations and the monotonicity of q on $(0, r_0)$ motivate the following.

Take V to be the open ball in $\mathscr{B}$ of radius $q(r_0)$ centered at θ. Given any $u \in V$ there is an $r_1 \in (0, r_0)$ such that $\|u\| \leq q(r)$ for all $r \in [r_1, r_0)$. Hence, by the contraction-mapping theorem, for any such u and all associated r there is a unique solution $x \in \beta(r)$ of the equation $x - CNx = u$. Also, for any open ball β_a in V centered at some point a, there is an $r \in (0, r_0)$ such that $\|u\| \leq q(r)$ for $u \in \beta_a$, and it easily follows that the solutions x_1 and x_2 corresponding to any u_1 and u_2 in β_a satisfy $\|x_1 - x_2\| \leq (1 - 3\rho r^2)^{-1} \|u_1 - u_2\|$.

Thus, with V_0 the intersection of the open ball $\beta_0(r_0)$ of radius r_0 centered at θ with the inverse image of V under $(I - CN)$, we see that V_0 is open and that $(I - CN)$ is a homeomorphism of V_0 onto the open c-star V about the point θ, with $(I - CN)^{-1}$ locally Lipschitz on V.*

AUTHOR

Irwin W. Sandberg, B.E.E., 1955, M.E.E., 1956, and D.E.E., 1958, Polytechnic Institute of Brooklyn; AT&T Bell Laboratories, 1958—. Mr. Sandberg has been concerned with analysis of radar systems for military defense, synthesis and analysis of active and time-varying networks, with several fundamental studies of properties of nonlinear systems, and with some problems in communication theory and numerical analysis. His more recent interests have included compartmental models, the theory of digital filtering, global implicit-function theorems, and functional expansions for nonlinear systems. IEEE Centennial Medalist, Former Vice Chairman IEEE Group on Circuit Theory, and Former Guest Editor IEEE Transactions on Circuit Theory Special Issue on Active and Digital Networks. Fellow and member, IEEE; member, American Association for the Advancement of Science, Eta Kappa Nu, Sigma Xi, Tau Beta Pi, National Academy of Engineering.

---

* A similar example can be given that addresses the more general case in which $(Nx)(t) = \eta[x(t)]$, $t \geq 0$ $(x \in \mathscr{B})$ with η an entire function such that $\eta(0) = 0$.