An Algebraic Approach to a Nonproduct Form Network

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By applying an algebraic approach to the method of stages, an explicit solution is derived for a closed network consisting of a nonexponential server and a service station with two identical nonexponential servers in parallel. There is a finite number of jobs and the queueing discipline is first-come-first-served in the closed network. The solution is described in a quasi-matrix-geometric form, which is a generalization of the matrix-geometric form.

I. INTRODUCTION

There have been many developments and researches in the queueing networks for modeling of computer systems in the last two decades. Most researchers believe, however, that it is very difficult to conceptualize how to derive steady state solutions for general nonproduct form queueing networks, which do not have a product form solution.[1] For instance, a steady state solution for a general queueing network with first-come-first-served (FCFS) queueing discipline and general service times is unavailable at present. Several recent empirical papers have shown that service time distribution can have a significant effect on performance with particular emphasis on the modeling of computer systems.[2,3] Recently several works describe steady state solutions for the restricted cases of the nonproduct form queueing networks.[4-8] Neuts investigated a single server queue with phase type distribution and also found the solution to the M/G/1 queue.[6] Carroll et al.

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approached M/G/1 and GI/M/1 algebraically and found an explicit solution.[5] Van de Liefvoort extended Carroll's work to G/G/1//N type loops.[7] The solutions of these works are described in Neut's book on matrix geometric forms.

This paper attempts to generalize these studies. It focuses on a closed network consisting of a nonexponential server and a service station with two identical nonexponential servers in parallel, which is a typical model of a computer system; a central processing unit and two input/output processors. There are a finite number of jobs, and the queueing discipline is FCFS at each node. An explicit steady state solution for this network is derived in a *quasi matrix geometric* form, where matrices are recursively defined in certain product spaces. Obviously this form is a generalization of the matrix geometric form. This result may provide insight for obtaining exact solutions for general closed networks that do not have a product-form solution.

Section II of this paper introduces notations and definitions for describing the algebraic description of the general distribution server. In Section III, both external states and internal states are introduced to represent the states of the network. In Section IV, the global balance equations of the network is derived in a matrix form. Section V gives the steady state solution of the global balance equation. The conclusions are summarized in Section VI.

II. DESCRIPTION OF THE NETWORK

One of the common approaches for representing general service time distributions is to use the method of stages. That is, each nonexponential service time distribution is replaced by a subnetwork of exponential stages with the constraint that the subnetwork can only be accessed by one job at a time. The principle on which the method of stages is based is the memoryless property of exponential distribution. Thus this method is both general and compatible with the definition of Markov processes. All works to be studied in this paper are based on Cox's statement that any server whose service time distribution function has a rational Laplace transform could be replaced exactly by a subnetwork of exponential distributions.[9]

In this paper, we consider a closed network consisting of two service stations, a nonexponential server (station 0) and a service station (station 1) containing two identical nonexponential servers (see Fig. 1).

Let N denote the total number of jobs (there can be only one job active at any time within each of the servers). When a job completes service at either server of station 1, it leaves station 1 and joins the queue of station 0, while another job (if any) in the queue of station 1 takes its place.
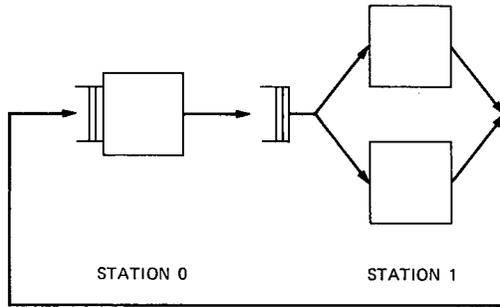
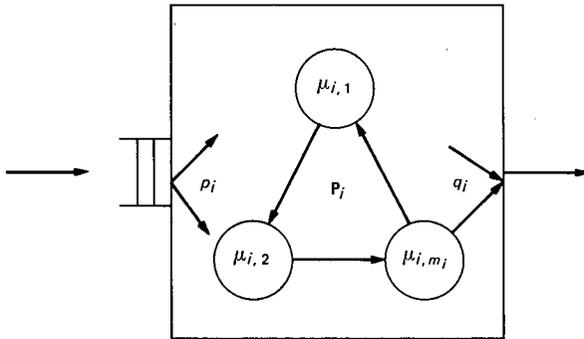Fig. 1—A closed network with nonexponential servers.

Fig. 2—Subnetwork of exponential stages.

Each of the nonexponential servers in Fig. 1 is represented by a subnetwork of exponential stages, as shown in Fig. 2. There are m_i exponential stages, whose service rates are $\mu_{i_1}, \ldots, \mu_{i,m_i}$, in the server of station i ($i = 0, 1$). The server of station i can be characterized by vectors and matrices. These vectors are denoted by lowercase letters, whereas matrices are denoted by uppercase and bold letters. This convention is followed without exception. The notation that will be used is consistent with that of Refs. 5, 7, and 8.

Definition 1: For each server of station i, define the following:

p_i = the entrance probability vector,
q_i = the departure probability vector,
$\mathbf{P}_i$ = the substochastic transition matrix, and
$\mathbf{M}_i$ = the service rate matrix = diag $(\mu_{i,1}, \ldots, \mu_{i,m_i})$.

For instance, $(p_i)_k$, the kth component of the vector p_i, is the conditional probability that a job, upon entering the server of station i, will first go to stage k. Similarly, $(q_i)_k$ is the conditional probability that a job, upon completing service at the stage k in the server of station i, will leave the server, and $(\mathbf{P}_i)_{kj}$ is the transition probability
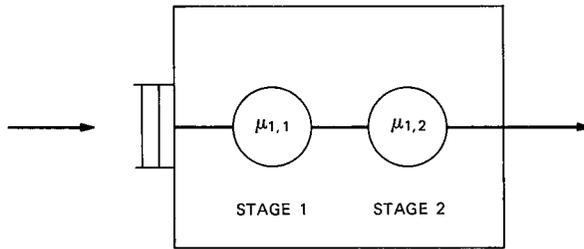
Fig. 3—Erlangian-2.

from stage k to stage j within the server of station i. One simple example is provided to show how to characterize the nonexponential server.

Example 1: If a server of station 1 is Erlangian-2 as shown in Fig. 3, then the server is characterized by the following vectors and matrices:

$$p_1 = (1 \quad 0), \quad q_1 = (0 \quad 1), \quad \mathbf{P}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{M}_1 = \begin{bmatrix} \mu_{1,1} & 0 \\ 0 & \mu_{1,2} \end{bmatrix}.$$

Lemma 1: For $i = 0, 1$, let $\mathbf{I}_i$ be an identity matrix of order m_i and e_i a vector containing m_i ones. The following relationships hold:

$$p_i e_i^T = 1 \tag{1}$$

and

$$q_i^T = (\mathbf{I}_i - \mathbf{P}_i) e_i^T, \tag{2}$$

where T denotes the transpose.

Proof: Using the law of total probability, we obtain the following relationships:

$$\sum_{j=0}^{m_i} (p_i)_j = 1, \quad \text{for} \quad i = 0, 1,$$

$$\sum_{k=1}^{m_i} (\mathbf{P}_i)_{jk} = 1 - (q_i)_j, \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad 1 \le j \le m_i.$$

Equations (1) and (2) are obtained from the above relationships. This proves the lemma.

Next, three important matrices in the paper are introduced to simplify the balance equations and express the properties of the server.

Definition 2: For $i = 0, 1$, define the following:

$$\mathbf{B}_i = \mathbf{M}_i(\mathbf{I}_i - \mathbf{P}_i), \tag{3}$$

$$\mathbf{V}_i = \mathbf{B}_i^{-1}, \quad \text{and} \tag{4}$$

$$\mathbf{Q}_i = e_i^T p_i, \tag{5}$$

where -1 denotes the matrix inverse.

Note that $\mathbf{Q}_i$ is an idempotent matrix of rank 1. We can assume that $\mathbf{I}_i - \mathbf{P}_i$, $i = 0, 1$, has an inverse since this expression means that a path exists from each stage out of the server of station i. Carroll et al.[5] and Neuts[4] have shown that traditional probability notations can be expressed by vectors and matrices.

Lemma 2: For $i = 0, 1$, let $b_i(x)$ be the probability density function (pdf) of the server of station i, $E_i(x^n)$ its nth moment, $E_i(x)$ its mean service time, and $B_i^*(s)$ its Laplace transform.
Then

$$B_i^*(s) = p_i(\mathbf{I}_i + s\mathbf{V}_i)^{-1}e_i^T, \tag{6}$$

$$E_i(x^n) = \int_0^\infty b_i(x)x^n dx = n! p_i \mathbf{V}_i^n e_i^T, \tag{7}$$

$$b_i(x) = p_i[\mathbf{B}_i \exp(-\mathbf{B}_i x)]e_i^T, \quad \text{and} \tag{8}$$

$$E_i(x) = p_i \mathbf{V}_i e_i^T. \tag{9}$$

Proof: Omitted.

III. THE STATES OF THE NETWORK

If there is only one job present ($N = 1$) in the system, queueing will never take place. The system is easily solved. If there is more than one job ($N > 1$), the state of the network is characterized by a number of jobs (called *external state*) and positions of active jobs (called *internal state*) at each station. The queueing problem is thus transformed from one involving the remaining service time for a job to one involving the position of the active job in the subnetwork.

In defining the states of the network, it is important to keep the internal state separate from the external state in order to make the balance equation easier to solve. Each external state has a set of internal states which we call the *internal state space*. Thus the state of the network can be represented by a pair, external state and internal state.

The set of possible external states, which we call the *external state space*, is determined as follows:

$$\{(N, 0), (N - 1, 1), \cdots, (N - n, n), \cdots, (0, N - 1)\},$$

where $1 \leq n \leq N$.

The cardinality of the external state space is $N + 1$. Next consider the internal state space. Since the number of internal states at station 0

is the same as the number of stages in the server, the internal state space of station 0 is defined as follows:

$$S_0 = \{1, 2, \cdots, m_0\}.$$

If there is only one job at station 1, it must be active at one of the servers. Thus the internal state space for one job at station 1 is defined as follows:

$$S_1 = \{1, 2, \cdots, m_1\}.$$

When there are two or more jobs at station 1, two of them must be active. In this case, a possible internal state would be a pair of integers (j, k), where one job is at stage j and the other is at stage k. Since the two nonexponential servers in station 1 are identical, (k, j) is the same as (j, k). Thus the internal state space for two or more jobs at station 1 is defined as follows:

$$S_2 = \{(i_1, i_2) \mid 1 \leq i_1 \leq i_2 \leq m_1\}.$$

Let $d(i)$ be the dimension (cardinality) of the state space S_i. There are $d(0) = m_0$ states in S_0, $d(1) = m_1$ states in S_1 and $d(2) = m_1(m_1 + 1)/2$ such states in S_2.

Definition 3: For $N > 2$, the *state of the network* is represented by a vector $[x_1, x_2]$, where x_1 is a vector representing an external state and x_2 is a vector representing the internal states of each station. That is

$[(N, 0), (i,)]$	= the state that all N jobs are at station 0, with internal state $i \in S_0$.
$[(N - 1, 1), (i, j)]$	= the state that $N - 1$ jobs are at station 0, with internal state $i \in S_0, j \in S_1$.
$[(N - n, n), (i, j)]$	= the state that $N - n$ jobs are at station 0, with internal state $i \in S_0$, and n jobs at station 1, with internal state $j \in S_2$, where $2 \leq n \leq N - 1$.
$[(0, N), (, j)]$	= the state that all N jobs are at station 1, with internal state $j \in S_2$.

Note that the cardinality of the state spaces is $d(0) + d(0)d(1) + (N - 2)d(0)d(2) + d(2)$. Next, the steady state probability vectors are defined similarly.

Definition 4: For $N > 2$, the steady state probability vectors, describing the internal states, are defined as follows:

$[b_0(N, 0)]_i$	= the steady state probability that the network is in the state $[(N, 0), (i,)]$, where $i \in S_0$.
$\pi_1(N, 0)$	= $b_0(N,0) \otimes p_1$, where $\otimes$ denotes Kronecker product[10] (see Appendix).

$[\pi_1(N - 1, 1)]_{i,j}$ = the steady state probability that the network is in the state $[(N - 1, 1), (i, j)]$, where $i \in S_0$ and $j \in S_1$.

$[\pi_2(N - n, n)]_{i,j}$ = the steady state probability that the network is in the state $[(N - n, n), (i, j)]$, where $2 \leq n \leq N - 1$, $i \in S_0$, and $j \in S_2$.

$[b_2(0, N)]_j$ = the steady state probability that the network is in the state $[(0, N), (, j)]$, where $j \in S_2$.

$\pi_2(0, N)$ = $p_0 \otimes b_2(0, N)$.

For $k = 1, 2$ and $n = 0, 1, \cdots, N$, the steady state probability $[\pi_k(N - n, n)]_{i,j}$ can be ordered lexicographically to create a vector $\pi_k(N - n, n)$. The subscript of these probability vectors denotes the dimension of the objects. For instance, the subscript k is used to denote the steady state probability vector of order $d(0)d(k)$. This vector is essentially decomposed into $d(0)$ subvectors, each of order $d(k)$. The vectors $b_0(N, 0)$ and $b_2(0, N)$ are of order $d(0)$ and $d(2)$.

Definition 5: For $N > 2$, the steady state probabilities, describing the external states, are defined as follows:

$\Pr(N, 0)$ = $\pi_1(N, 0)(e^{(1)})^T$
= the steady state probability that there are all N jobs at station 0.

$\Pr(N - 1, 1)$ = $\pi_1(N - 1, 1)(e^{(1)})^T$
= the steady state probability that there is one job at station 1.

$\Pr(N - n, n)$ = $\pi_2(N - n, n)(e^{(2)})^T$
= the steady state probability that there are n $(2 \leq n \leq N - 1)$ jobs at station 1.

$\Pr(0, N)$ = $\pi_2(0, N)(e^{(2)})^T$
= the steady state probability that there are all N jobs at station 1,

where $e^{(i)} = e_0 \otimes e_i = e_0 \hat{e}_i$, $i = 1, 2$, is a vector of order $d(0)d(i)$ containing all ones (see Appendix).

Next, objects of the state space S_2 are defined similarly. These are also necessary to connect between S_1 and S_2.

Definition 6: For the state space S_2, the following matrices are defined:

$\mathbf{M}_2$ = the diagonal matrix whose (i, i)th element is the probability rate of leaving state $i = (i_1, i_2) \in S_2$. That is, $(\mathbf{M}_2)_{ii} = \mu_{1,i_1} + \mu_{1,i_2}$.

$(\mathbf{P}_2)_{ij}$ = the probability of transition from state (k, i) to state (k, j), where $k \in S_0$, $i = (i_1, i_2) \in S_2$ and $j = (j_1, j_2) \in S_2$.

$$= \begin{cases} 0 & \text{if } \{i_1, i_2\} \cap \{j_1, j_2\} = \phi, \\ \dfrac{\mu_{1,\bar{\imath}}(\mathbf{P}_1)_{ij}}{(\mathbf{M}_2)_{ii}} & \text{if } \{i_1, i_2\} \cap \{j_1, i_2\} \neq \phi \quad \text{and} \quad i_1 \neq i_2, \\ (\mathbf{P}_1)_{\bar{\imath}\bar{\jmath}} & \text{if } \{i_1, i_2\} \cap \{j_1, j_2\} \neq \phi \quad \text{and} \quad i_1 = i_2, \end{cases}$$

where $\bar{\imath}$ is the other member of the i-pair, $\bar{\jmath}$ is the other member of the j-pair, and $\cap$ means set intersection.

$(\mathbf{R}_2)_{ij}$ = the probability that upon a job entering station 1, the state is changed from (k, i) to (k, j), where $k \in S_0$, $i \in S_1$, and $j = (j_1, j_2) \in S_2$.

$$= \begin{cases} (p_1)_j & \text{if } j_1 \text{ or } j_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$(\mathbf{Q}_2)_{ij}$ = the probability that upon a job completing at station 1, the state is changed from (k, i) to (k, j), where $k \in S_0$, $j \in S_1$, and $i = (i_1, i_2) \in S_2$.

$$= \begin{cases} \dfrac{\mu_{1,\bar{\imath}}(q_1)_{\bar{\imath}}}{(\mathbf{M}_2)_{ii}} & \text{if } i_1 \neq i_2, \\ (q_1)_{\bar{\imath}} & \text{if } i_1 = i_2. \end{cases}$$

A simple example is provided to show how to construct the objects of the state space S_2.

Example 2: If station 1 has two Erlangian servers ($m_1 = 2$) in parallel as shown in Fig. 4, then the internal state spaces and the dimension of the spaces are

$$S_1 = \{1, 2\},$$

$$S_2 = \{(1, 1), (1, 2), (2, 2)\},$$

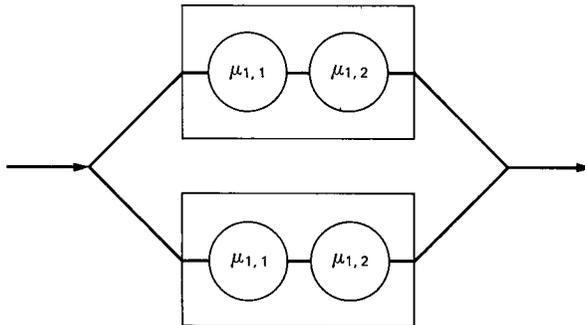$$d(1) = 2, \quad \text{and} \quad d(2) = 3.$$

Fig. 4—Station 1 containing two Erlangian servers in parallel.

Using definition 6, the objects of S_2 are determined by

$$\mathbf{M}_2 = \begin{bmatrix} \mu_{1,1} + \mu_{1,1} & 0 & 0 \\ 0 & \mu_{1,1} + \mu_{1,2} & 0 \\ 0 & 0 & \mu_{1,2} + \mu_{1,2} \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} 0 & (\mathbf{P}_1)_{12} & 0 \\ \dfrac{\mu_{1,2}(\mathbf{P}_1)_{21}}{\mu_{1,1} + \mu_{1,2}} & 0 & \dfrac{\mu_{1,1}(\mathbf{P}_1)_{12}}{\mu_{1,1} + \mu_{1,2}} \\ 0 & (\mathbf{P}_1)_{21} & 0 \end{bmatrix}$$

$$\mathbf{R}_2 = \begin{bmatrix} (p_1)_2 & (p_1)_2 & 0 \\ 0 & (p_1)_1 & (p_1)_2 \end{bmatrix}$$

$$\mathbf{Q}_2 = \begin{bmatrix} (q_1)_1 & 0 \\ \dfrac{\mu_{1,2}(q_1)_2}{\mu_{1,1} + \mu_{1,2}} & \dfrac{\mu_{1,1}(q_1)_1}{\mu_{1,1} + \mu_{1,2}} \\ 0 & (q_1)_2 \end{bmatrix}.$$

Note that $\mathbf{R}_2$ is $d(1) \times d(2)$ dimensional rectangular matrix and $\mathbf{Q}_2$ is a $d(2) \times d(1)$ dimensional rectangular matrix. Thus $\mathbf{Q}_2\mathbf{R}_2$ is the probability matrix that upon a completion at station 1, a job leaves and another takes its place. This event can happen only if there are more than two jobs at station 1.

Lemma 3: For the state space S_1 and S_2, let $\mathbf{I}_i$ be an identity matrix of order $d(i)$ and e_i a vector containing $d(i)$ ones. The following relationships hold:

$$\mathbf{R}_2 e_2^T = e_1^T, \tag{10}$$

$$(\mathbf{P}_2 + \mathbf{Q}_2\mathbf{R}_2)e_2^T = e_2^T, \tag{11}$$

$$\mathbf{Q}_2\mathbf{R}_2 e_2^T = \mathbf{Q}_2 e_1^T = (\mathbf{I}_2 - \mathbf{P}_2)e_2^T, \quad \text{and} \tag{12}$$

$$p_1 \mathbf{R}_2 e_2^T = 1. \tag{13}$$

Proof: This lemma follows immediately from lemma 1 and the law of total probability.

Next, note that we have encountered objects of different state spaces S_0, S_1, and S_2. They are of different dimensions $d(0)$, $d(1)$, and $d(2)$. Objects of order $d(0)d(1)$ and $d(0)d(2)$ are said to be on the *product spaces*. The product space of order $d(i)d(j)$ is represented by $F_{d_i x d_j}$. Before any operation can be performed, all objects have to be replaced by their images under embedding into product spaces. For instance, objects of order 1 are scalars. Since scalars constitute subspaces of

spaces S_0, S_1, and S_2 and of the product spaces, and there is a 1–1 correspondence between scalars and diagonal matrices all of whose diagonal elements are equal, scalars are isomorphic to scalar multiples of matrix I. Similarly, the spaces S_0, S_1, and S_2 are subspaces of the product spaces, so that objects from these spaces have a counterpart in the product spaces while algebraic relationships are preserved. These counterparts will be denoted by adding both superscripts and a hat, these counterparts are the image of an embedding (for details, see Appendix). For instance, A_0 is a matrix in the state space S_0, and A_1 is a matrix in the state space S_1. $\hat{A}_0^{(1)}$ is a matrix in the product space $F_{d_0 x d_1}$ and has characteristics inherited from matrix A_0. Also $\hat{A}_1^{(0)}$ is a matrix in the product space $F_{d_0 x d_1}$ and has characteristics from A_1. If there is no possibility of confusion, the superscript (0) is omitted. That is, $\hat{A}_1^{(0)}$ and $\hat{A}_2^{(0)}$ are denoted by $\hat{A}_1$ and $\hat{A}_2$.

IV. BALANCE EQUATION

In order for the network to be in steady state, the probability of leaving a particular state must be equal to the probability of entering that state. Thus the total flow into a particular internal state must be equal to the total flow out of that state. The steady state balance equations can then be derived for the internal states. All internal states belonging to one external state are combined in one mathematical entity, a vector. The balance equations can then be written in a matrix form, using Kronecker products and generalized embedding of vectorspaces (see Appendix).

Theorem 1: For the closed network with a nonexponential server and a station containing two identical nonexponential servers and $N > 2$, the global balance equations are:

$$\pi_1(N, 0)\hat{B}_0^{(1)} = \pi_1(N - 1, 1)\hat{B}_1\hat{Q}_1 \tag{14}$$

$$\pi_1(N - 1, 1)[\hat{B}_0^{(1)} + \hat{B}_1] = \pi_2(N - 2, 2)\hat{M}_2\hat{Q}_2$$
$$+ \pi_1(N, 0)\hat{B}_0^{(1)}\hat{Q}_0^{(1)} \tag{15}$$

$$\pi_2(N - 2, 2)[\hat{B}_0^{(2)} + \hat{B}_2] = \pi_2(N - 3, 3)\hat{M}_2\hat{Q}_2\hat{R}_2$$
$$+ \pi_1(N - 1, 1)\hat{B}_0^{(1)}\hat{Q}_0^{(1)}\hat{R}_2 \tag{16}$$

$$\pi_2(N - n, n)[\hat{B}_0^{(2)} + \hat{B}_2] = \pi_2(N - n - 1, n + 1)\hat{M}_2\hat{Q}_2\hat{R}_2$$
$$+ \pi_2(N - n + 1, n - 1)\hat{B}_0^{(2)}\hat{Q}_0^{(2)}, \quad \text{for} \quad n = 3, \cdots, N - 1 \tag{17}$$

$$\pi_2(0, N)\hat{B}_2 = \pi_2(1, N - 1)\hat{B}_0^{(2)}\hat{Q}_0^{(2)}, \tag{18}$$

$$\text{where} \quad \hat{B}_i = \hat{M}_i(\hat{I} - \hat{P}_i), \quad i = 1, 2. \tag{19}$$

Proof: The proof will be done in five different parts, one part for each of eqs. (14) through (18).

(1) By equating the flow out of and into a state $[(N, 0), (i,)]$ it follows that

$$[b_0(N, 0)]_i(\mathbf{M}_0)_{ii} = \sum_{k=1}^{d(0)} [b_0(N, 0)]_k(\mathbf{M}_0)_{kk}(\mathbf{P}_0)_{ki}$$

$$+ \sum_{k=1}^{d(1)} [\pi_1(N - 1, 1)]_{i,k}(\mathbf{M}_1)_{kk}(q_1)_k. \quad (20)$$

π_1 is a vector in the product space $F_{d_0 \times d_1}$, whereas the other factors are objects from space S_0. In order to write eq. (20) in a matrix form, these objects must be augmented into objects from product space $F_{d_0 \times d_1}$. By postmultiplying eq. (20) by $\hat{p}_1$ and using generalized embedding, eq. (20) can be written in a matrix form where vectors and matrices are in product space $F_{d_0 \times d_1}$:

$$\pi_1(N, 0)\hat{\mathbf{M}}_0^{(1)} = \pi_1(N, 0)\hat{\mathbf{M}}_0^{(1)}\hat{\mathbf{P}}_0^{(1)} + \pi_1(N - 1, 1)\hat{\mathbf{M}}_1\hat{q}_1^T\hat{p}_1. \quad (21)$$

It can be simplified by substituting $(\hat{\mathbf{I}}_i - \hat{\mathbf{P}}_i)\hat{e}_i^T$ for $\hat{q}_i^T$ and $\hat{\mathbf{B}}_i$ for $\hat{\mathbf{M}}_i(\hat{\mathbf{I}}_i - \hat{\mathbf{P}}_i)$:

$$\pi_1(N, 0)\hat{\mathbf{B}}_0^{(1)} = \pi_1(N - 1, 1)\hat{\mathbf{B}}_1\hat{e}_1^T\hat{p}_1. \quad (22)$$

By substituting $\hat{\mathbf{Q}}_1$ for $\hat{e}_1^T\hat{p}_1$, balance equation (14) is obtained.

(2) By equating the flow out of and into a state $[(N - 1, 1), (i, j)]$ it follows that

$$[\pi_1(N - 1, 1)]_{i,j}\{(\mathbf{M}_0)_{ii} + (\mathbf{M}_1)_{jj}\}$$

$$= \sum_{k=1}^{d(0)} [\pi_1(N - 1, 1)]_{k,j}(\mathbf{M}_0)_{kk}(\mathbf{P}_0)_{ki}$$

$$+ \sum_{k=1}^{d(1)} [\pi_1(N - 1, 1)]_{i,k}(\mathbf{M}_1)_{kk}(\mathbf{P}_1)_{kj}$$

$$+ \sum_{k=1}^{d(0)} [b_0(N, 0)]_k(\mathbf{M}_0)_{kk}(q_0)_k(p_0)_i(p_1)_j$$

$$+ \sum_{k=1}^{d(2)} [\pi_2(N - 2, 2)]_{i,k}(\mathbf{M}_2)_{kk}(\mathbf{Q}_2)_{kj}. \quad (23)$$

By simplifying eq. (23) and using generalized embedding, balance eq. (15) is obtained.

(3) By equating the flow out of and into a state $[(N - 2, 2), (i, j)]$ it follows that

$$[\pi_2(N - 2, 2)]_{i,j}\{(\mathbf{M}_0)_{ii} + (\mathbf{M}_2)_{jj}\}$$

$$= \sum_{k=1}^{d(0)} [\pi_2(N - 2, 2)]_{k,j}(\mathbf{M}_0)_{kk}(\mathbf{P}_0)_{ki}$$

$$+ \sum_{k=1}^{d(2)} [\pi_2(N - 2, 2)]_{i,k}(\mathbf{M}_2)_{kk}(\mathbf{P}_2)_{kj}$$

$$+ \sum_{k1=1}^{d(1)} \sum_{k0=1}^{d(0)} [\pi_1(N - 1, 1)]_{k0,k1}(\mathbf{M}_0)_{k0k0}(q_0)_{k0}(p_0)_i(\mathbf{R}_2)_{k1j}$$

$$+ \sum_{k=1}^{d(2)} [\pi_2(N - 3, 3)]_{i,k}(\mathbf{M}_2)_{kk}(\mathbf{Q}_2\mathbf{R}_2)_{kj}. \quad (24)$$

By simplifying eq. (24) and using generalized embedding, balance eq. (16) is obtained.

(4) By equating the flow out of and into a state $[(N - n, n), (i, j)]$ with $n = 3, \cdots, N - 1$, it follows that

$$[\pi_2(N - n, n)]_{i,j}\{(\mathbf{M}_0)_{ii} + (\mathbf{M}_2)_{jj}\}$$

$$= \sum_{k=1}^{d(0)} [\pi_2(N - n, n)]_{k,j}(\mathbf{M}_0)_{kk}(\mathbf{P}_0)_{ki}$$

$$+ \sum_{k=1}^{d(2)} [\pi_2(N - n, n)]_{i,k}(\mathbf{M}_2)_{kk}(\mathbf{P}_2)_{kj}$$

$$+ \sum_{k=1}^{d(0)} [\pi_2(N - n + 1, n - 1)]_{k,j}(\mathbf{M}_0)_{kk}(q_0)_k(p_0)_i$$

$$+ \sum_{k=1}^{d(2)} [\pi_2(N - n - 1, n + 1)]_{i,k}(\mathbf{M}_2)_{kk}(\mathbf{Q}_2\mathbf{R}_2)_{kj}. \quad (25)$$

By simplifying eq. (25) and using generalized embedding, balance eq. (17) is obtained.

(5) By equating the flow out of and into a state $[(0, N), (, j)]$ it follows that

$$[b_2(0, N)]_j(\mathbf{M}_2)_{jj} = \sum_{k=1}^{d(2)} [b_2(0, N)]_k(\mathbf{M}_2)_{kk}(\mathbf{P}_0)_{ki}$$

$$+ \sum_{k=1}^{d(0)} [\pi_2(1, N - 1)]_{k,j}(\mathbf{M}_0)_{kk}(q_0)_k. \quad (26)$$

By simplifying eq. (26) and using generalized embedding, balance eq. (18) is obtained. That completes the proof.

Next consider the entire Markov chain of this network. The state probability vector of the entire Markov chain can be represented by

$$\pi = (\pi_1(N, 0), \ \pi_1(N - 1, 1), \ \pi_2(N - 2, 2),$$

$$\cdots, \pi_2(N - n, n), \ \cdots, \pi_2(0, N)), \quad 2 \leq n \leq N.$$

This satisfies balance eq. (10) and the law of total probability. The infinitesimal generator $\mathbf{Q}^*$ of the Markov chain can be written as follows:

$$
\mathbf{Q}^* =
\begin{bmatrix}
\hat{\mathbf{B}}_0^{(1)} & -\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} \\
-\hat{\mathbf{B}}_1\hat{\mathbf{Q}}_1 & \hat{\mathbf{B}}_0^{(1)} + \hat{\mathbf{B}}_1 & -\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}\hat{\mathbf{R}}_2 & \cdots & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & -\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2 & \hat{\mathbf{B}}_0^{(2)} + \hat{\mathbf{B}}_2 & \cdots & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & -\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2 & \cdots & \mathbf{O} & \mathbf{O} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \hat{\mathbf{B}}_0^{(2)} + \hat{\mathbf{B}}_2 & -\hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2 & \hat{\mathbf{B}}_2
\end{bmatrix}
$$

Wallace also termed such processes *Quasi Birth* and *Death* (QBD) processes.[11] He has shown that if a QBD process is *boundary leading*, the process has a matrix geometric form solution.

The global balance equations were derived by equating the flow in and out of a certain state. Thus the flow into an external state must be equal to the flow out of the external state because the external state is a collection of states.

Lemma 4: (Flow-balance equations between the external states.) For $N > 2$, the flow out of a certain external state equals the flow into the external state. That is

$$\pi_1(N, 0)\hat{\mathbf{B}}_0^{(1)}(e^{(1)})^T = \pi_1(N - 1, 1)\hat{\mathbf{B}}_1(e^{(1)})^T \tag{27}$$

$$\pi_1(N - 1, 1)\hat{\mathbf{B}}_0^{(1)}(e^{(1)})^T = \pi_2(N - 2, 2)\hat{\mathbf{B}}_2(e^{(2)})^T \tag{28}$$

$$\pi_2(N - n, n)\hat{\mathbf{B}}_0^{(2)}(e^{(2)})^T = \pi_2(N - n - 1, n + 1)\hat{\mathbf{B}}_2(e^{(2)})^T,$$

$$for \quad n = 2, 3, \ldots, N - 1. \tag{29}$$

Proof: The proof will be done in three parts, one part for each of eqs. (27) through (29).

(1) By postmultiplying global balance eq. (14) by $(e^{(1)})^T$, eq. (27) is obtained.

(2) By postmultiplying balance eq. (15) by $(e^{(1)})^T$, we get

$$\pi_1(N, 0)\hat{\mathbf{B}}_0^{(1)}(e^{(1)})^T - \pi_1(N - 1, 1)\hat{\mathbf{B}}_1(e^{(1)})^T$$

$$= \pi_1(N - 1, 1)\hat{\mathbf{B}}_0^{(1)}(e^{(1)})^T - \pi_2(N - 2, 2)\hat{\mathbf{B}}_2(e^{(2)})^T. \tag{30}$$

By applying eq. (27) to eq. (30), we get

$$\pi_1(N - 1, 1)\hat{\mathbf{B}}_0^{(1)}(e^{(1)})^T = \pi_2(N - 2, 2)\hat{\mathbf{B}}_2(e^{(2)})^T,$$

which is eq. (28).

(3) By induction on n ($n = 2, 3, \cdots, N - 1$),

Basis step: By postmultiplying balance eq. (16) by $(e^{(2)})^T$, it follows that

$$\pi_2(N - 2, 2)\hat{\mathbf{B}}_0^{(2)}(e^{(2)})^T - \pi_2(N - 3, 3)\hat{\mathbf{B}}_2(e^{(2)})^T$$
$$= \pi_1(N - 1, 1)\hat{\mathbf{B}}_0^{(1)}(e^{(1)})^T - \pi_2(N - 2, 2)\hat{\mathbf{B}}_2(e^{(2)})^T. \quad (31)$$

By applying eq. (28) to eq. (31), we get

$$\pi_2(N - 2, 2)\mathbf{B}_0^{(2)}(e^{(2)})^T - \pi_2(N - 3, 3)\hat{\mathbf{B}}_2(e^{(2)})^T.$$

Inductive step: Suppose that

$$\pi_2(N - k, k)\hat{\mathbf{B}}_0^{(2)}(e^{(2)})^T = \pi_2(N - k - 1, k + 1)\hat{\mathbf{B}}_2(e^{(2)})^T,$$

$$2 \le k \le N - 2. \quad (32)$$

By postmultiplying balance eq. (17) by $(e^{(2)})^T$ and rearranging, we get

$$\pi_2(N - k - 1, k + 1)\hat{\mathbf{B}}_0^{(2)}(e^{(2)})^T$$
$$= \pi_2(N - k - 2, k + 2)\hat{\mathbf{B}}_2(e^{(2)})^T. \quad (33)$$

Thus, we have

$$\pi_2(N - n, n)\hat{\mathbf{B}}_0^{(2)}(e^{(2)})^T = \pi_2(N - n - 1, n + 1)\hat{\mathbf{B}}_2(e^{(2)})^T,$$

$$\text{for} \quad n = 2, 3, \cdots, N - 1,$$

which is eq. (29). This proves the lemma.

V. STEADY STATE SOLUTIONS

The solution of the global balance equations (Theorem 1) can be derived now. A definition is introduced to obtain the steady state solutions.

Definition 7: For $N > 2$, define recursively the following matrices:

$$\mathbf{U}_2(0) = \hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)}(\hat{\mathbf{B}}_2)^{-1}, \quad (34)$$

$$\mathbf{U}_2(n) = \hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)}[\hat{\mathbf{B}}_0^{(2)} + \hat{\mathbf{B}}_2 - \mathbf{U}_2(n - 1)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2]^{-1},$$

$$\text{for} \quad n = 1, 2, \ldots, N - 2, \quad (35)$$

$$\mathbf{U}_1(N - 1) = \hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}[\hat{\mathbf{B}}_0^{(1)} + \hat{\mathbf{B}}_1 - \hat{\mathbf{R}}_2\mathbf{U}_2(N - 2)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2]^{-1}. \quad (36)$$

Since $\pi_1(N, 0)$ depends on one unknown vector $b_0(N, 0)$, we can express all state probability vector π's in terms of it. With notations defined above we are ready to derive the steady state solutions.

Theorem 2: For the closed network with a nonexponential server and a station containing two identical nonexponential servers and $N > 2$, the steady state probability vectors can be described by

$$\pi_1(N, 0) = b_0(N, 0)\hat{p}_1$$

$$\pi_1(N - 1, 1) = \pi_1(N, 0)\mathbf{U}_1(N - 1) \tag{37}$$

$$\pi_2(N - 2, 2) = \pi_1(N, 0)\mathbf{U}_1(N - 1)\hat{\mathbf{R}}_2\mathbf{U}_2(N - 2) \tag{38}$$

$$\pi_2(N - n, n) = \pi_1(N, 0)\mathbf{U}_1(N - 1)\hat{\mathbf{R}}_2\mathbf{U}_2(N - 2)\mathbf{U}_2(N - 3)$$

$$\cdots \mathbf{U}_2(N - n), \quad for \quad n = 3, 4, \cdots, N. \tag{39}$$

Proof: The proof will be done in three parts, one part for each of eqs. (37) through (39).

(1) By induction on $N - n$ ($N - n = 0, 1, \cdots, N - 3$),

Basis step: From eq. (18) and definition 7, we get

$$\pi_2(0, N) = \pi_2(1, N - 1)\hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)}(\hat{\mathbf{B}}_2)^{-1} = \pi_2(1, N - 1)\mathbf{U}_2(0). \tag{40}$$

Inductive step: Suppose that

$$\pi_2(k, N - k) = \pi_2(k + 1, N - k - 1)\mathbf{U}_2(k),$$

$$0 \leq k < N - 3. \tag{41}$$

From eq. (17) and (41), we get

$$\pi_2(k + 1, N - k - 1)\{\hat{\mathbf{B}}_0^{(2)} + \hat{\mathbf{B}}_2\}$$

$$= \pi_2(k, N - k)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2$$

$$+ \pi_2(k + 2, N - k - 2)\hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)}$$

$$= \pi_2(k + 1, N - k - 1)\mathbf{U}_2(k)$$

$$\cdot \hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2 + \pi_2(k + 2, N - k - 2)\hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)}.$$

Thus, using definition 7, it follows that

$$\pi_2(k + 1, N - k - 1) = \pi_2(k + 2, N - k - 2)\hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)}$$

$$\cdot [\hat{\mathbf{B}}_0^{(2)} + \hat{\mathbf{B}}_2 - \mathbf{U}_2(k)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2]^{-1}$$

$$= \pi_2(k + 2, N - k - 2)\mathbf{U}_2(k + 1).$$

Therefore, we have

$$\pi_2(N - n, n) = \pi_2(N - n + 1, n - 1)\mathbf{U}_2(N - n),$$

$$for \quad N - n = 0, 1, 2, \cdots, N - 3. \tag{42}$$

(2) From eqs. (16) and (42), we get

$$\pi_2(N - 2, 2)(\hat{\mathbf{B}}_0^{(2)} + \hat{\mathbf{B}}_2) = \pi_2(N - 3, 3)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2$$
$$+ \pi_1(N - 1, 1)\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}\hat{\mathbf{R}}_2$$
$$= \pi_2(N - 2, 2)\mathbf{U}_2(N - 3)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2$$
$$+ \pi_1(N - 1, 1)\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}\hat{\mathbf{R}}_2$$

Thus, using definition 7, it follows that

$$\pi_2(N - 2, 2) = \pi_1(N - 1, 1)\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}\hat{\mathbf{R}}_2$$
$$\cdot [\hat{\mathbf{B}}_0^{(2)} + \hat{\mathbf{B}}_2 - \mathbf{U}_2(N - 3)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2]^{-1}$$
$$= \pi_1(N - 1, 1)\hat{\mathbf{R}}_2\hat{\mathbf{B}}_0^{(2)}\hat{\mathbf{Q}}_0^{(2)}$$
$$\cdot [\hat{\mathbf{B}}_0^{(2)} + \hat{\mathbf{B}}_2 - \mathbf{U}_2(N - 3)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2\hat{\mathbf{R}}_2]^{-1}$$
$$= \pi_1(N - 1, 1)\hat{\mathbf{R}}_2\mathbf{U}_2(N - 2). \tag{43}$$

(3) From eqs. (15) and (43), we get

$$\pi_1(N - 1, 1)(\hat{\mathbf{B}}_0^{(1)} + \hat{\mathbf{B}}_1) = \pi_2(N - 2, 2)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2 + \pi_1(N, 0)\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}$$
$$= \pi_1(N - 1, 1)\hat{\mathbf{R}}_2\mathbf{U}_2(N - 2)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2$$
$$+ \pi_1(N, 0)\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}.$$

Thus, using definition 7, it follows that

$$\pi_1(N - 1, 1) = \pi_1(N, 0)\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}$$
$$\cdot [\hat{\mathbf{B}}_0^{(1)} + \hat{\mathbf{B}}_1 - \hat{\mathbf{R}}_2\mathbf{U}_2(N - 2)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2]^{-1} = \pi_1(N, 0)\mathbf{U}_1(N - 1). \tag{44}$$

From eqs. (42), (43), and (44), the steady state vectors can be written as described in eqs. (37), (38), and (39). That completes the proof.

The solution of the network, as described in Theorem 2, is expressed in terms of the vector $b_0(N, 0)$ and the matrix $U_k(N - n)$'s. The vector $b_0(N, 0)$ is thus far unknown, while matrix $U_k(N - n)$'s can be generated. In general, the vector $b_0(N, 0)$ contains $d(0)$ unknowns. Definitions and a lemma are introduced to make the solution explicit.

Definition 8: For $N > 2$, define recursively the following matrix:

$$R_1(N - 1) = (\hat{\mathbf{B}}_0^{(1)} + \hat{\mathbf{B}}_1 - \hat{\mathbf{R}}_2\mathbf{U}_2(N - 2)\hat{\mathbf{M}}_2\hat{\mathbf{Q}}_2)^{-1} \tag{45}$$

so that $\mathbf{U}_1(N - 1) = \hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}R_1(N - 1)$.

Definition 9: For $N > 2$, define the following matrix of order $d(1) \times d(2)$:

$$\mathbf{K}(N) = \mathbf{R}_1(N-1)\{\hat{\mathbf{B}}_1\hat{\mathbf{Q}}_1(\hat{\mathbf{B}}_0^{(1)})^{-1}\hat{\mathbf{R}}_2 + \hat{\mathbf{R}}_2[\hat{\mathbf{I}}_2 + \mathbf{U}_2(N-2) + \cdots$$

$$+ \mathbf{U}_2(N-2)\mathbf{U}_2(N-3) \cdots \mathbf{U}_2(2)$$

$$+ \mathbf{U}_2(N-2)\mathbf{U}_2(N-3) \cdots \mathbf{U}_2(2)\mathbf{U}_2(1)$$

$$+ \mathbf{U}_2(N-2)\mathbf{U}_2(N-3) \cdots \mathbf{U}_2(2)\mathbf{U}_2(1)\mathbf{U}_2(0)]\}. \tag{46}$$

Lemma 5: The following relation holds:

$$b_0(N, 0)\mathbf{B}_0\mathbf{Q}_0\hat{p}_1\mathbf{K}(N)(e^{(2)})^T = 1. \tag{47}$$

Proof: Using the law of total probability and definition 4, we get

$$1 = \sum_{n=0}^{N} r(N-n, n) = \sum_{n=0}^{1} \pi_1(N-n, n)(e^{(1)})^T$$

$$+ \sum_{n=2}^{N} \pi_2(N-n, n)(e^{(2)})^T$$

$$= \{\sum_{n=0}^{1} \pi_1(N-n, n)\hat{\mathbf{R}}_2 + \sum_{n=2}^{N} \pi_2(N-n, n)\}(e^{(2)})^T$$

$$= \pi_1(N, 0)\hat{\mathbf{B}}_0^{(1)}\hat{\mathbf{Q}}_0^{(1)}\mathbf{K}(N)(e^{(2)})^T = b_0(N, 0)\mathbf{B}_0\mathbf{Q}_0\hat{p}_1\mathbf{K}(N)(e^{(2)})^T.$$

This proves the lemma.

To make the solution explicit, a normalization vector must be determined. The normalization vector can be derived now.

Theorem 3: The normalization vector can be determined by

$$b_0(N, 0)\mathbf{B}_0\mathbf{Q}_0 = c(N)p_0, \tag{48}$$

where $c(N) = \dfrac{1}{p_0\hat{p}_1\mathbf{K}(N)(e^{(2)})^T}$ *is a scalar.*

Proof: The normalization vector is proportional to p_0:

$$b_0(N, 0)\mathbf{B}_0\mathbf{Q}_0 = b_0(N, 0)\mathbf{B}_0 e_0^T p_0 = c(N)p_0, \tag{49}$$

where $c(N) = b_0(N, 0)\mathbf{B}_0 e_0^T$ is a scalar. Thus, the proportionality is normalization constant $c(N)$. By substituting eq. (49) for eq. (47), it follows that

$$c(N)p_0\hat{p}_1\mathbf{K}(N)(e^{(2)})^T = 1.$$

The proof is now completed.

VI. CONCLUSION

An explicit solution is derived for a closed network consisting of a nonexponential server and a service station with two identical nonexponential servers in parallel (e.g. CPU and I/O devices). There is a

finite number of jobs, and the queueing discipline at each node is FCFS. By applying an algebraic approach to the method of stages, the properties of each nonexponential server are represented by vectors and matrices in a space identified with that server. Product spaces are introduced to combine the properties of these servers and to allow their interactions to be described. Both external states and internal states are introduced to represent the states of the network. That is, the queueing problem is thus transformed from one involving the time of service remaining for a job to one involving the position of an active job in the subnetwork of exponential stages. In the mathematical view, the problem of integral equations (continuous) is transformed to one of algebraic equations (discrete) over a finite dimension.

From Theorems 1, 2, and 3, it turns out that the solution of this network is described in what I call a quasi matrix geometric form, in which the steady state vector is given by

$$\pi_k = \pi_0 \mathbf{U}(N-1)\mathbf{U}(N-2) \cdots \mathbf{U}(N-k), \quad \text{for} \quad 1 \le k \le N,$$

where π_0 is a normalization vector chosen to make the steady state probabilities sum to 1, and each $\mathbf{U}(n)$ is a matrix which is recursively defined by matrices involved. Obviously this form is a generalization of the matrix geometric form. All of techniques used in this paper are drawn from linear algebra. This algorithm is easy to implement, even for a person not familiar with queueing theory, because it involves only matrix multiplication and inversion (we can use commercial matrix packages).

VII. ACKNOWLEDGMENTS

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APPENDIX

A.1 Kronecker Products

The Kronecker product is the direct product of two disjoint operator spaces. In particular, if K is an $m \times n$ and L an $r \times s$ matrix, then the Kronecker product of K and L, denoted $K \otimes L$, is the matrix of size $mr \times ns$ which is obtained by multiplying each element $(K)_{ij}$ of matrix K by the full matrix L.

A.2 Embedding of Vectorspaces

In this paper, there are equations involving matrices of different dimensions. Before any operation can be performed, all matrices have to be replaced by their images under embedding into product spaces. According to the definitions in the previous sections, $d(i)$ denotes a dimension of state space $S_i (i = 0, 1)$. Let $F_{d_i \times d_j}$ be the additive group of $d(i) \times d(j)$ matrices.

Definition A.1: Define the following mappings between $F_{d_0 \times d_0}$, $F_{d_1 \times d_1}$, $\cdots F_{d_c \times d_c}$ and $F_{d_0 d_1 \times d_0 d_1}$, $F_{d_0 d_2 \times d_0 d_2}$ as follows:

$$\hat{} : \quad F_{d_i \times d_i} \rightarrow F_{d_0 d_i \times d_0 d_i} ; \quad A_i \rightarrow \hat{A}_i = I_0 \otimes A_i$$
$$\hat{}(i) : F_{d_0 \times d_0} \rightarrow F_{d_0 d_i \times d_0 d_i} ; \quad A_0 \rightarrow \hat{A}_0^{(i)} = A_0 \otimes I_i,$$

where $\otimes$ stands for the Kronecker product and $i = 1, 2, \ldots, c$.

These mappings are group homomorphisms and preserve all algebraic characteristics of the matrix. The proof is omitted.

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