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Analysis of Laser Beam Propagation in a Turbulent Atmosphere

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The beam propagation method, based on the parabolic approximation to the wave equation, is used in conjunction with Papoulis' redefinition for optical fields of Woodward's ambiguity function. A simple derivation is given of Tatarskii's formula for the lateral coherence function, and hence the mean intensity profile, of a laser beam propagating through a turbulent atmosphere. Statistics of the received signal and the effects of spatial nonstationarity of the turbulence can also be deduced using this technique, as can the effects of very large-scale variations in refractive index and receiver directivity.

I. INTRODUCTION

There has been a recent revival of interest in the propagation of laser beams through the atmosphere for communication purposes. King et al.[1] have conducted experiments that show that a laser is an effective standby substitute for a microwave link over a clear-air, line-of-sight path of several tens of kilometers. When the microwave link is subject to severe multipath fading, the laser signal is found to be much more stable. In these clear-air conditions the laser beam is mainly affected by the atmosphere's turbulence, which produces a spread of the propagating beam in excess of that expected due to diffraction. Over a 37-km path the lateral intensity profile of the laser beam is found to be random but to have an average Gaussian shape with a spread of about 6m between e^{-1} points.

More than two decades of research on the theory of optical propa-

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gation through random media has been very competently reviewed by Strohbehn and others;[2] the chapter by Ishimaru[3] is particularly relevant here. The wide variety of approaches taken by different authors is apparent, ranging from the purely physical to the highly mathematical. The present paper seeks to produce a reasonably simple theoretical picture, which is accurate both physically and mathematically, and which will also be useful for engineering purposes.

The starting point, as so often elsewhere, is the parabolic approximation to the wave equation,[4] but it is used here in a manner that has become known as the "beam propagation method."[5] Other names for the method are the "split-step Fourier technique" of Tappert and Hardin[6] and the "multiple random phase-screen method."[7,8] The essential idea that makes a simple solution possible is that, because the fluctuations in refractive index are so weak and their scale size is so large compared to the wavelength, the phenomena of diffraction and scattering can be artificially separated. The propagation path is divided into many short sections, so that the propagating wave is barely disturbed by each section, but their cumulative effect can be considerable.

In each of these sections the irregularities are effectively removed, in the form of an accumulated random phase, to one or another of the boundary planes. Free-space diffraction is then allowed to occur within the now uniform section between the planes. The essential next step is that the Fresnel diffraction, which occurs between the planes in each of the sections, is described by what Papoulis[9] has called the "ambiguity function," after the name given by Woodward[10] to a similar function of fundamental importance in radar. For an optical field the ambiguity function was redefined by Papoulis as the Fourier transform of the lateral mutual coherence function (i.e., the lateral autocorrelation function of the field). The "field ambiguity function" so defined has the very useful property that it propagates in a uniform medium without changing its functional form. What does change is the argument, in a manner reminiscent of a wave traveling along a transmission line.

On encountering the artificially accumulated random phase at each boundary plane, the field ambiguity function is modified appropriately but then propagates through the next section again without change. This procedure can continue as long as the propagating beam is essentially forward scattered, which is true for a laser beam propagating through atmospheric turbulence at least out to 50 km, if not farther. Then over the final plane the lateral field autocorrelation function is obtained by taking the Fourier transform of the field ambiguity function. The mean intensity of the beam is contained in the field autocorrelation function as a special case.

It is gratifying that application of this simple procedure reproduces precisely what Ishimaru[3] has described as "Tatarskii's exact result"[4] for the lateral mutual coherence function of a laser beam propagating through atmospheric turbulence. In fact, because of its underlying physical clarity, the present procedure allows one to go a little farther than Tatarskii and describe the statistics of the propagating field in more detail, and also to deal with spatially nonstationary turbulence.

II. ELECTROMAGNETIC BASIS: THE BEAM PROPAGATION METHOD

In attempting to solve propagation problems in which the scale size of the refractive-index irregularities is large compared with the wavelength, and the magnitude of these fluctuations is very small, it is often helpful to factor out the term $\exp(-jkz)$, assuming propagation in the general direction of the z axis. This procedure is analogous to factoring out the time dependence $\exp(j\omega t)$. Thus, any phasor component $f(\boldsymbol{r}, t)$ of the quasi-monochromatic propagating field can be written as

$$f(\boldsymbol{r}, t) = u(\boldsymbol{r}, t)\exp(-jkz), \tag{1}$$

where $u(\boldsymbol{r}, t)$ is a slowly varying phasor function of position $\boldsymbol{r}$ and time t. The propagation constant k is some convenient mean value.

As a consequence of the assumed large scale size and tenuous nature of the refractive index irregularities, it can be shown[4,11] that the function $u(\boldsymbol{r}, t)$, is governed by the parabolic equation approximation to the wave equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - j2k\frac{\partial u}{\partial z} + 2k^2 n_1 u = 0, \tag{2}$$

where $n_1(\boldsymbol{r}, t)$ is the departure of the refractive index from its mean value, which will be assumed to be unity. Time fluctuations will be ignored in what follows, although they can be incorporated easily if required.

Consider the time-invariant solution of eq. (2) for a wave launched from the plane $z = 0$. A well-established approach[5,6,11] is to solve the equation in two iterative steps, and will be referred to here as the beam propagation method. First assume that there are no variations in refractive index, so that $n_1 = 0$. Then the solution for the field over any plane z is given by Fresnel's diffraction formula[12]

$$f(x, y, z) = j\,\frac{\exp(-jkz)}{\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y', 0) \cdot\exp\left\{-\frac{jk}{2z}\left[(x - x')^2 + (y - y')^2\right]\right\} dx'dy' \tag{3}$$

in terms of the field $f(x, y, 0)$ over the aperture plane $z = 0$.

If diffraction now is ignored, by suppressing the first two terms in eq. (2), and if the irregularities in refractive index are restored, their first-order effect can be obtained from the solution of

$$\frac{\partial u}{\partial z} + jkn_1 u = 0, \tag{4}$$

which is, by straightforward integration,

$$u(x, y, z) = u(x, y, 0)\exp\{j\Phi(x, y)\}, \tag{5}$$

where

$$\Phi(x, y) = -k \int_0^z n_1(x, y, z)dz. \tag{6}$$

This is simply the phase induced by the refractive-index irregularities along straight-line paths, parallel to the z-axis, from 0 to z.

Thus the solution of eq. (2) is in the two artificially separated parts given by eqs. (3) and (5). The first part allows for diffraction but suppresses the effect of the irregularities, while the second part suppresses diffraction but allows for the effect of the irregularities. The two parts of the solution then must be combined in some suitable way, as illustrated in the following examples. The first is concerned with the nonrandom effect of the overall linear trend of refractive index with height, and the second considers the effect of the turbulence-induced, small-scale random irregularities in refractive index.

2.1 Effect of linear gradient in refractive index

If the refractive index varies linearly with height, with constant gradient g, then over a distance of Δz eq. (6) gives the phase variation with height x as linear also, namely as $\Phi(x) = -kgx\Delta z$. The mean linear trend in the atmosphere is usually negative, and so it can be seen that the effect of this negative gradient in refractive index will be to tilt the advancing wavefront forward through an angle $g\Delta z$. If this continous forward tilt is interpreted as bending the propagating beam, giving it a radius of curvature R, then the angle of tilt would be $\Delta z/R$. Hence $R = g^{-1}$, a well-known result.[13] But it should be emphasized that while the beam is being bent it is also experiencing diffraction, according to eq. (3), and so spreads as it bends as it propagates.

2.2 Effect of turbulent fine structure

Having seen that the beam-bending effect of the mean linear trend in refractive index can be treated separately, consider now a medium that is on average uniform but whose refractive index $n_1(\boldsymbol{r})$ is a zero-mean random process. The magnitude of these fluctuations in refractive index is typically 10^{-8}, for homogeneous turbulence conditions, with scale sizes of at least several millimeters, which is large compared to optical wavelengths. Hence the conditions for the beam propagation method to apply are fulfilled.

Consider a segment of the medium between the beam-launch plane $z = 0$ and the plane $z = \Delta z$. If the refractive-index irregularities are temporarily ignored, the field over the exit plane $z = \Delta z$ would be given by Fresnel's diffraction formula of eq. (3). Now restoring the irregularities but temporarily suppressing diffraction, their effect is accounted for in the accumulated random phase along parallel ray paths of length Δz:

$$\Phi(x, y) = -k \int_0^{\Delta z} n_1(x, y, z)dz. \tag{7}$$

The statistics of this random phase process will be important later and so are derived here.

Since the refractive-index fluctuation process $n_1(x, y, z)$ is zero mean,

$$\langle \Phi(x, y) \rangle = 0, \tag{8}$$

where the sharp brackets indicate taking the expectation.

If n_1 is wide-sense stationary, with autocovariance

$$B_{n_1}(\xi, \eta, \zeta) = \langle n_1(x, y, z) n_1(x + \xi, y + \eta, z + \zeta) \rangle \tag{9}$$

and variance

$$\sigma_{n_1}^2 = B_{n_1}(0, 0, 0), \tag{10}$$

then the autocovariance of the phase process

$$B_\Phi(\xi, \eta) = k^2 \int_0^{\Delta z} \int_0^{\Delta z} \langle n_1(x, y, z) n_1(x + \xi, y + \eta, z') \rangle dz dz'. \tag{11}$$

This is equivalent to[14]

$$B_\Phi(\xi, \eta) = k^2 \Delta z \int_{-\Delta z}^{\Delta z} B_{n_1}(\xi, \eta, \zeta) d\zeta. \tag{12}$$

Now it is convenient to assume that the width of the section $\Delta z \gg \zeta_0$, the scale size of the irregularities in the z direction, and so

$$B_\Phi(\xi, \eta) = k^2 \Delta z \int_{-\infty}^{\infty} B_{n_1}(\xi, \eta, \zeta) d\zeta. \tag{13}$$

It also follows from the condition $\Delta z \gg \zeta_0$ that the phase over the exit plane can always be taken to be Gaussian, whether the refractive index itself is Gaussian or not, as a consequence of the central limit theorem. Hence, the random phase process is completely described by the autocovariance of eq. (13), and its variance can be taken to be of the order

$$\sigma_\Phi^2 \sim k^2 \sigma_{n_1}^2 \zeta_0 \Delta z, \tag{14}$$

in which ζ_0 is sometimes referred to as the integral scale size of the refractive index in the direction of propagation.

Since the beam propagation method is, in essence, a perturbation technique, applied locally, it is important that over each section the phase variance $\sigma_\Phi^2 \ll 1$. In applications such as laser beam propagation through atmospheric turbulence, this condition is easily met, even with the constraint that $\Delta z \gg \zeta_0$.

III. STATISTICAL FIELDS: THE AMBIGUITY FUNCTION

The example of Section 2.2 will now be our main concern. So far we have the field distribution over the plane $z = \Delta z$ as $f_0(x, y, \Delta z) \exp\{j\Phi(x, y)\}$, where $f_0(x, y, \Delta z)$ is the free-space diffraction of the original aperture field and $\Phi(x, y)$ is the random phase process of eq. (7). If the field over the plane Δz is now allowed to diffract, Fresnel's diffraction formula would give the field over the next plane, assuming that the intervening region is free space. The irregularities could then be replaced and a different random phase process could account for them over this next plane. This new field then can be allowed to further diffract, and so on.

Over some plane z, well into the random medium, the phasor field component $f(x, y, z)$ will itself be random. The property of greatest utility would be its lateral mutual coherence function:

$$\Gamma(x, y; \xi, \eta; z) = \langle f^*(x - \xi/2, y - \eta/2, z) f(x + \xi/2, y + \eta/2, z)\rangle, \tag{15}$$

otherwise referred to as the lateral field autocorrelation function. (The asterisk denotes complex conjugate.) Hence the mean intensity

$$\langle I(x, y, z)\rangle = \langle |f(x, y, z)|^2\rangle = \Gamma(x, y; 0, 0; z). \tag{16}$$

However, $\Gamma(\ )$ is not simple to calculate,[2] whereas its Fourier transform is. Papoulis[9] introduced the Fourier transform of $\Gamma(\ )$, calling it the ambiguity function of the optical field, and showed that it greatly simplified the calculation of Fresnel diffracted fields. In particular, a very useful property of the ambiguity function is that it propagates without changing its functional form in a uniform medium; what does change is the argument of the function.

The definition of ambiguity function that will be used here, for the field over the plane z, is

$$A(\mu, \nu; \xi, \eta; z) = \frac{1}{\lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(x, y; \xi, \eta; z) \exp\{jk(\mu x + \nu y)\} dxdy. \quad (17)$$

If the medium between this plane z and the plane $z + \Delta z$ is uniform, it can be shown that, under conditions when Fresnel diffraction occurs, the ambiguity function over the plane $z + \Delta z$ is simply[9]

$$A(\mu, \nu; \xi, \eta; z + \Delta z) = A(\mu, \nu; \xi - \mu\Delta z, \eta - \nu\Delta z; z). \quad (18)$$

The way in which the ambiguity function and this relation are used is described in the next two sections.

IV. BEAM WAVE PROPAGATION THROUGH TURBULENCE

Consider a beam of radiation launched into a turbulent medium in which the conditions for the application of the beam propagation method are satisfied, namely that the magnitude of the fluctuations in refractive index is very small and their scale size is very large in comparison with the wavelength of the propagating beam. If the field is launched from an aperture plane at $z = 0$, over which it is $f(x, y, 0)$, the ambiguity function over the $z = 0$ plane is given by eqs. (15) and (17) as

$$A(\mu, \nu; \xi, \eta; 0) = A_0(\mu, \nu; \xi, \eta) \quad (19)$$

$$= \frac{1}{\lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^*(x - \xi/2, y - \eta/2, 0)$$

$$\cdot f(x + \xi/2, y + \eta/2, 0) \cdot \exp\{jk(\mu x + \nu y)\} dxdy. \quad (20)$$

If the propagation path is divided into short sections of width Δz_i, $i = 1, 2, \cdots, N$, and if it is assumed that over each section the medium is uniform, with the effect of the irregularities swept forward onto the exit face, then the ambiguity function just to the left of the plane $z = \Delta z_1$ is

$$A(\mu, \nu; \xi, \eta; \Delta z_1^-) = A_0(\mu, \nu; \xi - \mu\Delta z_1, \eta - \nu\Delta z_1) \quad (21)$$

with eq. (20) substituted.

The accumulated phase can be inserted at this point by writing

$$f(x, y, \Delta z_1^+) = f(x, y, \Delta z_1^-) \exp\{j\Phi(x, y)\}, \quad (22)$$

where the phase $\Phi(x, y)$ is given by eq. (7) and has the statistical properties derived in Section 2.2. Now assuming that the $\Phi(x, y)$ process is stationary, over the lateral extent of the beam within the first section, it follows that the ambiguity function at the exit face of the section is

$$A(\mu, \nu; \xi, \eta; \Delta z_1^+) = A_0(\mu, \nu; \xi - \mu\Delta z_1, \eta - \nu\Delta z_1)\exp\{B_{\Phi_1}(\xi, \eta) - B_{\Phi_1}(0, 0)\}, \quad (23)$$

where the phase autocovariance function $B_{\Phi_1}(\xi, \eta)$ is given by eq. (13), and the argument of the exponential is sometimes referred to as the phase structure function. So

$$B_{\Phi_1}(\xi, \eta) = \sigma_{\Phi_1}^2 \rho(\xi, \eta) \quad (24)$$

with the phase variance over this first section given by

$$\sigma_{\Phi_1}^2 = k^2 \sigma_{n_1}^2 \zeta_0 \Delta z_1, \quad (25)$$

where $\sigma_{n_1}^2$ is the variance of the refractive index fluctuations, ζ_0 is its scale size in the direction of propagation, and $\rho(\ )$ is the normalized phase autocovariance function. Thus eq. (23) is equivalently

$$A(\mu, \nu; \xi, \eta; \Delta z_1^+) = A_0(\mu, \nu; \xi - \mu\Delta z_1, \eta - \nu\Delta z_1)\exp\{-\sigma_{\Phi_1}^2[1 - \rho(\xi, \eta)]\}. \quad (26)$$

It will be recalled from Section 2.2 that while $\Delta z_1 \gg \zeta_0$, it is to be kept small enough for $\sigma_{\Phi_1}^2 \ll 1$.

In the next section, of width Δz_2, again artificially separating the phenomena of scattering and diffraction, by analogy with eq. (26)

$$A(\mu, \nu; \xi, \eta; \Delta z_1 + \Delta z_2^+) = A(\mu, \nu; \xi - \mu\Delta z_2, \eta - \nu\Delta z_2; \Delta z_1^+)\exp\{-\sigma_{\Phi_2}^2[1 - \rho(\xi, \eta)]\}, \quad (27)$$

in which it has been assumed that the turbulence is statistically uniform along the beam. (This condition can be relaxed, and clearly ought to be in some circumstances, but at the expense of greater complication.) Combining eqs. (26) and (27) gives

$$A(\mu, \nu; \xi, \eta; \Delta z_1 + \Delta z_2^+) = A_0(\mu, \nu; \xi - \mu[\Delta z_1 + \Delta z_2], \eta - \nu[\Delta z_1 + \Delta z_2])\exp\{-\sigma_{\Phi_2}^2[1 - \rho(\xi, \eta)]\} \cdot\exp\{-\sigma_{\Phi_1}^2[1 - \rho(\xi - \mu\Delta z_2, \eta - \nu\Delta z_2)]\}. \quad (28)$$

This argument can be continued out to a distance

$$z = \sum_{i=1}^{N} \Delta z_i \quad (29)$$

provided only that the propagation is essentially in the forward direction. This is ensured by the large scale size of the turbulence compared to the wavelength, and the small magnitude of the refractive index fluctuations. Then the ambiguity function at the plane z is

$$A(\mu, \nu; \xi, \eta; z) = A_0(\mu, \nu; \xi - \mu z, \eta - \nu z) \cdot \exp\left\{-\sum_{n=1}^{N} \sigma_{\Phi_n}^2 \left[1 - \rho\left(\xi - \mu\left\{z - \sum_{i=1}^{n} \Delta z_i\right\}, \eta - \nu\left\{z - \sum_{i=1}^{n} \Delta z_i\right\}\right)\right]\right\}, \quad (30)$$

in which

$$\sigma_{\Phi_n}^2 = k^2 \sigma_{n_1}^2 \zeta_0 \Delta z_n \quad (31)$$

and with eq. (20) substituted.

The ambiguity function of eq. (30) can be written in integral form if the Δz_n can be taken to be sufficiently small. In the case of a He-Ne laser beam propagating through a turbulent atmosphere, Δz_n can be of the order of a meter, which is small enough when it is realized that both μ and ν are always small. The assumption of the statistical uniformity of the turbulence throughout the length of the path is still maintained, and so

$$A(\mu, \nu; \xi, \eta; z) = A_0(\mu, \nu; \xi - \mu z, \eta - \nu z) \cdot \exp\left\{-k^2 \sigma_{n_1}^2 \zeta_0 \left[z - \int_0^z \rho(\xi - \mu z', \eta - \nu z') dz'\right]\right\}. \quad (32)$$

Finally, the lateral mutual coherence function is given by the inverse Fourier transform of eq. (17) as

$$\Gamma(x, y; \xi, \eta; z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\mu, \nu; \xi, \eta; z) \exp\{-jk(\mu x + \nu y)\} d\mu d\nu \quad (33)$$

with eq. (32) substituted. This result is identical in form to Tatarskii's exact solution of the differential equation for the mutual coherence function, quoted by Ishimaru.[3] This agreement is encouraging, in that Tatarskii's approach[4] is basically more rigorous but is physically somewhat obscure, whereas the present method keeps the physical meaning of the mathematics well to the fore. As an example, the method given here avoids Tatarskii's assumption that the refractive index is delta-function correlated in the direction of propagation, which is a physical impossibility. In fact it seems that this so-called Markov approximation means physically that the induced phase processes in each section are statistically independent. Also, the summation form of the ambiguity function of eq. (30) could be advantageous in applications to paths along which the turbulence is nonstationary.

Incidentally, the mean field obtained from eq. (22), by taking its expectation, is

$$\langle f(x, y, \Delta z_1)\rangle = f_0(x, y, \Delta z_1)\exp\left\{-\frac{1}{2}\sigma_{\Phi_1}^2\right\}, \tag{34}$$

where $f_0(\ )$ is the field diffracted from the original aperture field in the absence of irregularities. Carrying out the same procedure over the N sections of the path gives

$$\langle f(x, y, z)\rangle = f_0(x, y, z)\exp\left\{-\frac{1}{2}k^2\sigma_{n_1}^2\zeta_0 z\right\}, \tag{35}$$

which is also referred to as the coherent part of the field. Note that when the total accumulated phase variance

$$\sigma_{\Phi_T}^2 = k^2\sigma_{n_1}^2\zeta_0 z \tag{36}$$

becomes much larger than unity, the coherent field becomes negligible.

Equations (35) and (33) give the first and second statistical moments, respectively, of the propagating field. The physical mechanism also strongly suggests, as a consequence of the central limit theorem, that the field is complex-Gaussian distributed. The statistical description of the random propagating field is therefore complete.

V. PROPAGATION OF A LASER BEAM THROUGH ATMOSPHERIC TURBULENCE

To find the mean intensity and other characteristics of a laser beam propagating through a turbulent atmosphere, it will be assumed that the beam is launched in its fundamental mode with a plane wave front, that is,

$$f(x, y, 0) = f_0 \exp\left\{-\frac{x^2 + y^2}{w_0^2}\right\}, \tag{37}$$

where w_0 is the beamwaist parameter and f_0 is the complex amplitude at the center of the beam. The ambiguity function for this field is, from eq. (20),

$$A_0(\mu, \nu; \xi, \eta) = \frac{\pi w_0^2|f_0|^2}{2\lambda^2}\exp\left\{-\frac{\pi^2 w_0^2}{2\lambda^2}(\mu^2 + \nu^2)\right\} \cdot\exp\left\{-\frac{\xi^2 + \eta^2}{2w_0^2}\right\}. \tag{38}$$

Equations (32) and (33) then yield the lateral mutual coherence function in this case as

$$\Gamma(x, y; \xi, \eta; z) = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{- \frac{\pi^2 w_0^2}{2\lambda^2}(\mu^2 + \nu^2)\right\}$$
$$\cdot \exp\left\{- \frac{(\xi - \mu z)^2 + (\eta - \nu z)^2}{2w_0^2}\right\}$$
$$\cdot \exp\left\{- k^2 \sigma_{n_1}^2 \zeta_0 \left[z - \int_0^z \rho(\xi - \mu z', \eta - \nu z') dz'\right]\right\}$$
$$\cdot \exp\{- jk(\mu x + \nu y)\} d\mu d\nu, \quad (39)$$

and the mean intensity is obtained from this by setting ξ and η equal to 0.

It is easily verified as a check, using a standard integral, that eq. (39) in the absence of any irregularities in refractive index (i.e., $\sigma_{n_1} = 0$) gives the correct intensity formula for free-space propagation of a laser beam,[15] namely,

$$I_0(x, y, z) = \frac{|f_0|^2 w_0^2}{w^2(z)} \exp\left\{- \frac{2(x^2 + y^2)}{w^2(z)}\right\}, \quad (40)$$

where the beamwaist parameter

$$w(z) = w_0 \sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2} \quad (41)$$

depends on distance. The particular distance

$$z_F = \frac{\pi w_0^2}{\lambda} \quad (42)$$

usefully indicates the transition from the near field ($z \ll z_F$), when the beam is essentially collimated, to the far field ($z \gg z_F$), when the beam spreads out linearly with distance. In the experiment of King et al.,[1] for example, with the beamwaist at launch $w_0 = 8$ cm and the wavelength 0.63 μm, the distance $z_F = 32$ km.

To obtain some analytical results for the mean intensity, it will be necessary to resort to some approximation. One way of doing this is to replace the variable z' in the inner integral of eq. (39) by the constant z. Then the mean intensity, using eq. (16), is given approximately by

$$\langle I(x, y, z)\rangle = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{- \frac{\pi^2 w_0^2}{2\lambda^2}(\mu^2 + \nu^2)\right\}$$
$$\cdot \exp\left\{-z^2 \frac{\mu^2 + \nu^2}{2w_0^2}\right\} \exp\{-\sigma_{\Phi_T}^2[1 - \rho(-\mu z, -\nu z)]\}$$
$$\cdot \exp\{- jk(\mu x + \nu y)\} d\mu d\nu, \quad (43)$$

in which $\sigma^2_{\Phi_T}$ [see eq. (36)] is the total accumulated phase variance over the length of the path. Equation (43) can also be written equivalently, by substituting

$$\mu z = -\xi \quad \text{and} \quad \nu z = -\eta, \tag{44}$$

so that

$$\langle I(x, y, z)\rangle = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2 z^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{\pi^2 w_0^2}{2\lambda^2 z^2} + \frac{1}{2w_0^2}\right)(\xi^2 + \eta^2)\right\} \cdot \exp\{-\sigma^2_{\Phi_T}[1 - \rho(\xi, \eta)]\} \exp\left\{j\,\frac{k(\xi x + \eta y)}{z}\right\} d\xi d\eta. \tag{45}$$

If now the middle exponential in eq. (45) is written as the sum of two parts, as

$$\exp\{-\sigma^2_{\Phi_T}[1 - \rho(\xi, \eta)]\} = \exp\{-\sigma^2_{\Phi_T}\} + \exp\{-\sigma^2_{\Phi_T}\}[\exp\{\sigma^2_{\Phi_T}\rho(\xi, \eta)\} - 1], \tag{46}$$

then the mean intensity formula of eq. (45) conveniently splits into the sum of the coherent intensity and the incoherently scattered intensity, namely,

$$\langle I(x, y, z)\rangle = I_0(x, y, z)\exp\{-\sigma^2_{\Phi_T}\} + I_s(x, y, z), \tag{47}$$

in which $I_0(x, y, z)$ is given by eq. (40) and

$$I_s(x, y, z) = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2 z^2} \exp\{-\sigma^2_{\Phi_T}\} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{\pi^2 w_0^2}{2\lambda^2 z^2} + \frac{1}{2w_0^2}\right)(\xi^2 + \eta^2)\right\} \cdot [\exp\{\sigma^2_{\Phi_T}\rho(\xi, \eta)\} - 1] \exp\left\{j\,\frac{k(\xi x + \eta y)}{z}\right\} d\xi d\eta. \tag{48}$$

It is interesting to examine the two extreme cases of $\sigma^2_{\Phi_T} \ll 1$ and $\sigma^2_{\Phi_T} \gg 1$, which correspond respectively to short and long propagation paths.

5.1 Short paths

If, according to eq. (36), the path length z is short enough to make $\sigma^2_{\Phi_T} \ll 1$, then eq. (47) shows that the coherent part will predominate. Appropriate approximations in eq. (48) give the incoherent scattered power in this case as

$$I_s(x, y, z) = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2 z^2} \sigma^2_{\Phi_T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\xi, \eta) \cdot \exp\left\{-\left(\frac{\pi^2 w_0^2}{2\lambda^2 z^2} + \frac{1}{2w_0^2}\right)(\xi^2 + \eta^2)\right\} \exp\left\{j\,\frac{k(\xi x + \eta y)}{z}\right\} d\xi d\eta. \tag{49}$$

If it is assumed that the initial laser beam width $w_0 \gg a$, where a is the typical scale size of the turbulence (of the order of 1 cm in the atmosphere), then the first exponential term in eq. (48) can be ignored and the mean intensity profile of the scattered field is the Fourier transform of $\rho(\xi, \eta)$.

Thus, over short paths the laser beam will be only slightly diminished in comparison to the free-space situation, and the energy lost will be scattered. When $w_0 \gg a$ there will be an intense central spot surrounded by a faint halo of scattered light. The form of the scattered intensity profile will be determined by the lateral correlation of the turbulent refractive-index fluctuations. On the other hand, if $w_0 \ll a$ the laser beam will snake its way through the turbulence, preserving its original profile but continually changing its direction in a random manner.

5.2 Long paths

If the path length is long enough to make $\sigma^2_{\Phi_T} \gg 1$, then according to eq. (47) the coherent part will be negligible, and so eq. (45) can be used directly to describe the now completely scattered field. Examining the middle exponential term in the integrand of eq. (45) reveals that for very large $\sigma^2_{\Phi_T}$ its behavior will be dominated by the behavior of $\rho(\xi, \eta)$ near the origin.[16] For turbulence

$$\rho(\xi, \eta) = 1 - \frac{\xi^2 + \eta^2}{a^2} + \cdots , \tag{50}$$

where a is now to be interpreted as the dissipation scale size of the assumed uniform and isotropic turbulence. (This assumption may not be true of course, but any naturally occurring refractive-index fluctuations will have an autocorrelation function that behaves parabolically in the neighborhood of the origin.[17]) Substituting eq. (50) into eq. (45) gives the mean intensity profile as

$$\langle I(x, y, z)\rangle = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2 z^2} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\left(\frac{\pi^2 w_0^2}{2\lambda^2 z^2} + \frac{1}{2w_0^2} + \frac{\sigma^2_{\Phi_T}}{a^2}\right)(\xi^2 + \eta^2)\right\} \cdot \exp\left\{j\,\frac{k(\xi x + \eta y)}{z}\right\} d\xi d\eta. \tag{51}$$

So, if $w_0 \gg a$, the $\sigma^2_{\Phi_T}/a^2$ term dominates, and a standard integral yields

$$\langle I(x, y, z)\rangle = \frac{|f_0|^2 w_0^2}{w^2(z)} \exp\left\{-\,\frac{2(x^2 + y^2)}{w^2(z)}\right\}, \tag{52}$$

where now

$$w(z) = \frac{\sqrt{2}\lambda\sigma_{\Phi_T} z}{\pi a}. \tag{53}$$

Equation (52) gives the mean intensity profile as being Gaussian in shape, as observed in the experiment of King et al.[1] The beamwaist parameter depends on $z^{3/2}$, since σ_{Φ_T} varies as $\sqrt{z}$. Some care must be taken with the longitudinal integral scale ζ_0 of the turbulence, which is needed to evaluate σ_{Φ_T} (see eq. 36). It is tempting to identify it with the dissipation scale size a, used in eq. (50), but that is probably an underestimate. On the other hand, Ishimaru's identification of it with the outer scale of turbulence[18] seems like an overestimate. The cautionary remark at the end of Section 5.3 should also be noted.

5.3 Lateral field autocorrelation

The autocorrelation of the propagating field over a plane is given by the lateral mutual coherence function of eq. (39). When the same approximation as for the intensity (eq. 43) is made, namely, replacing z' in the inner integral of eq. (39) by z, the lateral mutual coherence function becomes

$$\begin{aligned}\Gamma(x, y; \xi, \eta; z) = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{\pi^2 w_0^2}{\lambda^2}(\mu^2 + \nu^2)\right\} \\ \cdot \exp\left\{-\frac{(\xi - \mu z)^2 + (\eta - \nu z)^2}{2w_0^2}\right\} \\ \cdot \exp\left\{-\sigma_{\Phi_T}^2[1 - \rho(\xi - \mu z, \eta - \nu z)]\right\} \exp\{-jk(\mu x + \nu y)\} d\mu d\nu. \end{aligned} \tag{54}$$

Making the substitutions $p = \xi - \mu z$, $q = \eta - \nu z$ gives

$$\begin{aligned}\Gamma(x, y; \xi, \eta; z) = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2 z^2} \\ \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{\pi^2 w_0^2}{2\lambda^2 z^2}[(\xi - p)^2 + (\eta - q)^2]\right\} \\ \cdot \exp\left\{-\frac{p^2 + q^2}{2w_0^2}\right\} \exp\left\{-\sigma_{\Phi_T}^2[1 - \rho(p, q)]\right\} \\ \cdot \exp\left\{-j\frac{k(\xi x + \eta y)}{z}\right\} \exp\left\{j\frac{k(px + qy)}{z}\right\} dp dq. \end{aligned} \tag{55}$$

In the long-path limit, when $\sigma_{\Phi_T}^2 \gg 1$, and making use of eq. (50), the lateral mutual coherence function now becomes

$$\Gamma(x, y; \xi, \eta; z) = \frac{\pi w_0^2 |f_0|^2}{2\lambda^2 z^2} \exp\left\{-j\frac{k(\xi x + \eta y)}{z}\right\}$$

$$\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{\pi^2 w_0^2}{2\lambda^2 z^2}[(\xi - p)^2 + (\eta - q)^2]\right\}$$

$$\cdot \exp\left\{-\left(\frac{1}{2w_0^2} + \frac{\sigma_{\Phi_T}^2}{a^2}\right)(p^2 + q^2)\right\} \exp\left\{j\frac{k(px + qy)}{z}\right\} dpdq. \quad (56)$$

Performing the integration, and expressing the result in terms of the long-path mean intensity of eq. (51), gives

$$\Gamma(x, y; \xi, \eta; z) = \exp\left\{-j\frac{k(\xi x + \eta y)}{z}\right\}$$

$$\cdot \exp\left\{-\frac{\pi^2 w_0^2}{2\lambda^2 z^2}(\xi^2 + \eta^2)\right\} \langle I(x, y, x)\rangle. \quad (57)$$

It should be remembered, however, that this result is based on the approximation made in deriving eq. (54) from eq. (39). The physical significance of this approximation is now clear and is the following.

In effect, the accumulated random phase along the path has been incorporated in a single random phase-changing screen placed just in front of the radiating laser aperture. This observation follows from a very useful result given in Ratcliffe,[16] which is that the angular correlation function for the far field is the Fourier transform of the *magnitude* squared of the aperture field distribution. Using this result, one would expect the scattered field to be initially correlated over angles of the order of $\lambda/(\pi w_0)$. But that with increasing z, as the magnitude of the field becomes more finely divided, the lateral correlation scale size would be less than the $\lambda z/(\pi w_0)$ indicated by eq. (57). This speculation is borne out by some computer simulations.[19] A better approximation might be to replace the inner integral of eq. (39) by $z\rho(\xi - \mu z/2, \eta - \nu z/2)$. This would move the single equivalent random phase-changing screen to a point midway along the path. However, neither approximation is particularly satisfactory, and it is obviously safer to use the unapproximated eq. (39), although this would probably require numerical evaluation.

VI. CONCLUSIONS

The beam propagation method, based on the parabolic approximation to the wave equation, has been applied to the propagation of a laser beam through the clear but turbulent atmosphere. Papoulis' extension to optical fields of Woodward's ambiguity function was used. The resulting formula for the lateral mutual coherence function of the propagating laser beam, and hence its mean intensity profile, agrees with that of Tatarskii. The method has the advantage over alternative approaches of greater physical clarity. Incorporation of the effects of very-large-scale irregularities and of receiver directivity is then very simple, as also is the estimation of signal statistics and allowing for the consequences of spatial nonstationarity of the turbulence.

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