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Criteria for the Global Existence of Functional Expansions for Input/Output Maps*

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Much has been learned in recent years about the existence, determination, and properties of power-series-like expansions for expressing a nonlinear system's outputs in terms of its inputs. In particular, the existence and local convergence of expansions, and of certain "associated expansions," for important large classes of systems are now well established. While the focus of attention has been on questions such that the *size* of the inputs for which convergence is guaranteed is not the main issue, some related material has appeared that bears on the problem of determining the extent of the region of convergence. The result most closely related to this paper is a recent theorem that gives necessary and sufficient conditions under which f^{-1} has a generalized power-series expansion when f is an invertible locally-Lipshitz map between certain general subsets of two complex Banach spaces. In applications involving nonlinear models, ordinarily only *real* spaces of inputs and outputs are of direct interest. A "complexification" involving a certain solvability condition in complex spaces has to be able to be carried out to use the theorem referred to above. This paper reports on pertinent general results concerning invertible maps between subsets of real Banach spaces, with their complex extensions, and with generalized power-series expansions in both real and complex spaces. It focuses on questions concerning expansions for inverses of maps defined in real spaces. The results show that for a very large class of systems that have input/output maps, the ability to complexify is not just a useful sufficient

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condition for expandability, but is in fact the *key* condition for an input/output map to be representable by a generalized power-series expansion.

I. INTRODUCTION

Convolution operator input/output representations for linear systems are well understood and are widely used. With regard to corresponding representations for nonlinear systems, much has been learned in recent years about the existence, determination, and properties of power-series-like expansions for expressing a system's outputs in terms of its inputs (see, for example, Refs. 1–7). In particular, the existence and local convergence of expansions, and of certain "associated expansions,"[3] are now well established for important large classes of systems.

While the focus of attention in Refs. 1 through 6 has been on questions such that the *size* of the inputs for which convergence is guaranteed is not the main issue, some related material has appeared that bears on the problem of determining the extent of the region of convergence. The result most closely related to this paper is a theorem in Ref. 7 which gives necessary and sufficient conditions under which f^{-1} has a generalized power-series expansion (in the sense of our Section 2.1) when f is an invertible locally-Lipshitz map between certain general subsets of two complex Banach spaces. Another theorem in Ref. 7 yields an algorithm for obtaining the expansion whenever it exists, and these two theorems are used therein to prove results concerning a certain system model considered in Ref. 2 and in earlier papers.

In applications involving nonlinear models, ordinarily only *real* spaces of inputs, outputs, and intermediate signals are of direct interest. A "complexification" involving the existence of a certain inverse map defined on a complex space has to be able to be carried out to use the theorems in Ref. 7. One of the main applications of the results in this paper is a proof that in an important general setting this complexification condition is always met when certain invertibility and expandability conditions are satisfied in the underlying real space. As a consequence, for a very large class of systems that have input/output maps, the ability to complexify emerges as the *key* condition for an input/output map to be representable by a generalized power-series expansion. (Under certain reasonable assumptions these expansions reduce to Volterra-like series.)[2,8]

To be more explicit, models of the kind mentioned above are characterized by five operators: a nonlinear operator N, and four linear operators a, b, c, and d. They have an input v and an output w, which belong to a space X of functions. Here X is taken to be a real Banach space; a, b, and c are assumed to be bounded maps of X into X, and we suppose that N is defined on all of X and takes X into X. One has $w = dv + bN(I - cN)^{-1}av$ (I the identity map on X) subject to some natural qualifications, from which it is clear that the study of such models* often involves the study of maps of the form $(I - cN)$. The X of particular interest to us is the space of real Lebesgue-measurable, n-vector-valued functions x defined on $[0, \infty)$, with the norm in X given by $\| x \| = \max_j \sup_{t \geq 0} | x_j(t) |$, where $x_j(t)$ is the jth component of $x(t)$.

Let A_0 and A be subsets of X such that $(I - cN)$ restricted to A_0 is an invertible map of A_0 onto A. Assume that both A_0 and A are open sets, and that A contains the zero element of X. Under these conditions, w is well defined for each v such that $av \in A$, the zero function is an allowed input, and the set of allowed inputs is open. The question that we ask is this: With $(I - cN)^{-1}$ the inverse of the restriction of $(I - cN)$ to A_0, assumed to be continuous, when is it true that $(I - cN)^{-1}u$ has a generalized power-series expansion that converges for $u \in A$? When it *is* true, the map from v to w has an expansion that converges whenever $av \in A$, assuming (and this is frequently very reasonable) that N is such that the existence of the expansion for $(I - cN)^{-1}u$ implies the existence of an expansion for $N(I - cN)^{-1}u$; see, for example, Corollary 1 in Appendix A or Theorems 1 and 7.

Theorem 2 in Section II provides an answer to the question, under the assumption that N has an extension into a complex space $\mathscr{B}$ associated with X, with this extension a certain type of globally convergent generalized power series. For the X of particular interest, this assumption is a reasonable one, and the corresponding $\mathscr{B}$ turns out to be just the natural complex associate of X. The answer given by Theorem 2 is that there must be two open subsets V_0 and V of $\mathscr{B}$ such that: $A \subset V$ (meaning that $u + i0 \in V$ for each $u \in A$; see Section 2.1), $A_0 \subset V_0$, V is a "star" in the sense that $zq \in V$ when $q \in V$ and z is a scalar such that $|z| \leq 1$, and the map $(I - cN)$ extended into $\mathscr{B}$ (see Section 2.2), and restricted to V_0, must be a homeomorphism of V_0 onto V, with the inverse of the restriction of the extended $(I - cN)$ locally Lipshitz on V. While this necessary and sufficient condition† may look complicated at first glance, its interpretation is straightforward: $(I - cN)^{-1}u$ has a power-series expansion that converges for $u \in A$ if and only if the equation $x - cNx = u$, when

* There is an error in the corresponding equation in Ref. 7, where B in (11) should be replaced with BN. This does not change the conclusion drawn there from Theorem 3; see, for example, our Theorem 7.

† The *sufficiency* of this type of condition is discussed in Ref. 7, Section 2.4.2. Also, with regard to the system model in Ref. 7 (p. 84), note that the existence of an expansion for w (in terms of v) implies the existence of an expansion for y and thus for x if, for example, B is the identity operator.

extended into the complex space $\mathscr{B}$, is, so-to-speak, uniquely locally-Lipschitz solvable in some open subset of $\mathscr{B}$ containing the points of A_0 for all right sides belonging to V, where V is any open star in $\mathscr{B}$ that contains the elements of A. (See Section 2.3.1. In this connection, notice that an open ball centered at the origin is an example of a star.)

As is suggested by the application described above, the results in this paper are concerned with invertible maps between subsets of real Banach spaces, with their complex extensions, and with generalized power-series expansions in both real and complex spaces, with the focus on questions concerning global expansions for inverses of maps defined in real spaces. Preliminaries are introduced in Section 2.1, and Sections 2.2 through 2.6 contain the paper's principal results.

There are several natural applications of the material in Section II other than the one already discussed. For example, consider again the five-operator model described above, and assume that the assumptions introduced are met. Assume in addition that an expansion representation for $(I - cN)^{-1}u$ does exist for $u \in A$. Suppose that this expansion also converges for $u \in B$, where B is some open subset of X for which $A \subset B$. Theorem 3 shows that then the map from v to w is in fact both well defined and has a generalized power-series expansion for $av \in B$.

It will become clear that the theorems in Section II are considerably more general than the applications discussed above are able to illustrate. For instance, they bear on cases in which the underlying function space is a set of functions of more than one independent variable. Also, in Section 2.5 corresponding results are given for certain implicitly defined maps. These latter results are useful in, for example, studies of globally convergent generalized power-series expansions for solutions of differential equations.

II. COMPLEX EXTENSIONS AND EXPANSION REPRESENTATIONS

2.1 Preliminaries

Throughout the paper X denotes a real Banach space. We associate with X (see Ref. 9, p. 312 and Ref. 10, p. 665) a complex Banach space $\mathscr{B}$ defined as follows: the elements of $\mathscr{B}$ are ordered pairs (x_1, x_2) of elements of X, addition and multiplication obey

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$(\alpha + i\beta)(x_1, x_2) = (\alpha x_1 - \beta x_2, \alpha x_2 + \beta x_1),$$

and the norm of an element of $\mathscr{B}$ is given by

$$\| (x_1, x_2) \| = \sup_{\|\xi\|=1} [\xi^2(x_1) + \xi^2(x_2)]^{1/2},$$

where ξ denotes a general, real, bounded linear functional on X. We sometimes use $x_1 + ix_2$ to denote an element (x_1, x_2) of $\mathscr{B}$.

The map $x \to x + i0$ of elements of X into elements of $\mathscr{B}$ isometrically* imbeds X into the complex space $\mathscr{B}$. In particular, in this sense, $\mathscr{B}$ is a complex extension of X. For example, if X is the space of bounded, Lebesgue-measurable, real n-vector-valued functions x defined on $[0, \infty)$, with $\| x \| = \max_j \sup_t | x_j(t) |$, then the elements v of $\mathscr{B}$ are bounded, Lebesgue-measurable, complex n-vector-valued functions defined on $[0, \infty)$, and (see Appendix B) one simply has $\| v \| = \max_j \sup_j | v_j(t) |$.

A *star* in $\mathscr{B}$ means a subset S of $\mathscr{B}$ such that $zv \in S$ for $v \in S$ and any complex scalar z with $|z| \leq 1$. A subset S of $\mathscr{B}$ is *c-convex* if for any bounded open set Δ of complex numbers, we have $(v + \Delta u) \subset S$ whenever $(v + \Gamma u) \subset S$, where Γ is the boundary of Δ.

In the paper, X_0 denotes a second real Banach space and $\mathscr{B}_0$ stands for its complex extension. We allow the possibility that $X_0 = X$.

Now let Y and W be any two Banach spaces, both real or both complex.

Given any positive integer m, by an m-linear map q from Y^m into W we mean that $q(y_1, \cdots, y_m)$ is linear (i.e., additive and homogeneous) separately in each y_j. Such a map is *symmetric* if $q(y_1, \cdots, y_m)$ is symmetric in the variables $y_1, \cdots, y_m$. A map h from Y into W is called a homogeneous polynomial of degree m if there exists an m-linear q from Y^m to W such that $h(y) = q(y, \cdots, y)$ for all y.† A homogeneous polynomial of degree zero is a constant map.

For S a subset of Y, let $\mathscr{P}(S, W)$ denote the set of all maps p from S into W such that there are homogeneous polynomials h_m of degree m $(m = 0, 1, \cdots)$ from Y to W, with the properties that $\sum_{m=0}^{\infty} h_m(s)$ converges in W for each $s \in S$, and

$$p(s) = \sum_{m=0}^{\infty} h_m(s), \quad s \in S. \tag{1}$$

The set $\mathscr{P}(S, W)$ is, of course, a set of maps p that admit a generalized power-series expansion in the sense indicated. If S contains an open ball in Y centered at the origin, then the expansion (1) for any $p \in \mathscr{P}(S, W)$ is unique in the sense that if

$$p(s) = \sum_{m=0}^{\infty} g_m(s), \quad s \in S,$$

with each g_m a homogeneous polynomial of degree m, then $g_m = h_m$ for all m (see Ref. 11, p. 174 and Ref. 1, Section 2.7).

Finally, we say that p belongs to $\mathscr{P}_F(S, W)$ if $p \in \mathscr{P}(S, W)$ and for

* By the Hahn-Banach theorem, $\| x \| = \sup\{\xi(x): \| \xi \| = 1\}$.

† The same class of maps is obtained if "m-linear" is replaced with "symmetric m-linear."

each positive m there is a *continuous* symmetric m-linear q_m from Y^m into W such that $h_m(s) = q_m(s, \cdots, s)$ for all s. In particular, then each h_m is bounded in the sense that there is a constant $\rho_m > 0$ such that $\| h_m(s) \| \leq \rho_m \| s \|^m$ for all m and s, and every h_m is Fréchet differentiable on Y.

2.2 Inverse maps and necessary conditions for the existence of series representations

Throughout this section, and in Sections 2.3, 2.4, and 2.6, f is a map from X_0 into X, A and A_0 are open subsets of X and X_0, respectively, with $0 \in A$, f restricted to A_0 is a homeomorphism of A_0 onto A, and $g: A \to A_0$ is the inverse of the restriction of f. It is assumed that there is an $f^* \in \mathscr{P}_F(\mathscr{B}_0, \mathscr{B})$ such that $f(x) = f^*(x + i0)$ for $x \in X_0$. [Of course, by $f(x) = f^*(x + i0)$ we mean that $f(x) + i0 = f^*(x + i0)$.]

The following extension theorem is this paper's main result.

Theorem 1: If g has a power-series representation in the sense that $g \in \mathscr{P}(A, X_0)$, then there are open sets V and V_0 in $\mathscr{B}$ and $\mathscr{B}_0$, respectively, together with a map $g^*: V \to V_0$ such that V is a c-convex star, $A \subset V$, $A_0 \subset V_0$, and

1. *the restriction of f^* to V_0 is a homeomorphism of V_0 onto V with inverse g^**
2. $g^* \in \mathscr{P}_F(V, \mathscr{B}_0)$
3. $g(x) = g^*(x + i0)$, $x \in A$.

2.2.1 Proof of Theorem 1

Two lemmas are used in the proof. The first of these follows.

Lemma 1: Let D be an open subset of X with $0 \in D$, let $h \in \mathscr{P}(D, X_0)$, and assume that h is continuous on D. Then there are an open c-convex star $Z \subset \mathscr{B}$ and a map h^* from Z to $\mathscr{B}_0$ such that $D \subset Z$, $h^* \in \mathscr{P}_F(Z, \mathscr{B}_0)$, and $h(x) = h^*(x + i0)$ for $x \in D$.

Proof of Lemma 1: We have

$$h(s) = \sum_{m=0}^{\infty} h_m(s), \quad s \in D, \tag{2}$$

where each h_m is a homogeneous polynomial of degree m. Let q_m be the unique symmetric m-linear map such that $h_m(s) = q_m(s, \cdots, s)$ for $s \in X$ and positive m (see Ref. 12, pp. 762–3). Let $h_0^* = h_0 + i0$, and define $h_m^*: \mathscr{B} \to \mathscr{B}_0$ for each $m \geq 1$ by

$$h_m^*(x_1 + ix_2) = \sum_{k=0}^{m} i^{(m-k)} \binom{m}{k} q_m(x_1, \cdots, x_1, x_2, \cdots, x_2), \tag{3}$$

with kx_1's and $(m - k)x_2$'s on the right side. It is not difficult to verify that the h_m^* are homogeneous polynomials of degree m (see Ref. 9, p.

313 and Ref. 13, p. 71). By Theorem 5.7 of Ref. 13 there is an open subset Z_1 of $\mathscr{B}$ such that $D \subset Z_1$ and the series

$$\sum_{m=0}^{\infty} h_m^*(v) \tag{4}$$

converges for $v \in Z_1$. Since D is open and h is continuous, it follows (see Ref. 13, Theorems 4.4 and 6.6) that h is analytic in D in the sense of Ref. 13, p. 75. Thus, using the hypothesis that $0 \in D$, h has a power-series expansion valid in a neighborhood of the origin, with the terms in the expansion continuous homogeneous polynomials. At this point the uniqueness result mentioned in Section 2.1 shows that the h_m in (2) are continuous.

By the continuity of the h_m, the q_m in (3) are continuous. Using (3) and the fact that $\| x_1 \| \leq 1$ and $\| x_2 \| \leq 1$ are implied by $\| x_1 + ix_2 \| \leq 1$ [see (7), below], we see that each h_m^* is bounded in the ball $\| x_1 + ix_2 \| \leq 1$. This shows (see Ref. 12, Theorem 26.2.4) that the h_m^* are continuous.

Let Z denote the interior of the region of convergence of the series (4), and let $h^*(v)$ be the sum (4) for any $v \in Z$. Obviously $Z_1 \subset Z$. Since the h_m^* are continuous, it follows (see Ref. 12, Theorem 26.6.1) that Z is a c-convex star. Since it is clear that $h_m^*(x + i0) = h_m(x)$ for $x \in X$, the proof of the lemma is complete.

Continuing with the proof of the theorem, by Lemma 1 there are an open c-convex star $V \subset \mathscr{B}$ and a map $g^*: V \rightarrow \mathscr{B}_0$ such that $A \subset V$, $g^* \in \mathscr{P}_F(V, \mathscr{B}_0)$, and $g(x) = g^*(x + i0)$ for $x \in A$.

We now turn to our second lemma.

Lemma 2: Let h be a Fréchet-differentiable map (*Ref.* 14, *p.* 149) *from an open connected subset D of $\mathscr{B}$ into $\mathscr{B}_0$. Assume that there is a point p in D and an open ball Q in X centered at the origin such that $(p + Q) \triangleq \{s \in \mathscr{B}: s = p + (q + i0), q \in Q\} \subset D$ and h maps $(p + Q)$ into the origin in $\mathscr{B}_0$. Then h vanishes everywhere in D.*

Proof of Lemma 2: Since h is Fréchet differentiable in a neighborhood of p, there are (see Ref. 12, Theorems 3.16.2 and 26.3.5) homogeneous polynomials H_m of degree m ($m = 1, 2, \cdots$) and a $\sigma > 0$ such that $s \in D$ and

$$h(s) = \sum_{m=1}^{\infty} H_m(s - p)$$

when $\| s - p \| < \sigma$. Thus, for some positive $\rho < \sigma$, we have

$$\sum_{m=1}^{\infty} H_m(q + i0) = \theta$$

for $q \in \{x \in X: \| x \| < \rho\}$, where θ is the zero element of $\mathscr{B}_0$. It easily follows (see Ref. 1, Section 2.7) that $H_m(x + i0) = \theta$ for each m and all $x \in X$.

Consider $H_m(x_1 + ix_2)$ with m, x_1, and x_2 arbitrary, and let P_m denote the polar form (see Ref. 12, pp. 762–3) associated with H_m. We see that $H_m(x_1 + ix_2)$ can be written as a finite sum of terms of the form $cP(y_1 + i0, \cdots, y_m + i0)$, in which $c \in \{\pm 1, \pm i\}$ and each y_j is either x_1 or x_2. On the other hand, $P_m(y_1 + i0, \cdots, y_m + i0)$ can be expressed (see Ref. 9, p. 306) as

$$(m!)^{-1} \sum_{\epsilon_1, \cdots, \epsilon_m = 0}^{1} (-1)^{m-(\epsilon_1 + \cdots + \epsilon_m)} H_m[\epsilon_1(y_1 + i0) + \cdots + \epsilon_m(y_m + i0)].$$

Therefore, using $H_m(x + i0) = \theta$ for $x \in X$, one has $H_m(x_1 + ix_2) = \theta$. This shows that $h(s) = \theta$ for s in an open ball in D. Since D is connected and h is G-differentiable in the sense of Ref. 12 (pp. 109–10), it follows (see Ref. 12, Theorem 3.16.4) that $h(s) = \theta$ throughout D, as claimed. We now return to the proof of the theorem.

Let $E(v)$ denote $f^*[g^*(v)] - v$ ($v \in V$). Since $f^* \in \mathscr{P}_F(\mathscr{B}_0, \mathscr{B})$ and $g^* \in \mathscr{P}_F(V, \mathscr{B}_0)$, f^* and g^* are Fréchet differentiable on $\mathscr{B}_0$ and V, respectively (see Ref. 12, Theorems 26.6.4 and 3.17.1). Thus, using a version of the chain rule for differentiating a composite map, E is Fréchet differentiable on V. In addition, the set V is connected (because it is a star), and we have

$$E(x + i0) = f^*[g^*(x + i0)] - (x + i0) = 0, \quad x \in A.$$

Choose any point $p_1 \in A$, and let Q be an open ball in X centered at the origin such that $(p_1 + Q) \subset A$. Let $p = (p_1 + i0)$, and observe that $E[p + (q + i0)] = 0$ for $q \in Q$. By Lemma 2 (with $\mathscr{B} = \mathscr{B}_0$), $E(v) = \theta$ (the zero element of $\mathscr{B}$) for $v \in V$. This gives

$$f^*[g^*(v)] = v, \quad v \in V. \tag{5}$$

Since V is connected and g^* is continuous on V, $g^*(V)$ is connected. The continuity of f^* implies that $f^{*-1}(V)$ is open. From (5) it is clear that $g^*(V) \subset f^{*-1}(V)$. Let V_0 denote the maximal connected subset (i.e., the component) of $f^{*-1}(V)$ that contains $g^*(V)$. Since $f^{*-1}(V)$ is open, so is V_0. The map f^* obviously takes V_0 into V.

Now let $F(w)$ denote $g^*[f^*(w)] - w$ ($w \in V_0$). It follows from Lemma 2 and $F(x + i0) = 0$ for $x \in A_0$ that

$$g^*[f^*(w)] = w, \quad w \in V_0. \tag{6}$$

Since (5) and (6) hold, f^* restricted to V_0 is a homeomorphism of V_0 onto V. The observation that $A_0 \subset V_0$ because $A_0 = g(A) \subset g^*(V) = V_0$ completes the proof of the theorem.

Here we use Theorem 1 to obtain necessary and sufficient conditions for the global expandability of the map g introduced at the beginning of Section 2.2. With regard to our result, Theorem 2 (below), we say that a map h from an open subset D of $\mathscr{B}$ into $\mathscr{B}_0$ is *locally Lipschitz* on D if for each $a \in D$ there are a positive number c_a and an open ball $\beta_a \subset D$ centered at a such that $\| h(v_1) - h(v_2) \| \leq c_a \| v_1 - v_2 \|$ for v_1 and v_2 in β_a.

Theorem 2: We have $g \in \mathscr{P}(A, X_0)$ if and only if (i) there are open sets V and V_0 in $\mathscr{B}$ and $\mathscr{B}_0$, respectively, with V a star, $A \subset V$, and $A_0 \subset V_0$ such that the restriction of f^* to V_0 is a homeomorphism of V_0 onto V, with the inverse of the restriction of f^* to V_0 locally Lipschitz on V, and (ii) the spaces $\mathscr{B}$ and $\mathscr{B}_0$ are homeomorphic, in the sense that there is a linear homeomorphism of $\mathscr{B}$ onto $\mathscr{B}_0$.

Proof: The necessity of (i) follows from Theorem 1 and the observation that g^* in Theorem 1, which belongs to $\mathscr{P}_F(V, \mathscr{B}_0)$, is Fréchet differentiable, hence continuously Fréchet differentiable (see Ref. 7, Lemma 2), and thus locally Lipschitz. Similarly, the necessity of (ii) is a consequence of Theorem 1 and the fact that the conclusion of Theorem 1 implies (see Ref. 15, p. 175, Problem 6) that the F-derivative (i.e., the Fréchet derivative) of f^* at any point in V_0 is an invertible map of $\mathscr{B}_0$ onto $\mathscr{B}$.

On the other hand, if (i) and (ii) are met, then, using Lemma 1 of Ref. 7, the inverse H of the restriction of f^* to V_0 is F-differentiable [and thus G-differentiable (see Ref. 12, pp. 109–10)] on V. It follows that $H \in \mathscr{P}(V, \mathscr{B}_0)$ (Ref. 12, Theorems 3.16.2 and 26.3.4). Let the series for H be given by

$$H(v) = \sum_{m=0}^{\infty} H_m(v), \quad v \in V.$$

By the conditions on f, for any $x \in A$ there is a $y \in A_0$ such that $f^*(y + i0) = (x + i0)$. Therefore, using $H[f^*(v)] = v$ ($v \in V_0$) and $A_0 \subset V_0$, we see that H takes A into A_0. Thus, since $f^*[H(v)] = v$ ($v \in V$), we have

$$g(x) = \sum_{m=0}^{\infty} H_m(x + i0), \quad x \in A.$$

Of course $H_0(0 + i0) \in X_0$. We claim that for each positive m there is an m-linear map Q_m from X^m into X_0 such that $Q_m(x, \cdots, x) = H_m(x + i0)$, $x \in A$. Since the H_m are homogeneous polynomials, we need only show that each H_m maps X into X_0, and we do that as follows.

The norm in $\mathscr{B}_0$ has the property that $\| x_1 + ix_2 \| < \delta$ implies that $\| x_2 \| < \delta$, because, using the Hahn-Banach theorem,

$$\| x_2 \| = \sup_{\|\xi\|=1} |\xi(x_2)| \leq \sup_{\|\xi\|=1} [\xi(x_1)^2 + \xi(x_2)^2]^{1/2} = \| x_1 + ix_2 \|. \quad (7)$$

Thus, the convergence of $\sum_{m=0}^{\infty} H_m(x + i0)(x \in A)$ implies that for each $x \in A$, the series $\sum_{m=0}^{\infty} KH_m(x + i0)$ converges in X_0, and that it converges to the zero element, where K is the map from $\mathscr{B}_0$ to X_0 defined by $x_2 = K(x_1 + ix_2)$. In particular, with β any open ball in A centered at the origin, one has $rx \in \beta$ and

$$0 = \sum_{m=0}^{\infty} KH_m(rx + i0) = \sum_{m=0}^{\infty} r^m KH_m(x + i0)$$

for $x \in \beta$ and $|r| < 1$. It follows (see Ref. 11, proof of Theorem 6) that $KH_m(x + i0) = 0$ for each m and any $x \in X$, showing that the H_m map X into X_0. This completes the proof.

2.3.1 Comments

Since the inverse image of an open set under a continuous map is open, we see that (i) is equivalent to the condition that there be an open subset S of $\mathscr{B}_0$ and an open star $V \subset \mathscr{B}$ with the following properties: $A_0 \subset S$, $A \subset V$, for each $v \in V$ there is a unique $w \in S$ that satisfies $f^*(w) = v$, and the map $v \rightarrow w$ is locally Lipschitz. This more sharply focuses attention on how machinery for proving existence, such as fixed-point techniques, might be used to establish expandability. A pertinent example can be found in Ref. 7, Appendix B. A simple, related additional example follows.

Let X be the space of real numbers, with the absolute value norm, and observe that the corresponding $\mathscr{B}$ is the usual space of complex numbers. Take $X_0 = X$, let $f(x) = x + x^3$ for all real x, and take A and A_0 to be $\{x: |x| < r\}$ and $f^{-1}(A)$, respectively, for some positive r. Notice that any $r > 0$ will do, and that our f^* is given by $f^*(z) = z + z^3$ for all complex z.

An easy contraction-mapping argument* shows that given $\rho \in (0, \sqrt{3}^{-1}]$ and any complex number a with $|a| < (\rho - \rho^3)$, there is a unique complex number z with $|z| < \rho$ such that $z + z^3 = a$, that z is real whenever a is, and that the map from a to z is locally Lipschitz. It follows from Theorem 2 that we have $g \in \mathscr{P}(A, X_0)$ for $r = r_0$, where $r_0 = 2(3\sqrt{3})^{-1}$.

Theorem 2 also can be used to show that g *does not* belong to $\mathscr{P}(A, X_0)$ if $r > r_0$: Suppose, for the purpose of obtaining a contradiction, that $g \in \mathscr{P}(A, X_0)$ for some $r > r_0$. Then for some V and V_0 as described in the theorem, f^* restricted to V_0 is a homeomorphism of V_0 onto V. By the proof of Theorem 2, the inverse g^* of f^* is differ-

* A good general source of information on the use of the contraction-mapping theorem is Ref. 16.

entiable. Using $g^*[f^*(z)] = z$ for $z \in V_0$, we have $g^{*\prime}[f^*(z)]f^{*\prime}(z) = 1$ ($z \in V_0$), where "′" denotes the ordinary derivative. Since $f^{*\prime}(z) = 0$ at $z = z_0 \triangleq i(\sqrt{3})^{-1}$, V_0 cannot contain z_0. Using this fact and the continuity of g^*, it is not difficult to show that V cannot contain the point $f^*(z_0) = 2i(3\sqrt{3})^{-1}$. Since V is a star and $A \subset V$, $2(3\sqrt{3})^{-1} \notin A$, which is the contradiction sought. This finishes the discussion of the example.

Using Theorem 1, it follows at once that Theorem 2 remains true if the word "star" is replaced by "c-convex star."

The hypothesis concerning f^* at the beginning of Section 2.2 is equivalent to the condition that $f \in \mathscr{P}_F(X_0, X)$; see the proof of Lemma 1 and Ref. 13 (top of p. 75).

2.4 Convergence and the extent of invertibility

In this section we prove a result that shows, in particular, that if g is expandable on A, and if its expansion converges on a larger open set B, then there is a set B_0 that contains the points of A_0 such that f is in fact an invertible map of B_0 onto B.

Theorem 3: Let g belong to $\mathscr{P}(A, X_0)$, and let it have the generalized power-series representation

$$g(x) = \sum_{m=0}^{\infty} g_m(x) \tag{8}$$

for $x \in A$. Suppose that the right side of (8) converges in X_0 for $x \in B$, where B is an open subset of X such that $A \subset B$. Then there is an open subset B_0 of X_0 such that (i) $A_0 \subset B_0$, and f is a homeomorphism of B_0 onto B; and (ii) the inverse G of the restriction of f to B_0 has the representation

$$G(x) = \sum_{m=0}^{\infty} g_m(x), \quad x \in B.$$

Proof: Since $g \in \mathscr{P}(A, X_0)$ and g is continuous, $g \in \mathscr{P}_F(A, X_0)$ (see the proof of Lemma 1). Thus, by Ref. 13, Theorem 6.2, the function $h: B \to X_0$ defined by

$$h(x) = \sum_{m=0}^{\infty} g_m(x), \quad x \in B$$

is analytic in the sense of Ref. 13 and hence continuous. Using Lemma 1, there is an open connected set $W \subset \mathscr{B}$ and a Fréchet-differentiable map $h^*: W \to \mathscr{B}_0$ such that $B \subset W$ and $h(x) = h^*(x + i0)$ for $x \in B$.

By Lemma 2 and the hypothesis that $f[g(x)] = x$ ($x \in A$), we find that

$$f^*[h^*(v)] = v, \quad v \in W. \tag{9}$$

Again using Lemma 2, and proceeding as in the proof of Theorem 1, one finds that

$$h^*[f^*(v)] = v, \quad v \in W_0, \tag{10}$$

where W_0 is the component of $f^{*-1}(W)$ that contains $h^*(W)$. Therefore, f^* is a homeomorphism of W_0 onto W.

We have, from (9), $f[h(x)] = x$ $(x \in B)$. Now let $B_0 = f^{-1}(B) \cap R_0$, where $R_0 = \{x \in X_0 : x + i0 = z, z \in W_0\}$. Since W_0 is open, R_0 is open in X_0. Thus, B_0 is an open subset of X_0, and from (10) one has $h[f(x)] = x$ for $x \in B_0$. This shows that f is a homeomorphism of B_0 onto B, with h the inverse of the restriction of f to B_0. Finally, using $A_0 \subset f^{-1}(A) \subset f^{-1}(B)$, and $A_0 = h(A) \subset h(B) \subset h^*(W) \subset W_0$ (which implies that $A_0 \subset R_0$), as well as the definition of B_0, it is clear that $A_0 \subset B_0$. This proves the theorem.

2.5 Results for implicit functions

Theorems along the lines of Theorems 2 and 3 are given here for maps that are defined implicitly in the sense of the implicit function theorem. In this section, X_1 stands for a third real Banach space, $\mathscr{B}_1$ denotes its complex extension in the sense of Secton 2.1, and $X_0 \times X$ and $\mathscr{B}_0 \times \mathscr{B}$ are product Banach spaces constructed from X_0 and X and $\mathscr{B}_0$ and $\mathscr{B}$, respectively.* We say that a map h defined on an open subset D of $\mathscr{B}$ into $\mathscr{B}_0$ is *Gâteaux differentiable* on D (see Ref. 12, pp. 109–10) if for each $v \in D$ and arbitrary $w \in \mathscr{B}$ the limit $\lim_{z \to 0} z^{-1}[h(v + zw) - h(v)]$ exists, in which z is a complex scalar.

As in Section 2.2, A is an open subset of X, with $0 \in A$. Here F is a map from $X_0 \times X$ into X_1 such that there is a continuous map $G: A \to X_0$ with the property that

$$F[G(x), x] = 0, \quad x \in A.$$

Assume that there is an $F^* \in \mathscr{P}_F(\mathscr{B}_0 \times \mathscr{B}, \mathscr{B}_1)$ such that $F(y, x) = F^*(y + i0, x + i0)$ for $(y, x) \in X_0 \times X$.

Theorem 4: $G \in \mathscr{P}(A, X_0)$ if and only if there is an open star $V \subset \mathscr{B}$, with $A \subset V$, and a continuous Gâteaux-differentiable map $G^*: V \to \mathscr{B}_0$ such that we have $G(x) = G^*(x + i0)$ $(x \in A)$ as well as

$$F^*[G^*(v), v] = 0, \quad v \in V. \tag{11}$$

Proof: First suppose that $G \in \mathscr{P}(A, X_0)$. By Lemma 1 there is an open star $V \subset \mathscr{B}$ and a map $G^* \in \mathscr{P}_F(V, \mathscr{B}_0)$ such that $A \subset V$ and $G(x) =$

* Except where indicated to the contrary, the choice of the norms in $X_0 \times X$ and $\mathscr{B}_0 \times \mathscr{B}$ is not important for our purposes. It would suffice to let the norm $\|\cdot\|$ in $X_0 \times X$ be given by $\|(x_0, x)\| = \max(\|x_0\|, \|x\|)$, and similarly for $\mathscr{B}_0 \times \mathscr{B}$.

$G^*(x + i0)$ for x in A. Since $G^* \in \mathscr{P}_F(V, \mathscr{B}_0)$, G^* is F-differentiable on V and thus continuous and G-differentiable in V.

It follows from a version of the chain rule (Ref. 15, pp. 171–2) that h defined by

$$h(v) = F^*[G^*(v), v], \quad v \in V$$

is F-differentiable on V. [Notice that $h(v) = (F^*Q)(v)$, where Q takes $v \in V$ into the point $(G^*(v), v)$ in $\mathscr{B}_0 \times \mathscr{B}$.] Since V is connected, and $F^*[G^*(\theta + x + i0), x + i0] = F[G(x), x] = 0$ (θ is the zero element of $\mathscr{B}$) for x in some open ball in X centered at the origin, by Lemma 2, one has (11).

Assuming, on the other hand, that we have a V and a G^* as indicated in the theorem, G^* is F-differentiable (see Ref. 12, Theorem 3.17.1) and an obvious modification of the part of the proof of Theorem 2 that concerns H shows that $G \in \mathscr{P}(A, X_0)$. This proves the theorem.

Theorem 5: Assume that $G \in \mathscr{P}(A, X_0)$, and that G has the generalized power-series representation

$$G(x) = \sum_{m=0}^{\infty} G_m(x) \tag{12}$$

for $x \in A$. Suppose that the right side of (12) *converges for $x \in B$, where B is an open subset of X such that $A \subset B$. Then $F[G(x), x] = 0$ $(x \in B)$, where G is defined for all $x \in B$ by* (12).

Proof: Paralleling the beginning of the proof of Theorem 3, there is an open connected set $W \subset \mathscr{B}$ and a Fréchet-differentiable $h^*: W \rightarrow \mathscr{B}_0$ such that $B \subset W$ and $G(x) = h^*(x + i0)(x \in B)$. Using Lemma 2 and the observation in the proof of Theorem 5 concerning the applicability of a version of the chain rule, we have $F^*[h^*(v), v] = 0$ for $v \in W$, which implies that $F[G(x), x] = 0$ $(x \in B)$.

Remarks: Theorem 4 bears directly on problems concerning the existence of generalized power-series expansions for the solutions of nonlinear differential equations, because, as is well known, these equations can frequently be put in the form $F[G(x), x] = 0$, $x \in A$, where x takes into account inputs and/or initial conditions, and $G(x)$ is the corresponding solution. For related earlier work, see Ref. 8 and the references cited therein; the work includes, in particular, a description of the specific type of expansions that arise.

A result similar to Theorem 2 in Section 2.3 can be obtained for equations of the form $H(y, x) = w$, where y is a solution that depends on both x and w.* Specifically, suppose that H is a map from $X_0 \times X$ into X_1 such that $H(y, x) = H^*(y + i0, x + i0)$ for all x and y for some

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* See Ref. 8, p. 75, for an example of how such an equation arises.

$H^* \in \mathscr{P}_F(\mathscr{B}_0 \times \mathscr{B}, \mathscr{B}_1)$. Assume now that the norm $\| \cdot \|$ in $X_0 \times X$ is defined by $\| (x_0, x) \| = \max(\| x_0 \|, \| x \|)$, and similarly for $(X_1 \times X)$, $(\mathscr{B}_0 \times \mathscr{B})$, and $(\mathscr{B}_1 \times \mathscr{B})$. Assume also that $\mathscr{B}_0$ and $\mathscr{B}_1$ are homeomorphic in the sense of (ii) of Theorem 2.

Let A_{wx} and S, respectively, be open subsets of $(X_1 \times X)$ and $(X_0 \times X)$, with $(0, 0) \in A_{wx}$, such that for each $(w, x) \in A_{wx}$ there is a unique $y \in X_0$ for which $(y, x) \in S$ and $H(y, x) = w$. Assume that the map from (w, x) to y is continuous. Define $f:(X_0 \times X) \to (X_1 \times X)$ by $f(y, x) = [H(y, x), x]$ for all y and x, and let A_{yx} denote the open set $S \cap f^{-1}(A_{wx})$. Notice that f restricted to A_{yx} is a homeomorphism of A_{yx} onto A_{wx}. Thus, using Theorem 2 and the observation in the footnote in Appendix B, we see that the map from (w, x) to y described above belongs to $\mathscr{P}(A_{wx}, X_0)$ if and only if $(\mathscr{B}_1 \times \mathscr{B})$ and $(\mathscr{B}_0 \times \mathscr{B})$, respectively, contain open subsets V and V_0 with $A_{wx} \subset V$, $A_{yx} \subset V_0$, V a star, and $[H^*(y^*, x^*), x^*] \in V$ for $(y^*, x^*) \in V_0$, such that for each $(w^*, x^*) \in V$, there is a unique $y^* \in \mathscr{B}_0$ that satisfies $(y^*, x^*) \in V_0$ and $H^*(y^*, x^*) = w^*$, with the map from (w^*, x^*) to y^* locally Lipschitz.

2.6 Construction of the series for g of Section 2.2

Here we return to the setting introduced at the beginning of Section 2.2. Theorem 6 (in this section) provides an algorithm for determining the expansion of g whenever it exists. The theorem is a version of a result in Ref. 1 concerning complex spaces. We shall first prove a proposition that establishes the existence of certain derivatives that play a central role in the theorem.

Proposition: For each $m = 1, 2, \cdots$ the mth order Fréchet derivative (Ref. 14, pp. 179–81) $d^m f[g(0)]$ [of f at the point $g(0) \in X_0$] exists, and one has

$$f[g(0) + x] = f[g(0)] + \sum_{m=1}^{\infty} (m!)^{-1} d^m f[g(0)] x^m, \quad x \in X_0. \tag{13}$$

Proof: Using the hypothesis that $f^* \in \mathscr{P}_F(\mathscr{B}_0, \mathscr{B})$, the Fréchet derivative $df^*(w)$ and hence the derivatives $d^m f^*(w)$ $(m = 2, 3, \cdots)$ exist for $w \in \mathscr{B}_0$, and we have

$$f^*[g^*(0) + w] = f^*[g^*(0)] + \sum_{m=1}^{\infty} (m!)^{-1} d^m f^*[g^*(0)] w^m, \quad w \in \mathscr{B}_0 \tag{14}$$

(see Ref. 12, Theorems 26.6.3 and 3.16.2; Ref. 15, Lemma 3.6.1; Ref. 7, Lemma 2), where $g^*(0) = g(0) + i0$. Since df^* exists on $\mathscr{B}_0$, and the norm in $\mathscr{B}$ has the property that $\| x_1 + ix_2 \| < \delta$ implies that $\| x_1 \| < \delta$ and $\| x_2 \| < \delta$ [see (7)], it is easy to see that df exists on X_0 and that one has $df(a)x = df^*(a + i0)(x + i0)$ for a and x in X_0. A simple inductive argument shows that $d^m f$ exists on X_0, with

$$d^m f(a)x_1 \cdots x_m = d^m f^*(a + i0)(x_1 + i0) \cdots (x_m + i0) \quad (15)$$

for $a \in X_0$, $(x_1, \cdots, x_m) \in X_0^m$, and each m. This proves the proposition, since it is clear that $f^*[g^*(0)] = f[g(0)]$. The relation (13) directs attention to an interpretation of the $d^m f[g(0)]$; it is not used otherwise.

Theorem 6: Let $g \in \mathscr{P}(A, X)$. Then $df[g(0)]$ is a homeomorphism of X_0 onto X, and

$$g(x) = g(0) + \sum_{m=1}^{\infty} g_m(x), \quad x \in A,$$

where the g_m are the homogeneous polynomials defined by

$$g_1(x) = df[g(0)]^{-1}x$$

and

$$g_m(x) = -df[g(0)]^{-1} \sum_{\ell=2}^{m} (\ell!)^{-1} \cdot \sum_{\substack{k_1+\cdots+k_\ell=m \\ k_j>0}} d^\ell f[g(0)] g_{k_1}(x) \cdots g_{k_\ell}(x), \quad m \geq 2.$$

2.6.1 Proof of Theorem 6

The inverse of $df[g(0)]$ exists because f is a homeomorphism of A_0 onto A with f and g, respectively, Fréchet differentiable on A_0 and A (Ref. 15, p. 175, Problem 6). Let

$$g(x) = g(0) + \sum_{m=1}^{\infty} g_m(x), \quad x \in A,$$

in which each g_m is a homogeneous polynomial on X of degree m.

With g^* and V associates of g and A, respectively, in accordance with Theorem 1, we have $g^* \in \mathscr{P}_F(V, \mathscr{B}_0)$ and $v = f^*[g^*(v)]$ for $v \in V$. Let $g_1^*, g_2^*, \cdots$ be continuous homogeneous polynomials such that

$$g^*(v) = g^*(0) + \sum_{m=1}^{\infty} g_m^*(v) \quad (16)$$

for $v \in V$. By the part of the proof of Theorem 2 concerning H, $g_m(x) = g_m^*(x + i0)$ for each m and each $x \in X$. Using (14) and $f^*[g^*(0)] = 0$,

$$v = f^*[g^*(v)] = \sum_{\ell=1}^{\infty} (\ell!)^{-1} d^\ell f^*[g^*(0)] \left(\sum_{k_1=1}^{\infty} g_{k_1}^*(v) \cdots \sum_{k_\ell=1}^{\infty} g_{k_\ell}^*(v) \right), \quad v \in V.$$

Since $g^* \in \mathscr{P}_F(V, \mathscr{B}_0)$, there is a $\sigma > 0$ such that the right side of (16) is *absolutely* convergent for $\| v \| < \sigma$ (Ref. 12, Theorem 26.6.6). Thus, using the boundedness of the $d^\ell f^*[g^*(0)]$ and an easily proved generalization of Theorem 5.5.3 of Ref. 14,

$$v = \sum_{\ell=1}^{\infty} (\ell!)^{-1} \sum_{k_1,\cdots,k_\ell=1}^{\infty} d^\ell f^*[g^*(0)](g^*_{k_1}(v) \cdots g^*_{k_\ell}(v)) \tag{17}$$

for $\| v \| < \sigma$, in which the sum over $(k_1, \cdots, k_\ell)$ is absolutely convergent.

At this point we use the proposition that there are positive constants M, β, K, and α such that

$$\| d^\ell f^*[g^*(0)]w_1 \cdots w_\ell \| \leq \ell^\ell M\beta^\ell \| w_1 \| \cdots \| w_\ell \|$$

and

$$\| g^*_k(v) \| \leq K(\alpha \| v \|)^k$$

for $\ell \geq 1$, $k \geq 1$, $v \in \mathscr{B}$, and $w_1, \cdots, w_\ell$ in $\mathscr{B}_0$ (see Ref. 7, Appendix A). It is clear that

$$\| d^\ell f^*[g^*(0)][g^*_{k_1}(v) \cdots g^*_{k_\ell}(v)] \| \leq \ell^\ell M(\beta K)^\ell (\alpha \| v \|)^{(k_1+\cdots k_\ell)}.$$

Consider the sum

$$\sum_{\ell=1}^{\infty} \sum_{k_1,\cdots,k_\ell=1}^{\infty} (\ell!)^{-1} \ell^\ell M(\beta K)^\ell (\alpha \| v \|)^{(k_1+\cdots+k_\ell)}. \tag{18}$$

Notice that

$$\sum_{k_1,\cdots,k_\ell=1}^{\infty} (\alpha \| v \|)^{(k_1+\cdots+k_\ell)} = [\alpha \| v \| (1 - \alpha \| v \|)^{-1}]^\ell \tag{19}$$

for $\alpha \| v \| < 1$. Using (19) and Stirling's formula for $n!$, which gives $n! > (2\pi)^{1/2} n^{1/2} n^n e^{-n}$, it easily follows that the sum (18) converges for $\| v \|$ sufficiently small. Thus (see Ref. 14, Theorem 5.3.4) for such v the sum in (17) over $(\ell, k_1, \cdots, k_\ell)$, which equals

$$\sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{\substack{k_1+\cdots+k_\ell=m \\ k_j>0}} (\ell!)^{-1} d^\ell f^*[g^*(0)](g^*_{k_1}(v) \cdots g^*_{k_\ell}(v)), \tag{20}$$

can be written as (see Ref. 14, Theorem 5.3.6)

$$\sum_{m=1}^{\infty} \sum_{\ell=1}^{m} \sum_{\substack{k_1+\cdots+k_\ell=m \\ k_j>0}} (\ell!)^{-1} d^\ell f^*[g^*(0)](g^*_{k_1}(v) \cdots g^*_{k_\ell}(v)). \tag{21}$$

By the uniqueness result for generalized power series mentioned in Section 2.1, and the fact that (21) equals v for v of sufficiently small norm,

$$df^*[g^*(0)]g^*_1(v) = v$$

and

$$df^*[g^*(0)]g_m^*(v)$$

$$= -\sum_{\ell=2}^{m} \sum_{\substack{k_1+\cdots+k_\ell=m \\ k_j>0}} (\ell!)^{-1} d^\ell f^*[g^*(0)][g_{k_1}^*(v) \cdots g_{k_\ell}^*(v)] \quad (m \geq 2)$$

for $\nu \in \mathscr{B}$. This, with $v = x + i0$ and (15), completes the proof.

2.6.2 Comments

For the case in which the expansion (14) for f^* has only a finite number of terms, the proof of Theorem 6 simplifies considerably, because then the equivalence of (20) and (21) is a consequence of just the absolute convergence of the sum over $(k_1, \cdots, k_\ell)$ in (17). A related result for this case is given in Ref. 17, p. 29.

For a different approach to the problem of determining the expansion of g, see Ref. 18.

The proof of Theorem 6 provides an alternative proof of Theorem 2 of Ref. 1, which is an analogous result concerning only complex spaces. It also yields a proof of the following "substitution theorem" (an earlier version proved in a different way appears in Ref. 19).*

Theorem 7: Take W_1, W_2, and W_3 to be three complex Banach spaces, and let S_1 and S_2 be nonempty open subsets of W_1 and W_2, respectively, with S_1 a star. Let $G \in \mathscr{P}_F(S_1, W_2)$ and let F be a Fréchet-differentiable map of S_2 into W_3. Assume that $G(S_1) \subset S_2$. Then $(FG)(\cdot) \in \mathscr{P}_F(S_1, W_3)$, the Fréchet derivatives $d^mG(0)$ and $d^mF[G(0)]$ exist for $m \geq 1$, and we have

$$(FG)(v) = F[G(0)] + \sum_{m=1}^{\infty} H_m(v), \quad v \in S_1, \tag{22}$$

where the H_m are the homogeneous polynomials given by

$$H_m(v) = \sum_{\ell=1}^{m} (\ell!)^{-1} \sum_{\substack{k_1+\cdots+k_\ell=m \\ k_j>0}} d^\ell F[G(0)](k_1!)^{-1} d^{k_1}G(0)v^{k_1} \cdots (k_\ell!)^{-1} d^{k_\ell}G(0)v^{k_\ell}.$$

Theorem 7 and Lemma 1 can be used to obtain results along the lines of Theorem 7 for cases in which the spaces of interest are real. This is discussed briefly in Appendix A.

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* Concerning the proof of Theorem 7, $(FG)(\cdot) \in \mathscr{P}_F(S_1, W_3)$ because G and hence $(FG)(\cdot)$ are Fréchet differentiable on the star S_1. Also, since F is Fréchet differentiable in a neighborhood of $G(0)$, F has a locally convergent expansion about that point.

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APPENDIX A

Substitution Results for Real Spaces

This appendix presents two useful corollaries of Theorem 7. Proofs are omitted because the corollaries can be proved using direct modifications of material already discussed.

Corollary 1: Assume that W_1, W_2, and W_3 are real Banach spaces, that S_1 and S_2 are open subsets of W_1 and W_2, respectively, and that $0 \in S_1$. Let G be a map of S_1 into W_2 such that the Fréchet derivative $d^mG(0)$ exists for $m = 1, 2, \cdots$, and G has the representation

$$Gx = G(0) + \sum_{m=1}^{\infty} (m!)^{-1} d^m G(0) x^m, \quad x \in S_1.$$

Suppose that $G(S_1) \subset S_2$. Let F map S_2 into W_3 such that $d^mF[G(0)]$ exists for each $m > 0$, and

$$F[y + G(0)] = F[G(0)] + \sum_{m=1}^{M} (m!)^{-1} d^m F[G(0)] y^m$$

for $y + G(0) \in S_2$, *where* M *is a positive integer (and thus* F *is assumed to be a polynomial). Then* $(FG)(\cdot) \in \mathscr{P}(S_1, W_3)$ *and (22) holds.*

Corollary 2: Suppose that W_1, W_2, *and* W_3 *are three real Banach spaces. Let* $G \in \mathscr{P}_F(S_1, W_2)$ *for some open subset* S_1 *of* W_1 *containing the point 0, and let* $F \in \mathscr{P}_F(S_2, W_3)$, *where* S_2 *is an open subset of* W_2 *containing* $G(0)$. *Then the Fréchet derivatives* $d^m G(0)$ *and* $d^m F[G(0)]$ *exist for* $m \geqslant 1$, *and there is an open subset* T_1 *of* S_1, *containing 0, such that* $G(T_1) \subset S_2$, $(FG)(\cdot) \in \mathscr{P}_F(T_1, W_3)$, *and one has (22), with* S_1 *replaced with* T_1.

APPENDIX B

A Comparison of Norms on Complex Spaces

Consider the Banach space $\mathscr{B}$ described in Section 2.1, and let $\mathscr{C}$ denote a Banach space consisting of the same set of points with a possibly different norm $\|\cdot\|_{\mathscr{C}}$.

Proposition: Let the norm $\|\cdot\|_{\mathscr{C}}$ *have the property that* $\|x_1\| \leqslant \|x_1 + ix_2\|_{\mathscr{C}}$ *for* $(x_1 + ix_2) \in \mathscr{C}$, *in which* $\|x\|$ *is the* X *norm of* x. *Then* $\|x_1 + ix_2\| \leqslant \|x_1 + ix_2\|_{\mathscr{C}}$ *for* $(x_1 + ix_2) \in \mathscr{B}$.

Proof: Assume, for the purpose of obtaining a contradiction, that $\|x_1 + ix_2\| > \|x_1 + ix_2\|_{\mathscr{C}}$ for some $(x_1 + ix_2)$. Then there is a ξ with $\|\xi\| = 1$ such that

$$\xi(x_1)^2 + \xi(x_2)^2 > \|x_1 + ix_2\|_{\mathscr{C}}^2.$$

For this ξ, choose real α and β so that not both are zero and

$$\alpha\xi(x_2) + \beta\xi(x_1) = 0.$$

Using $(\alpha^2 + \beta^2)[\xi(x_1)^2 + \xi(x_2)^2] > (\alpha^2 + \beta^2)\|x_1 + ix_2\|_{\mathscr{C}}^2$ and the observation that $[\xi(ax - by)]^2 + [\xi(bx + ay)]^2 = (a^2 + b^2)[\xi(x)^2 + \xi(y)^2]$ for real a and b, and x and y in X, one has

$$|\xi(\alpha x_1 - \beta x_2)| > \|(\alpha x_1 - \beta x_2) + i(\beta x_1 + \alpha x_2)\|_{\mathscr{C}}.$$

Since the left side is at most $\|\alpha x_1 - \beta x_2\|$, we have a contradiction.

Comments: For X the space of bounded n-vector-valued functions described in Section 2.1, and $\mathscr{C}$ the corresponding complex Banach space with $\|v\|_{\mathscr{C}} = \max_j \sup_t |v_j(t)|$, the equality $\|\cdot\| = \|\cdot\|_{\mathscr{C}}$ holds, where $\|\cdot\|$ is the norm in $\mathscr{B}$. Indeed, $\|\cdot\| \leqslant \|\cdot\|_{\mathscr{C}}$ by the proposition, while with arbitrary $t \geqslant 0$ and $j \in \{1, \cdots, n\}$,

$$\xi(x_1)^2 + \xi(x_2)^2 = [x_{1j}(t)]^2 + [x_{2j}(t)]^2$$

for ξ the linear functional of unit norm on X defined by $\xi(x) = x_j(t)$, showing that $\| \cdot \| \geqslant \| \cdot \|_{\mathscr{G}}$.*

However, we have $\| \cdot \| \leqslant \| \cdot \|_{\mathscr{G}}$ but *not* $\| \cdot \| = \| \cdot \|_{\mathscr{G}}$ whenever X is a Hilbert space, $\| x_1 + ix_2 \|_{\mathscr{G}} = (\| x_1 \|^2 + \| x_2 \|^2)^{1/2}$, and X is typical in the sense that it has a pair of nonzero elements x and y that are orthogonal. This follows from the inner-product representation of linear functionals in a Hilbert space, and the fact that for X, x, and y, as indicated above, one can show that

$$\sup_{\|v\|=1} \frac{(v, x)^2 + (v, y)^2}{\| x \|^2 + \| y \|^2} < 1,$$

where $(\cdot, \cdot)$ is the inner product in X.

Finally, we mention that the norm in $\mathscr{B}$ *cannot* be replaced with

$$\| (x_1, x_2) \| = (\| x_1 \|^2 + \| x_2 \|^2)^{1/2}, \tag{23}$$

because (23) does not define a norm in $\mathscr{B}$ unless X is a Hilbert space (see Ref. 20). It is not difficult to see that (23) does not suffice: If it did, we would have $\| ax - by \|^2 + \| bx + ay \|^2 = (a^2 + b^2)(\| x \|^2 + \| y \|^2)$ for any real numbers a and b, and arbitrary elements x and y of X. This would give $\| x - y \|^2 + \| x + y \|^2 = 2(\| x \|^2 + \| y \|^2)$, which is not valid unless X is a Hilbert space (see Ref. 21, p. 211).

AUTHOR

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* This type of argument also shows that the norm $\| (a_0, a) + i(b_0, b) \|$ of an arbitrary element $(a_0, a) + i(b_0, b)$ of the complex extension of the Banach space $X_0 \times X$ with norm $\max(\| \cdot \|, \| \cdot \|)$ is $\max(\| a_0 + ib_0 \|, \| a + ib \|)$, where $\| a_0 + ib_0 \|$ and $\| a + ib \|$ are the $\mathscr{B}_0$ and $\mathscr{B}$ norms of $(a_0 + ib_0)$ and $(a + ib)$, respectively.