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A Conditional Response Time of the M/M/1 Processor-Sharing Queue

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In this paper we examine the distribution of the response time of an arriving customer conditioned on the number of customers present in an M/M/1 processor-sharing queue. We show that the rth moment of the distribution is a polynomial in the number of customers present, and obtain a recursion for its determination. The Laplace-Stieltjes transform of the conditional distribution is also obtained. We find an expansion in powers of the arrival rate, which permits accurate computation when the utilization is not too close to one, and also give an asymptotic expansion when the number of customers in the system is large. This permits assessment of response time in a heavily loaded system. We present numerical results using these methods and discuss their relative merits.

I. INTRODUCTION

The behavior of many computer systems can be approximated by the processor-sharing discipline. In this discipline, the server (CPU) operates at a rate of μ, and whenever i customers are present, each customer receives service at a rate of μ/i. We assume that the arrivals occur according to a Poisson process at a rate of λ and that each customer's service requirement is exponentially distributed.

Of interest to us is the response time conditioned on the number of customers seen by an arriving customer, including itself. The response time is the elapsed time between arrival and departure for a customer. In this paper, we show that the rth moment of the conditional response

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time is a polynomial of degree r in the number of customers seen by the arrival, including itself. Further, we show several methods of obtaining the distribution of the conditional response time in the form of a Laplace-Stieltjes transform.

The results presented in this paper may be useful in the design of a computer or switching system. For example, the designer of such a system may ask, "What is the minimum number of customers that must be present in the system before the 95th percentile of the response time exceeds some prespecified threshold?" The answer to this question gives some indication of the performance of the system under overload. It may be instrumental in deciding whether any special overload control mechanism is needed. If an overload control mechanism is needed (i.e., the arrivals are blocked if a certain number of customers are in the system), the analysis of such a queue is straightforward and will not be discussed in this paper.

This problem has been studied in greater generality by Coffman et al.,[1] who obtain the waiting time distribution conditioned on the number seen by an arrival and the amount of service required by the arriving customer. However, it is difficult to obtain all the results of this paper directly from those in Ref. 1. Related work in this area was done by Sakata et al.,[2] who characterized a solution for the M/G/1 processor-sharing model. Recently, Ott[3] and Ramaswami[4] have provided methods for characterizing the unconditional distributions of the response time for the M/G/1 and GI/M/1 queues, respectively. The unconditional distribution of the response time for the M/M/1 queue has also been solved by Morrison.[5] Rege and Sengupta[6] have solved a version of the M/M/1 processor-sharing model, which includes multiprogramming.

In Section II of this paper we derive the conditional moments of the response time. In Section III, we obtain the Laplace-Stieltjes transform of the distribution of conditional response time. In Section IV, we obtain an expansion for the transform in powers of λ, which is useful for computation when λ is not large. In Section V, the asymptotic expansion is given for $x \to \infty$. This is especially useful for computation in a heavily loaded system. In Section VI, we discuss the numerical issues of this problem.

II. MOMENTS OF THE CONDITIONAL RESPONSE TIME

Let X denote the number of customers seen by an arrival, including itself, and let T be the response time of this arrival. Let $u(x, s)$ be the Laplace-Stieltjes transform of the distribution of the conditional response time, i.e., $u(x, s) = E(e^{-sT} | X = x)$ for $x = 1, 2, \cdots$. By conditioning on the first event (defined as the next arrival or departure, whichever occurs first), we obtain

$$u(x+1, s) = \left(\frac{\lambda+\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+\mu} u(x+2, s)\right.$$

$$\left. + \frac{\mu}{\lambda+\mu} \frac{x}{x+1} u(x, s) + \frac{\mu}{\lambda+\mu} \frac{1}{x+1}\right) \quad (1)$$

and

$$u(1, s) = \left(\frac{\lambda+\mu}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+\mu} u(2, s) + \frac{\mu}{\lambda+\mu}\right). \quad (2)$$

Let $\mu_r(x)$ be the rth moment of T conditioned on $X = x$, i.e., $\mu_r(x) = E(T^r \mid X = x)$. Then by taking the rth derivative of (1) and (2) with respect to s, multiplying by $(-1)^r$ and setting s to zero, we obtain

$$(x+1)\mu_r(x+2) - (1+b)(x+1)\mu_r(x+1) + bx\mu_r(x)$$

$$= -r(x+1)\mu_{r-1}(x+1)/\lambda \quad (3)$$

for $x = 1, 2 \cdots$

$$\mu_r(2) - (1+b)\mu_r(1) = -r\mu_{r-1}(1)/\lambda \quad (4)$$

and

$$\mu_0(\cdot) = 1.$$

In eqs. (3) and (4), $b = \mu/\lambda$.

Theorem 1: The solution of (3) *and* (4) *has the form:*

$$\mu_r(x) = \sum_{j=0}^{r} a_{rj} x^j \quad \text{for} \quad x = 1, 2, \cdots; \qquad r = 1, 2, \cdots$$

in which the coefficients a_{rj} *satisfy*:

$$a_{rj} = \left\{\sum_{k=j}^{r-1} (-ra_{r-1,k}b_{k,j}/\lambda - a_{r,k+1}c_{k+1,j}) - ra_{r-1,j-1}/\lambda\right\} c_{jj}^{-1}$$

$$\textit{for} \quad j = r-1, \cdots, 1; \qquad r = 1, 2, \cdots$$

$$a_{r0} = \left\{\sum_{k=0}^{r-1} (ra_{r-1,k}/\lambda + (a_{r,k+1}(2^{k+1} - 1 - b))\right\} b^{-1}$$

$$\textit{for} \quad r = 1, 2, \cdots$$

$$a_{00} = 1$$

and

$$b_{kj} = \frac{(k+1)!}{j!(k-j+1)!},$$

$$c_{kj} = b_{kj}\left\{\left(\frac{k+j+1}{k+1}\right)2^{k-j} - (1+b)\right\}$$

for $j = 1, \cdots, r; \quad k = j, \cdots, r; \quad r = 1, 2 \cdots.$

Proof: We first write $\mu_r(x)$ as a power series of the form

$$\mu_r(x) = \sum_{j=0}^{\infty} a_{rj}x^j$$

and note that

$$\mu_r(x+m) = \sum_{j=0}^{\infty} x^j \sum_{k=j}^{\infty} a_{rk}k!m^{k-j}/(j!(k-j)!).$$

We substitute this in (3) and arrange the left- and right-hand sides as a power series in x. For $j > r$, we find that the terms containing x^j on the left-hand side vanish. For any $j > r$, the coefficient of x^j contains a_{rk} for $k \geq j$, and these coefficients can be chosen arbitrarily. It is, therefore, not inconsistent to choose $a_{rj} = 0$ whenever $j > r$. The rest of the theorem follows by equating coefficients of x^j on the left- and right-hand sides of (3) for $j = r, \cdots, 0$ and verifying that the boundary condition is satisfied. □

Corollary 1: For $r = 1$ *and* 2,

$$\mu_1(x) = \frac{(x+1)}{\lambda(2b-1)}$$

and

$$\mu_2(x) = \frac{2(x+1)\left(x + \dfrac{4b}{2b-1}\right)}{\lambda^2(2b-1)(3b-2)}$$

for

$$x = 1, 2, \cdots.$$

We observe that the mean conditional response time is a linear function of x. This particular result is true for the M/M/1 First Come First Served (FCFS) queue as well, for which $\mu_1(x) = x/\mu$. It is readily seen that the mean conditional response time for the processor-sharing queue is smaller than that for the FCFS queue if and only if

$$x - 1 > L,$$

where L is the mean number in the system. Thus, the processor-sharing queue has a smaller mean response time whenever the number of customers seen by an arrival (excluding itself) is *greater* than the mean number in the system.

This result parallels the known result about the mean response time conditioned on the service requirement (see Ref. 7). There, the processor-sharing queue has a smaller mean response time whenever the service requirement is *less* than the mean service time.

III. DISTRIBUTION OF THE CONDITIONAL RESPONSE TIME

The difference equation for the Laplace-Stieltjes transform of the response time distribution, $u(s, x)$, is repeated here for convenience:

$$(x-1)u(x) - a(x-1)u(x-1) + b(x-2)u(x-2) = -b, \qquad x \geq 2,$$

$$b = \frac{\mu}{\lambda}, \qquad a = b + 1 + \frac{s}{\lambda}. \tag{5}$$

The explicit indication of the dependence of $u(s, x)$ on s has been suppressed. An exact solution for the Laplace transform suitable for numerical inversion will be obtained.[8] Only one boundary condition is given explicitly in (5), namely, for $x = 2$, one has

$$u(2) - au(1) = -b; \tag{6}$$

however, another boundary condition is available through the requirement that $u(s, x)$ be a transform as a function of s. This, of course, will be satisfied by the solution to be obtained.

Designate by Lu the left-hand side of (5) so that the difference equation to be solved is

$$Lu = -b; \tag{7}$$

this will be referred to as the *complete* equation. A function $v(x)$ will now be found satisfying

$$Lv = 0. \tag{8}$$

Equation (8) is referred to as the *homogeneous* equation and $v(x)$ as a complementary solution. Laplace's method will be used to solve (8).[9]

Assume a representation of $v(x)$ of the form

$$v(x) = \int \tau^{x-1} g(\tau) d\tau \tag{9}$$

in which the function $g(\tau)$ and a path of integration in the complex τ-

plane are to be chosen. Substitution into (8) and subsequent integration by parts yields

$$Lv = (\tau^2 - a\tau + b)\tau^x g(\tau)\,|$$
$$+ \int \tau^{x-1}[(\tau^3 - a\tau^2 + b\tau)\dot{g}(\tau) + \tau^2 g(\tau)]d\tau. \quad (10)$$

The vertical bar designates evaluation of the concomitant on the path to be chosen, and the dot indicates $d/d\tau$. To satisfy (8) the differential equation

$$(\tau^2 - a\tau + b)\dot{g}(\tau) + \tau g(\tau) = 0 \quad (11)$$

is to be solved for $g(\tau)$. Let the roots of

$$\tau^2 - a\tau + b = 0 \quad (12)$$

be

$$\gamma = \frac{a - \sqrt{a^2 - 4b}}{2}, \qquad \gamma_1 = \frac{a + \sqrt{a^2 - 4b}}{2} \quad (13)$$

and let

$$\alpha = \frac{\gamma_1}{\gamma_1 - \gamma} = \frac{1}{2}\left(1 + \frac{a}{\sqrt{a^2 - 4b}}\right); \quad (14)$$

then the solution of (11) may be written as

$$g(\tau) = (\gamma - \tau)^{\alpha-1}(\gamma_1 - \tau)^{-\alpha}. \quad (15)$$

Since, for real s, $0 < \gamma < \gamma_1$, the path of integration in (9) is chosen as the segment $(0, \gamma)$ of the real axis; for this choice, the concomitant term of (10) vanishes and, hence, (8) is satisfied. Thus, one has

$$v(x) = \int_0^\gamma \tau^{x-1}(\gamma - \tau)^{\alpha-1}(\gamma_1 - \tau)^{-\alpha}d\tau. \quad (16)$$

Setting $z = \gamma^2/b = \gamma/\gamma_1$ and changing the variable of integration allows $v(x)$ to be written in the form

$$v(x) = \gamma^{x-1}z^\alpha \int_0^1 \tau^{x-1}(1 - \tau)^{\alpha-1}(1 - \tau z)^{-\alpha}d\tau. \quad (17)$$

Comparison of (17) with the integral form of the hypergeometric function, $F(a, b; c; z)$ shows that (see Ref. 10)

$$v(x) = \gamma^{x-1}\frac{\Gamma(\alpha)\Gamma(x)}{\Gamma(\alpha + x)}z^\alpha F(\alpha, x; \alpha + x; z). \quad (18)$$

This is particularly advantageous since one has

$$F(\alpha, x; \alpha + x; z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j (x)_j z^j}{(\alpha + x)_j j!}, \tag{19}$$

in which

$$(a)_0 = 1, \qquad (a)_j = a(a + 1) \cdots (a + j - 1), \qquad j \geq 1. \tag{20}$$

It may be observed that the series of (19) converges absolutely for $|z| < 1$, that is, for $\gamma^2 < b$ which holds.

The computation of F in (19) may be simply carried out by the recursions

$$F_0 = 1, \qquad F_j = F_{j-1} + T_j,$$

$$T_0 = 1, \qquad T_j = \frac{(\alpha + j - 1)(x + j - 1)}{(\alpha + x + j - 1)j} z T_{j-1}. \tag{21}$$

Similarly, since

$$\frac{\Gamma(\alpha)\Gamma(x)}{\Gamma(\alpha + x)} = \frac{(x - 1)!}{(\alpha)_x}, \tag{22}$$

this too may be easily computed recursively. It may be considered, therefore, that the computations that will be needed in the inversion procedure to be used may be conveniently performed for the function $v(x)$.

To solve the complete eq. (7), its order will be depressed. It is convenient to write it in the form

$$Lu = (x + 1)u(x + 2) - a(x + 1)u(x + 1) + bxu(x) = -b. \tag{23}$$

Let

$$u(x) = v(x)\tau(x); \tag{24}$$

then

$$Lu = L(v\tau) = \tag{25}$$

$(x + 1)v(x + 2)\tau(x + 2) - a(x + 1)v(x + 1)\tau(x + 1) + bxv(x)\tau(x)$.
Use of the formulae

$$\tau(x + 1) = \tau(x) + \Delta\tau(x),$$

$$\tau(x + 2) = \tau(x) + 2\Delta\tau(x) + \Delta^2\tau(x) \tag{26}$$

in (25) yields

$$Lu = (x + 1)v(x + 2)\Delta^2\tau + [2(x + 1)v(x + 2) - a(x + 1)v(x + 1)]\Delta\tau. \tag{27}$$

Setting

$$w(x) = \Delta\tau(x) \tag{28}$$

in (27) now provides the first-order equation

$$w(x+1) - R(x)w(x) = -\frac{b}{(x+1)v(x+2)},$$

$$R(x) = a\,\frac{v(x+1)}{v(x+2)} - 1. \tag{29}$$

Direct substitution verifies that

$$w(x) = \sum_{j=1}^{\infty} \frac{b}{(x+j)v(x+j+1)R(x)R(x+1)\cdots R(x+j-1)} \tag{30}$$

is a solution of (29).

To show that $w(x)$ converges, one has

$$(1-\tau z)^{-\alpha} \sim (1-z)^{-\alpha}, \qquad \tau \to 1-; \tag{31}$$

hence, from (17),

$$v(x) \sim \gamma^{x-1}\left(\frac{z}{1-z}\right)^{\alpha} \int_0^1 \tau^{x-1}(1-\tau)^{\alpha-1}d\tau, \tag{32}$$

$$v(x) \sim \gamma^{x-1}\left(\frac{z}{1-z}\right)^{\alpha} \frac{\Gamma(\alpha)\Gamma(x)}{\Gamma(\alpha+x)}, \qquad x \to \infty. \tag{33}$$

Further, since

$$\frac{\Gamma(x)}{\Gamma(\alpha+x)} \sim x^{-\alpha}, \qquad x \to \infty, \tag{34}$$

one has

$$v(x) \sim \gamma^{x-1}\left(\frac{z}{1-z}\right)^{\alpha} \Gamma(\alpha)x^{-\alpha}, \qquad x \to \infty; \tag{35}$$

thus

$$R(x) \to \frac{a}{\gamma} - 1, \qquad x \to \infty. \tag{36}$$

Evaluation of the jth term comprising $w(x)$ now shows that it is of the order

$$j^{\alpha-1}\left(\frac{\gamma}{b}\right)^{j}. \tag{37}$$

Since $\gamma\gamma_1 = b$ and $\gamma_1 > \gamma$, one has $\gamma < \sqrt{b}$, thus $\gamma < b$ if $b > 1$, that is, if the offered load $b^{-1} = \lambda/\mu < 1$. In this case the series for $w(x)$ converges.

A one-parameter family of solutions of (7) obtained from (28) and (24) is

$$u(x) = v(x)\left[D + \sum_{j=1}^{x-1} w(j)\right], \tag{38}$$

in which D is a constant that is determined by use of the boundary condition (6). The final result is

$$u(x) = v(x)\left[\sum_{j=1}^{x-1} w(j) - \frac{b + w(1)v(2)}{v(2) - av(1)}\right]. \tag{39}$$

In order that $u(x)/s$ be a Laplace transform, one must have

$$\lim_{s\to\infty} \frac{u(x)}{s} = 0, \tag{40}$$

which is verified by (39). Accordingly, (40) is the second boundary condition required to specify a unique solution of (7). That this is true follows from an examination of the complete solution of (7), which will not be done here.

IV. PERTURBATION IN λ

For this purpose (5) is written in the form

$$\lambda(x + 1)u(x + 2) - (\mu + s + \lambda)(x + 1)u(x + 1) + \mu x u(x) = -\mu, \tag{41}$$

and the boundary condition in (6) takes the form

$$\lambda u(2) - (\mu + s + \lambda)u(1) = -\mu. \tag{42}$$

Fortunately this constitutes a singular perturbation. Writing $u(x)$ in the form

$$u(x) = \sum_{j=0}^{\infty} u_j(x)\lambda^j \tag{43}$$

and substituting into (41), (42) yields the equations

$$(x + 1)u_0(x + 1) - \left(1 + \frac{s}{\mu}\right)^{-1} x u_0(x) = \left(1 + \frac{s}{\mu}\right)^{-1},$$

$$u_0(1) = \left(1 + \frac{s}{\mu}\right)^{-1}, \tag{44}$$

and

$$(x+1)u_j(x+1) - \left(1+\frac{s}{\mu}\right)^{-1} xu_j(x)$$

$$= \frac{1}{\mu}\left(1+\frac{s}{\mu}\right)^{-1} (x+1)\Delta u_{j-1}(x+1),$$

$$u_j(1) = \frac{1}{\mu}\left(1+\frac{s}{\mu}\right)^{-1} \Delta u_{j-1}(1). \tag{45}$$

These are first-order equations for $u_0(x)$ and $u_j(x)$, which present no difficulty of solution. One obtains

$$u_0(x) = \frac{\mu}{sx}\left[1 - \left(1+\frac{s}{\mu}\right)^{-x}\right],$$

$$u_j(x) = \frac{1}{\mu x}\sum_{\ell=1}^{x}\left(1+\frac{s}{\mu}\right)^{\ell-x-1} \ell\Delta u_{j-1}(\ell), \qquad j \geq 1. \tag{46}$$

V. ASYMPTOTICS FOR $x \to \infty$

The derivation of the asymptotic expansion of $u(x)$ depends on the operational method of Boole,[11] Jagerman,[12] and Milne Thomson (see Ref. 9); it is somewhat involved, so the details will be omitted but the results may be easily stated. One may write

$$u(x) \sim \sum_{j=1}^{\infty} \frac{\alpha_j}{x(x+1)\cdots(x+j-1)}, \tag{47}$$

in which the coefficients are given by

$$\alpha_1 = \frac{\lambda b}{s}, \qquad \alpha_2 = -\frac{\lambda^2 b}{s^2},$$

$$\alpha_3 = -\frac{\lambda^3 b(b-2)}{s^3}, \qquad \alpha_4 = -\frac{\lambda^4 b(b-2)(2b-3)}{s^4} - \frac{2\lambda^3 b^2}{s^3}, \tag{48}$$

and, in general, by the recursion

$$\alpha_j = -\frac{\lambda}{s}((j-2)(2-\alpha)+1)\alpha_{j-1} + \frac{\lambda}{s}(j-2)^2\alpha_{j-2}, \qquad j \geq 2. \tag{49}$$

Let $F(\tau, x)$ be the complementary distribution corresponding to $u(x)$; then

$$F(\tau, x) \sim 1 - \frac{\mu\tau}{x}\left[1 - \frac{1}{2}\frac{\lambda\tau}{x+1} - \frac{1}{6}\frac{\lambda^2\tau^2(b-2)}{(x+1)(x+2)} - \frac{\frac{1}{24}\lambda^3\tau^3(b-2)(2b-3) + \frac{1}{3}\lambda^2\tau^2 b}{(x+1)(x+2)(x+3)}\right], \quad x \to \infty. \qquad (50)$$

VI. NUMERICAL RESULTS

Our paper was concerned with exact answers for the moments (Section II), answers in the form of Laplace-Stieltjes transforms (Sections III and IV) for the distribution function, and asymptotic answers (Section V) for large x. One issue that we were concerned with was the applicability of these methods for ranges of parameter values of the problem.

In Table I, we present the first and second moments of the sojourn time calculated by three different methods (Sections II, III, and IV). The value of μ was chosen to be 1, x was taken to be 3, and λ was varied from 0.2 to 0.9. We numerically inverted the transforms obtained in Sections III and IV by the method of Ref. 8 and computed the moments from the distribution. As can be seen from the results, the results from the two distributions are accurate for the first moment for moderate and low value of λ. The method of Section III seems to be slightly better than the method of perturbations for a large value of λ. For the second moment, the results seem to be slightly less accurate. We state that we calculate the moments by numerically integrating the complementary distribution. Thus, truncation will cause the calculated moments to be underestimated. This fact is borne out in all the examples. In Fig. 1, we show the complementary distribution of the sojourn time by inverting the transforms from eq. (39).

In Table II, we show the accuracy of the asymptotic results presented in Section V. Here we calculate the complementary distribution when $\lambda = 0.5$, $\mu = 1$, and $x = 10$. As can be seen, there is good correspondence between the perturbation method (which uses the numerical inversion technique of Ref. 8) and the asymptotic solution.

Table I—Numerical results for the first and second moments ($x = 3$, $\mu = 1$)

	Mean = $\mu_1(x)$			Second Moment = $\mu_2(x)$		
λ	Exact (Section II)	From eq. (39)	From eq. (46)	Exact (Section II)	From eq. (39)	From eq. (46)
0.2	2.222	2.216	2.216	8.927	8.659	8.659
0.5	2.667	2.636	2.636	15.111	13.758	13.766
0.9	3.636	3.619	3.384	40.220	35.664	25.243

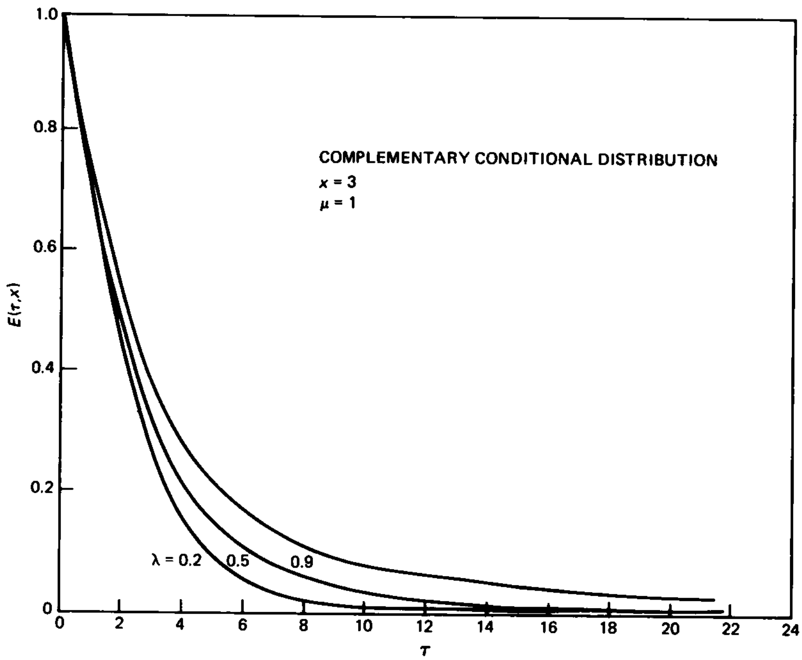

Fig. 1—Complementary conditional distribution of the sojourn time.

Table II—Numerical results for the asymptotic solution ($\lambda = 0.5$, $\mu = 1$, $x = 10$)

	$F(\tau, x) = P(T > \tau \mid X = x)$	
τ	From Asymptotic Solution (50)	From eq. (Perturbation) (46)
1.0	0.9023	0.9023
2.0	0.8092	0.8092
3.0	0.7207	0.7209
4.0	0.6370	0.6376
5.0	0.5580	0.5596
6.0	0.4839	0.4874
7.0	0.4147	0.4215
8.0	0.3504	0.3621
9.0	0.2912	0.3094
10.0	0.2370	0.2630

In conclusion, we would recommend the method of Section III for small values of x; the asymptotic solution of Section V for large values of x; and the perturbation method of Section IV, where x takes a wide range of values and when λ is not large.

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