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A Note on Discrete Representation of Lines

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In raster graphics a line must be drawn as a "discrete segment", a set of integer grid points that lie close to the line. Equivalence classes of identically drawn lines are described in terms of Farey series; this treatment notably simplifies previous work of Dorst and Smeulders. A log n algorithm serves to identify a line's equivalence class. Using it to choose among precomputed n-pixel discrete segments, we may draw lines in n-pixel blocks rather than the customary single-pixel steps.

I. INTRODUCTION

The problem of approximating a straight line by points of an integer grid may be reduced to that of finding the extreme grid points in a closed half-plane.[1,2] By reflections and translations the problem may be standardized to the form: For each integer x maximize y over the integers subject to

$$y \leqslant mx + b,$$

where m and b are restricted to the region

$$S = \{(m, b) \mid 0 \leqslant m \leqslant 1, 0 \leqslant b < 1\}.$$

An equivalent formulation is: For each integer x find the unique integer y that satisfies

$$mx + b - 1 < y \leqslant mx + b.$$

An explicit solution is $y = \lfloor mx + b \rfloor$, where $\lfloor b \rfloor$ denotes the largest

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integer that does not exceed b. The conventions of the preceding equations will be followed throughout:

n, p, q, x, y are integers,

s, t, u, v, w are rationals, and

b, m are reals.

In particular, range restrictions, such as $0 \leqslant x \leqslant n$ and $0 \leqslant m \leqslant 1$, must be understood in the domain of the variables involved.

A *discrete segment of length n* is a set $L(m, b, n)$ given by

$$L(m, b, n) = \{(x, \lfloor mx + b \rfloor) \mid 0 \leqslant x \leqslant n\}.$$

Usually n is fixed, so we may speak simply of a *discrete segment*.

Section II investigates the equivalence classes of lines that are approximated by identical discrete segments. These equivalence classes induce an interesting polygonal pattern in the space of parameters (m, b). Farey sequences abound in the pattern, which is accordingly dubbed a "Farey fan". To each region, or "facet", of a Farey fan there corresponds a distinct discrete segment. Thus the facet in which the parameters of a given line lie tells what discrete segment approximates that line. A canonical characterization for facets and an $O(\log n)$ algorithm for locating the facet for a given line are developed in Section III. The algorithm may be used in raster graphics to choose among precomputed discrete segments, so that lines may be drawn by block transfers in n-pixel chunks. The main results were first given by Dorst and Smeulders.[3] The contributions of this paper are (1) recognition of the role of Farey series, (2) simplified analysis based on the Farey fan, and (3) algorithms.

II. FAREY FANS

To each discrete segment L of length n there corresponds an equivalence class $C(L)$ of points in S given by

$$C(L) = \{(m, b) \mid (m, b) \in S \text{ and } L(m, b, n) = L\},$$

or

$$C(L) = \{(m, b) \mid mx + b - 1 < y \leqslant mx + b, (x, y) \in L\}.$$

Defined by linear inequalities, the equivalence classes are polygonal subregions of S, called *facets*. For each point (x, y) in L there is a pair of parallel bounding *rays*, $R(x, y)$ and $R(x, y + 1)$, where

$$R(x, y) = \{(m, b) \mid b = -xm + y\}, \qquad 0 < y \leqslant x \leqslant n.$$

$R(x, y)$ has slope $-x$, b-intercept y, and m-intercept y/x. The m-

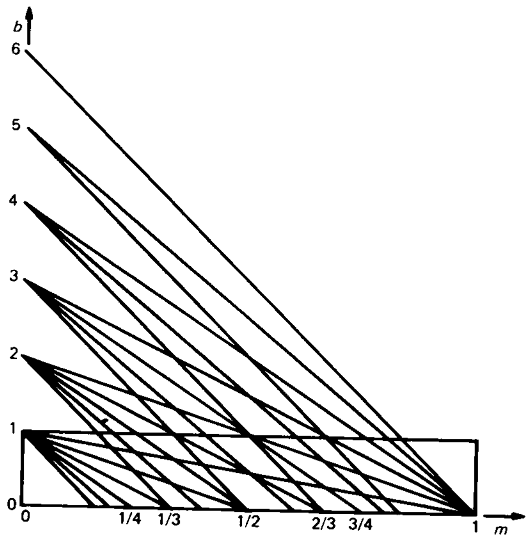

Fig. 1—Farey fan of order 6. Heavy lines delimit region S.

intercepts $\{y/x \mid 0 \leq y \leq x, 1 \leq x \leq n\}$ constitute a Farey series.* (See Chapter 3 of Ref. 4.) The rays make a pattern like Fig. 1. The part of this figure that is contained in the closure of S is called a *Farey fan of order n*, often shortened simply to "Farey fan".

A duality holds between the m-b space of Fig. 1 and the original x-y space where we are approximating. To every point $(x, y)^\dagger$ in original space there corresponds a line $b = -xm + y$ in dual space, whose points (m, b) are the slope and intercept parameters of members of a pencil of lines through (x, y) in original space. Similarly, to every point (m, b) of dual space there corresponds a line $y = mx + b$ in original space, whose points (x, y) are slope-intercept parameters of members of a pencil of lines through (m, b) in dual space.

A side of a facet is either a side of S or a segment of a ray. To include the top and bottom sides of S in the Farey fan, we admit $R(0, 0)$ and $R(0, 1)$ as well:

$$R(x, y) = \{(m, b) \mid b = -xm + y\}, \qquad \begin{cases} 0 \leq y \leq x \leq n \\ \text{or } x = 0, y = 1. \end{cases} \tag{1}$$

* The Farey series of order n is the ordered set of rational numbers p/q, $1 \leq q \leq n$, in the interval [0, 1]. For example, the Farey series of order 7 is:

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}.$$

† In this paragraph x and y are real variables.

The vertex where two rays, $R(x, y)$ and $R(x', y')$, $x > x'$, intersect is the point

$$(m, b) = \left(\frac{y - y'}{x - x'}, \frac{xy' - x'y}{x - x'}\right). \tag{2}$$

To simplify the analysis further, we extend the domain of $R(x, y)$ again, to include $0 \leqslant x \leqslant n$ and all integer y. None of the extra rays passes through S, so this extension has no effect on the facets. In the strip $0 \leqslant m \leqslant 1$, the m-coordinates of the intersections of $R(x, y)$ with other rays are then given by rational numbers

$$m = \frac{p}{q}, \qquad 0 \leqslant p \leqslant q, \qquad 1 \leqslant q \leqslant \max(x, n - x). \tag{3}$$

Taken in arithmetic order, these values constitute a segment of a Farey series of order $\max(x, n - x)$. From (1) any ray $R(x, y)$ must cross $m = p/q$ at $b = y - px/q$, that is, at a multiple of $1/q$. We have proved

Theorem 1: The abscissas of intersections of a given ray of a Farey fan with other rays constitute a Farey series.

Corollary 1: The abscissas of adjacent vertices of a facet are adjacent members of a Farey series. Moreover, if the abscissa of a vertex is p/q, the ordinate is a multiple of $1/q$.

Figure 1 suggests that every facet is triangular or quadrilateral. This observation is true in general:

Theorem 2: A facet of a Farey fan has at most four sides.

Proof, by induction: The statement is obviously true for a Farey fan of order 1. Assume it true for Farey fans of order $n - 1$, and assume also that all four-sided facets are *diamonds* that have two *middle vertices* with identical m-coordinates. Recalling that all rays have nonpositive slope, we see that each facet of the Farey fan of order $n - 1$ has one of the shapes exemplified in Fig. 2. Each shape will be considered separately.

Figure 2a shows a triangular facet bounded by $m = 0$. The only new line that crosses (enters the interior of) the facet in question is as shown in Fig. 3a. It divides the facet into two triangles.

Figure 2b shows a general triangular facet with long side upward. Any new ray that crosses the facet has a slope more negative than that of any of the sides and hence must cross side uw (Fig. 3b). Now u and w are adjacent members of a Farey series of some order q. From eq. (3) u, s, w must be members of a Farey series of order $q + 1$. Since at most one new member can appear between any two adjacent members of a Farey series in going from order q to order $q + 1$, at most one new ray can cross side uw. Thus u, s, w are consecutive

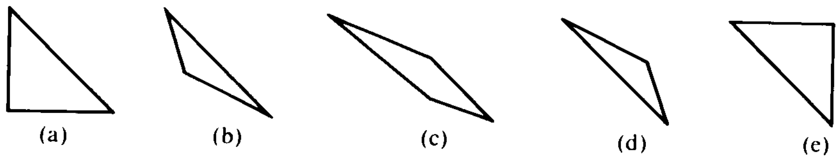

Fig. 2—Possible facets of a Farey fan: (a) triangular facet bounded by $m = 0$; (b) general triangular facet with long side upward; (c) diamond facet; (d) general triangular facet with long side downward; (e) triangular facet bounded by $m = 1$.

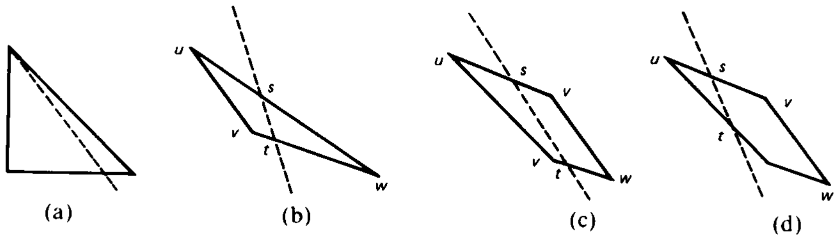

Fig. 3—Adding lines to a Farey fan of order $n - 1$ to make a Farey fan of order n. Vertices are labeled with their m-coordinates.

members of some Farey series. By Corollary 1, there is, for each pair drawn from $\{u, v, w\}$, a Farey series in which that pair is adjacent. Hence u, v, w are also consecutive members of some Farey series. Consequently $s = v$. The facet has been partitioned into one triangle and one diamond.

Figure 2c shows a diamond facet. A new ray that crosses two opposite sides as in Fig. 3c would join points s and t, where $s < v < t$ and where v belongs to a lower-order Farey series than do either s or t. Thus s and t cannot be adjacent members of any Farey series, contradicting Corollary 1. A ray that crosses two adjacent sides (Fig. 3d) would introduce new terms into both Farey series. Since the flanking terms, u and v, are the same in both series, the new terms must be the same: $s = t$. But this is impossible because the new ray would have infinite, not negative, slope. Because a new ray that crosses a diamond can cross neither two opposite nor two adjacent sides, it must pass through one of the two middle vertices and partition the diamond into a triangle and a diamond.

Figures 2d and e are analogous to 2b and a.

In every case the addition of rays to convert a Farey fan of order $n - 1$ into a Farey fan of order n yields triangular or diamond facets. The induction is complete.

III. IDENTIFYING FACETS

Since distinct discrete segments of length n correspond one-to-one with facets of the Farey fan of order n, the Farey fan makes a natural

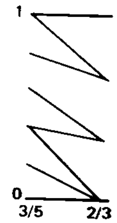

Fig. 4—A ladder from Fig. 1.

index to discrete segments. As Sproull has suggested,[5] one may use such an index to generate arbitrary approximate lines in gulps of length n, possibly beating the usual point-by-point methods. The words of a display memory make natural gulps.

To look entries up in the index, we need a way to identify in which facet a given (m, b) pair lies. Let f_i be the ith term of the Farey series of order n. Since all ray intersections in the Farey fan occur at points whose m-coordinates belong to the series, in each interval $f_i \leq m \leq f_{i+1}$ the fan is a simple *ladder* of *rungs* that never cross each other in the interior (Fig. 4). This observation suggests Method M below, a workable method for locating the facet in which a given (m, b) lies.

Method M:

1. Find the interval to which (m, b) belongs.
2. In that ladder locate the highest rung that lies on or below (m, b).
3. From that rung read off a precomputed discrete segment of length n.

An easy way to do Step 1 is binary search in the Farey series, or, since Farey series are fairly smooth, interpolation search. Step 2 can be done by binary search in the ladder. The total running time is $O(\log n)$: Step 1 searches a table of $O(n^2)$ Farey series members (see Ref. 4, Theorem 231); Step 2 searches a table of n rungs (Theorem 3).

The suggested implementation of Method M requires preprocessing and storage for $O(n^2)$ intervals times n rungs of tabulated information. This information is dispensable. Step 1, identification of an interval, can be done by an algorithm of Papadimitriou in $O(\log n)$ time.[6] The equation of a rung through $(p/q, r/q)$ for use in Step 2 can be generated by substituting $(m, b) = (p/q, r/q)$ in (1):

$$qy - px = r,$$

and solving for x and y by an extended Euclidean algorithm in $O(\log n)$ time.[7] One solution is required at each of the $O(\log n)$ stages of binary search. Thus facets can be identified in $O(\log^2 n)$ time, single-shot, with no preprocessing. Preprocessing, however, reduces the problem to simple binary search; it is the technique of choice for modest, fixed values of n.

Any scheme for uniquely identifying facets may be used for a canonical description for discrete segments. Before we demonstrate one, it will be helpful to have another theorem.

Theorem 3: A ladder in a Farey fan of order n has exactly one rung of each slope $-x$, $1 \leq x \leq n$. *Moreover, adjacent rungs that meet at a ladder edge,* $m = p/q$, *where* $\gcd(p, q) = 1$, *have slopes that differ by q.*

Proof: By eq. (1) a rung of slope $-x$ must meet the vertical line $m = m_0$ at an ordinate b such that there exists an integer y satisfying

$$0 \leq b = -xm_0 + y < 1. \tag{4}$$

Equation (4) reflects the fact that S is open along the top edge, $b = 1$. The closure of a rung, however, may meet that edge; we replace $<$ with $\leq$ in (4) to capture such rung ends. The solution

$$y = \lceil xm_0 \rceil \tag{5}$$

satisfies the constraints of (1) since $0 \leq m_0 \leq 1$. It is unique unless $xm_0 = 0$, in which case $y = 0$ and $y = 1$ are both solutions. But only one of two parallel rungs incident on $(m_0, 0)$ and $(m_0, 1)$ actually lies in S. The first claim of the theorem is established.

To prove the second claim, let two rungs $R(x, y)$ and $R(x'\ y')$, $x > x'$, meet at $m = p/q$. From eq. (2),

$$m = \frac{p}{q} = \frac{y - y'}{x - x'},$$

showing that $q \mid x - x'$. Thus we must have

$$y = y' + kp, \qquad x = x' + kq \tag{6}$$

for some positive integer k. Substitution in (2) yields

$$b = \frac{qy' - px'}{q}.$$

Since this value of b is independent of k, all rungs $R(x' + kq, y' + kp)$ meet $R(x'\ y')$ at the same point. If any value of k in (6) yields values of x and y that satisfy (1), then $k = 1$ certainly does. Thus, if $-x'$ is the larger slope of two rungs incident on a vertex at $m = p/q$, there is another rung through that vertex with slope $-(x' + q)$, and no rung with an intermediate slope. The second claim has been proved.

We now give the Dorst-Smeulders characterization of a discrete segment. Here the notion of "middle vertex" is extended from diamonds to all facets, meaning any but the upper left and lower right vertices.

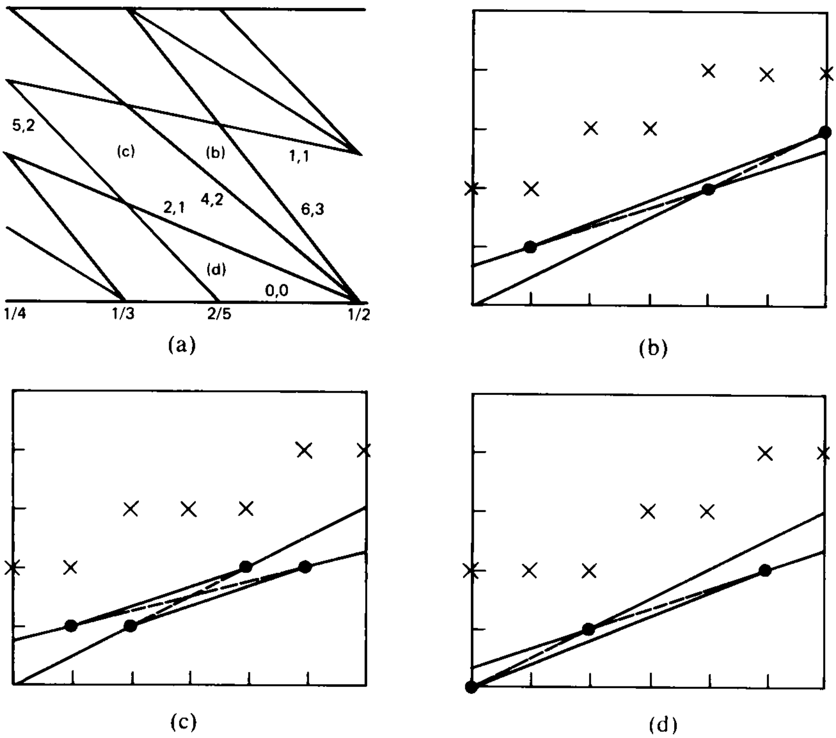

Fig. 5—(a) A fragment of Fig. 1. Certain rays are labeled with negated slope x and intercept y. (b) through (d) Frontier positions of lines in original space that are dual to facets labeled (b) through (d) in (a). The coordinates of breakpoints (●) are the same as the labels of the rays in (a) that are dual to the breakpoints. The discrete segment (X) for each facet is shown shifted up two units.

D-S Representation: A facet of a Farey fan and its corresponding discrete segment may be characterized by four integers, n, q, p, and x, where

n is the length of the discrete segment,

p/q, where $gcd(p, q) = 1$, is the abscissa of a middle vertex, and

$-x$ is the slope of the lower of the two rungs incident on the lower right vertex.

Theorem 3 justifies the representation: the slope, $-x$, of the lower rung serves to distinguish among the various facets that have middle vertices on $m = p/q$.

The four parameters have simple interpretations. Each facet is a convex combination of its vertices, which represent limiting positions of lines in the original space. Figure 5 shows the lines in those limiting positions; they form broken-line upper and lower *frontiers.* Any line that falls between the frontiers and does not touch the upper one will be approximated by the same discrete segment. If a frontier has two breakpoints, the slope of the line joining them is p/q.

Exactly one line of slope p/q can touch the lower frontier. Parameter x specifies the x-coordinate of the leftmost point where it touches; by (5) the y-coordinate is $\lceil px/q \rceil$. This line, with slope p/q and intercept $\lceil px/q \rceil - px/q$, may be taken to be the canonical representative of the equivalence class of all lines approximated by the same discrete segment. Because the canonical representative has rational slope, the discrete approximation to its infinite extension is invariant under translation by the vector (q, p). Hence the first differences ("chain-code," in computer graphics parlance) of the discrete approximation are periodic with period q. Dorst and Smeulders took this periodicity to be the defining property of q.

IV. DISCUSSION

Rob Pike first brought the motivating question to my attention: How may one quickly identify the discrete segment of length n that approximates a given line? The apparent complexity of a quick sketch of the Farey fan discouraged further consideration—a cautionary experience. In this subject one well-drawn picture is worth a pot of algebra. In fact, computer graphics helped prove Theorem 2: the induction hypothesis came from playing with Theo Pavlidis's PED graphics editor to make a Farey fan.[8] The conclusion of Theorem 2 fairly shrieks from Fig. 1, and Theorem 1 isn't far behind. In the absence of well-drawn diagrams, however, it took the work of Dorst and Smeulders to reveal the basic simplicity of the diagram. Their treatment involved intricate discrete analysis in the original space. The route through dual space, signposted by Farey series, turns out to be much easier. Moreover, the newer analysis solves the original problem: Method M may be used to select, from among a precomputed list, the correct discrete segment to approximate any given line. The method provides an exact alternative to a heuristic proposed by Sproull.[5]

The search scheme in Method M may be recognized as a variation on Shamos's slab test for the inclusion of a point in a polygon.[9,10] The asymptotic performance of Method M is the same as that of Lipton and Tarjan for locating a point in a general polygonal tiling.[11] The single-shot algorithm, at $O(\log^2 n)$, is much better than general single-shot methods, which require $O(n^3)$ time in this special case.[10]

I am indebted to Jon Bentley for pointers from his mental encyclopedia of computational geometry.

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