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Bandwidth-Error Exchange for a Simple Fading Channel Model

By B. F. LOGAN, JR.*

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It is assumed that a data carrier signal is transmitted over a fading channel whose frequency response can be closely approximated over the transmission band by a first-order polynomial in frequency, the coefficients being slowly varying functions of time. An equivalent baseband model is obtained wherein the transmitted signal is of the form $s(t) = \sum_{-\infty}^{\infty} a_k f(t - kT)$, and the received signal is of the form $r(t) = s(t) + x(t)s'(t)$, where $x(t)$ is an unknown (e.g., random) function of time. The problem solved in this paper is that of finding the function f of prescribed bandwidth that minimizes the mean-square error, $E\{|r(nT) - a_n|^2\}$, under the assumption that the a_k are independent random variables of zero mean and unit variance, and $E\{|x(t)|^2\} = \alpha^2$. The results also apply to the sometimes more realistic hypothesis, $|x(t)| \leq \alpha$.

I. INTRODUCTION

A common method of transmitting data $\{a_k\}$ and $\{b_k\}$ is via a carrier signal of the form

$$s_c(t) = s_1(t) \cos \omega_c t + s_2(t) \sin \omega_c t, \tag{1}$$

where

$$s_1(t) = \sum_{-\infty}^{\infty} a_k f(t - kT), \qquad s_2(t) = \sum_{-\infty}^{\infty} b_k f(t - kT). \tag{2}$$

In the usual mathematical model, $f(t) = (\sin \Omega t)/\Omega t$, $\Omega = \pi/T$, so that

* AT&T Bell Laboratories.

$$s_1(nT) = a_n; \quad s_2(nT) = b_n.$$

Assuming then that $s_c(t)$ is transmitted over an ideal noiseless channel, $s_1(t)$ and $s_2(t)$ can be recovered by synchronous demodulation of the received signal, and then the data obtained by sampling s_1 and s_2 at the times nT.

A problem, communicated by G. Foschini, arises when $s_c(t)$ is transmitted over a channel having a relatively slow-varying fading characteristic. L. J. Greenstein and B. A. Czekaj have found that, in many cases, the fading channel response can be fairly well approximated over the transmission band by a first-order polynomial in frequency ω,

$$A_0(t) + A_1(t)\{\omega - \omega_c\} + B_1(t)i\{\omega - \omega_c\},$$

with slowly varying real coefficients.[1] Then, to a good approximation, synchronous demodulation of the received signal gives, instead of $s_1(t)$,

$$r_1(t) = A_0(t)s_1(t) + B_1(t)s_1'(t) + A_1(t)s_2'(t),$$

and a similar expression for the alteration of $s_2(t)$.

We assume that $A_0(t)$, the center-frequency channel gain, is positive and can be determined at the receiver (e.g., by measuring the average power in a narrow band about the center frequency), so that, by the use of automatic gain control we have available,

$$r_1^*(t) = s_1(t) + x_1(t)s_1'(t) + x_2(t)s_2'(t), \tag{3}$$

where $x_1(t)$ and $x_2(t)$ are unknown, for example, random, slowly varying functions of time. Then

$$r_1^*(nT) = a_n + x_1(nT)s_1'(nT) + x_2(nT)s_2'(nT). \tag{4}$$

It has been suggested,[2] as an alternative to using adaptive channel compensation, that error-free reception may be obtained by doubling the bandwidth of $f(t)$ in (2), i.e., by taking $f(t) = (\sin \Omega t)^2/(\Omega t)^2$, $\Omega = \pi/T$, so that $s_1'(nT) = s_2'(nT) = 0$. Here we want to determine the best attainable trade-off between mean-square error and bandwidth under certain assumptions on $\{a_k\}$, $\{b_k\}$, $x_1(t)$, and $x_2(t)$.

We have, from (2),

$$r_1^*(nT) - a_n = a_n\{f(0) - 1\} + \sum_{k\neq n} a_k f(nT - kT)$$

$$+ x_1(nT) \sum_{-\infty}^{\infty} a_k f'(nT - kT) + x_2(nT) \sum_{-\infty}^{\infty} b_k f'(nT - kT). \tag{5}$$

Now we assume that the a_k and b_k are independent random variables of zero mean and unit variance. Then the expected mean-square error over $\{a_k\}$ and $\{b_k\}$ is

$$E\{|r_1^*(nT) - a_n|^2\} = |1 - f(0)|^2 + \sum_{k\neq 0} |f(kT)|^2$$

$$+ \{x_1^2(nT) + x_2^2(nT)\} \sum_{-\infty}^{\infty} |f'(kT)|^2 + 2x_1(nT) \sum_{k\neq 0} f'(kT)f(kT)$$

$$+ 2x_1(nT)[f(0) - 1]f'(0). \quad (6)$$

[Note that the cross product $x_1(nT)x_2(nT)$ does not enter here because of the assumptions on $\{a_k\}$ and $\{b_k\}$.]

Now let us suppose that $x_1(t)$ and $x_2(t)$ are random (continuous) functions satisfying

$$E\{|x_1(t)|^2\} = \alpha_1^2, \qquad E\{x_1(t)\} = 0, \quad (7a)$$

$$E\{|x_2(t)|^2\} = \alpha_2^2. \quad (7b)$$

Then we have, for the expected, or mean-square error, ϵ^2,

$$\epsilon^2 = |1 - f(0)|^2 + \sum_{k\neq 0} |f(kT)|^2 + \alpha^2 \sum_{-\infty}^{\infty} |f'(kT)|^2, \quad (8)$$

where $\alpha^2 = \alpha_1^2 + \alpha_2^2$.

As an alternative to the statistical assumptions, (7a) and (7b), let us suppose that

$$|x_1(t)| \leq \alpha_1, \quad (9a)$$

$$|x_2(t)| \leq \alpha_2. \quad (9b)$$

These assumptions may be more relevant in practice than the statistical assumptions. Note that if the coefficient of $x_1(nT)$ in (6) vanishes, which will be the case if $f(t)$ is even; then, under the assumptions (9a) and (9b), (8) will hold with the equality sign replaced by $\leq$. We wish to minimize the quantity on the right in (8) over bandlimited functions f of prescribed bandwidth. As we shall see, the minimum will be obtained for a real-valued even function, so that, indeed, the minimum will be an upper bound for ϵ^2 under the assumptions (9a) and (9b).

Note that, in the end, the quantity to be minimized depends only on how large $|x_1(t)|$ and $|x_2(t)|$ may be, slowly varying or not. The slowly varying hypothesis was used only to obtain (3) from the fading channel model. The same minimization problem is obtained, under the previous assumption on $\{a_k\}$, if

$$s(t) = \sum_{-\infty}^{\infty} a_k f(t - kT) \quad (10)$$

is transmitted over a channel such that the received signal is simply

$$r(t) = s(t) + x(t)s'(t), \quad (11)$$

where $x(t)$ is an unknown (continuous) function satisfying, if $x(t)$ is random,

$$E\{|x(t)|^2\} = \alpha^2, \qquad E\{x(t)\} = 0, \tag{11a}$$

or (say) if not, then

$$|x(t)| \leq \alpha. \tag{11b}$$

It is convenient, and sufficient, to consider the case $T = 1$ in (8), so that the quantity of interest is

$$\epsilon^2 = |1 - f(0)|^2 + \sum_{k\neq 0} f(k)|^2 + \alpha^2 \sum_{-\infty}^{\infty} |f'(k)|^2. \tag{12}$$

This is to be minimized over functions f in $B_2(\Omega)$; i.e., functions in L_2 of the form

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega)e^{i\omega t}d\omega. \tag{13}$$

We need only consider the case $0 < \Omega \leq 2\pi$, since for $\Omega \geq 2\pi$ we can make $\epsilon^2 = 0$ by taking $f(t) = (\sin \pi t)^2/(\pi t)^2$. The transmission system, of course, is useless if $\epsilon^2 \geq 1$, but we can always make $\epsilon^2 < 1$ by appropriate choice (or scaling) of f, ($f \equiv 0$ giving $\epsilon^2 = 1$).

In general, we will not have $f(0) = 1$ for the optimal f. This is easily seen by defining the governing quantity in the problem, viz.,

$$\mu(\Omega;\, \alpha) = \inf_{\substack{f\in B_2(\Omega)\\ f(0)=1}} \left\{ \sum_{k\neq 0} |f(k)|^2 + \alpha^2 \sum_{-\infty}^{\infty} |f'(k)|^2 \right\}. \tag{14}$$

It follows from this definition that

$$\sum_{k\neq 0} |f(k)|^2 + \alpha^2 \sum_{-\infty}^{\infty} |f'(k)|^2 \geq \mu(\Omega;\, \alpha)|f(0)|^2, \qquad f \text{ in } B_2(\Omega). \tag{15}$$

Thus from (12) and (15) we have

$$\epsilon^2 \geq |1 - f(0)|^2 + |f(0)|^2\mu(\Omega;\, \alpha), \qquad f \text{ in } B_2(\Omega). \tag{16}$$

The quantity on the right is minimized by taking

$$f(0) = \gamma = \gamma(\Omega;\, \alpha) = \{1 + \mu(\Omega,\, \alpha)\}^{-1}, \tag{17}$$

giving

$$\epsilon^2 \geq \frac{\mu(\Omega;\, \alpha)}{1 + \mu(\Omega;\, \alpha)}. \tag{18}$$

Thus if the infimum in (14) is attained for $f = f(t;\, \Omega,\, \alpha)$, then equality will hold in (18) for the optimal function

$$f_0(t;\, \Omega,\, \alpha) = \gamma f(t;\, \Omega,\, \alpha). \tag{19}$$

In cases of practical interest, μ will be small, and hence $f_0(0)$ will be slightly less than 1.

II. RESULTS

The results are summarized in the following:

Theorem: Define for $\alpha > 0$, $\Omega > 0$,

$$\rho(\Omega;\ \alpha) = \frac{\pi}{\Omega} \cdot \frac{\alpha\Omega}{\arctan(\alpha\Omega)}, \tag{20}$$

where $\arctan(\cdot)$ *is between* 0 *and* $\pi/2$. *Then in (14) we have, for* $0 < \Omega \leqslant \pi$,

$$\mu(\Omega;\ \alpha) = \rho(\Omega;\ \alpha) - 1, \tag{21}$$

and for $\pi < \Omega \leqslant 2\pi$,

$$\mu(\Omega;\ \alpha) = \left\{(1-\beta) + \beta \cdot \frac{\arctan(\alpha\beta\pi)}{\alpha\beta\pi}\right\}^{-1} - 1, \tag{22}$$

where

$$\beta = 2 - (\Omega/\pi). \tag{22a}$$

For $\Omega \geqslant 2\pi$,

$$\mu(\Omega;\ \alpha) = 0. \tag{23}$$

Furthermore, for $0 < \Omega \leqslant \pi$, *the infimum in (19) is attained only for*

$$f(t) = f(t;\ \Omega,\ \alpha) = \rho(\Omega;\ \alpha)\ \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{\cos \omega t}{1 + \alpha^2\omega^2}\, d\omega \tag{24}$$

and for $\pi < \Omega \leqslant 2\pi$, $\alpha > 0$, *only for*

$$f(t) = (1-\lambda) f(t;\ \beta\pi,\ \alpha) + \lambda\left\{\phi(t;\ \Omega)\cos \pi t - \frac{1}{\pi}\,\phi'(t;\ \Omega)\sin \pi t\right\}, \tag{25}$$

where $f(t;\ \cdot,\ \alpha)$ *is defined in (24),* β *is defined in (22a), and*

$$\lambda = \frac{1-\beta}{(1-\beta) + \beta \cdot \dfrac{\arctan(\alpha\beta\pi)}{\alpha\beta\pi}}, \tag{25a}$$

$$\phi(t;\ \Omega) = \frac{\sin(\Omega - \pi)t}{(\Omega - \pi)t}. \tag{25b}$$

III. DISCUSSION

In the simple fading channel model we have fixed the sampling interval T to be unity, so that $\Omega = \pi$ corresponds to Nyquist-rate transmission, and solved the minimization problem for $0 < \Omega \leqslant 2\pi$. The case $0 < \Omega < \pi$ might be considered uninteresting, for in this case the bandwidth is too small for the data rate. However, the solution for this "uninteresting" case enters in the solution for the interesting case, $\pi < \Omega < 2\pi$.

From the results stated in the Theorem, the optimal function in $B_2(\Omega)$ for minimizing the mean-square error is found to be

$$f_0(t;\,\Omega,\,\alpha) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \frac{\cos\omega t}{1+\alpha^2\omega^2}\,d\omega, \qquad 0 < \Omega < \pi, \tag{26}$$

$$f_0(t;\,\Omega,\,\alpha) = f_0(t;\,2\pi - \Omega,\,\alpha) + h_0(t;\,\Omega), \qquad \pi < \Omega \leqslant 2\pi, \tag{27}$$

where

$$h_0(t;\,\Omega) = \left\{\frac{\sin(\Omega-\pi)t}{\pi t}\right\}\cos\pi t - \frac{1}{\pi}\left\{\frac{d}{dt}\,\frac{\sin(\Omega-\pi)t}{\pi t}\right\}\sin\pi t$$

$$= \frac{1}{2\pi}\int_{\Omega_1<|\omega|<\Omega}\left(1 - \frac{|\omega|}{2\pi}\right)\cos\omega t\,d\omega, \ (\Omega_1 = 2\pi - \Omega). \tag{28}$$

The resulting minimum mean-square error is

$$\epsilon_0^2(\Omega;\,\alpha) = 1 - \frac{\Omega}{\pi}\,\frac{\arctan(\alpha\Omega)}{\alpha\Omega}, \qquad 0 < \Omega \leqslant \pi, \tag{29}$$

$$\epsilon_0^2(\Omega;\,\alpha) = \left(2 - \frac{\Omega}{\pi}\right)\left\{1 - \frac{\arctan[\alpha(2\pi-\Omega)]}{\alpha(2\pi-\Omega)}\right\}, \qquad \pi < \Omega \leqslant 2\pi, \tag{30}$$

where $0 \leqslant \arctan(\cdot) < \pi/2$.

The main interest attaches to this quantity for $\pi \leqslant \Omega \leqslant 2\pi$ and α small, in which case

$$\epsilon_0^2(\Omega;\,\alpha) \doteq \frac{\alpha^2\pi^2}{3}\,\{2 - (\Omega/\pi)\}^3. \tag{31}$$

So there is an interesting trade-off between error and bandwidth in this case.

In the other direction we have

$$\lim_{\alpha\to\infty} \epsilon_0^2(\Omega;\,\alpha) = 2 - (\Omega/\pi), \qquad \pi \leqslant \Omega \leqslant 2\pi. \tag{32}$$

Thus, if α is very large, Ω must be very near 2π in order for the error to be small; i.e., one may as well take $\Omega = 2\pi$ for very large α. However, other practical considerations enter in the case of large α; e.g., the necessities of very accurate sampling and very close approximation of the extremal function. Also, additive noise, which has been neglected in this analysis, will be magnified by automatic gain control during periods of deep fading. So the results are deemed of no practical value in the case of large α.

Notice that the extremal function for the case $\pi < \Omega < 2\pi$ has two distinct components, the low-frequency component being the extremal function for the frequency $(2\pi - \Omega)$. The other component, $h_0(t; \Omega)$ is a bandpass function that does not depend on α, since as may be seen from the first line in (28), its derivative vanishes at the integers. From the second line in (28), it is seen that $h_0(t; \Omega)$ is a bandpass version (center frequency π, upper frequency Ω) of $(\sin \pi t)^2/(\pi t)^2$, the extremal function for $\Omega = 2\pi$. Graphs of $F_0(\omega; \Omega, \alpha)$, the Fourier transform of $f_0(t; \Omega, \alpha)$, are shown in Fig. 1 for several values of Ω and α^2.

It is doubtful that detailed statistics on fading channels, even if available, would be useful in practice, except to the extent that they could give a rough idea of what might be expected. Generally speaking, large errors cannot be tolerated over long intervals (negating, to some

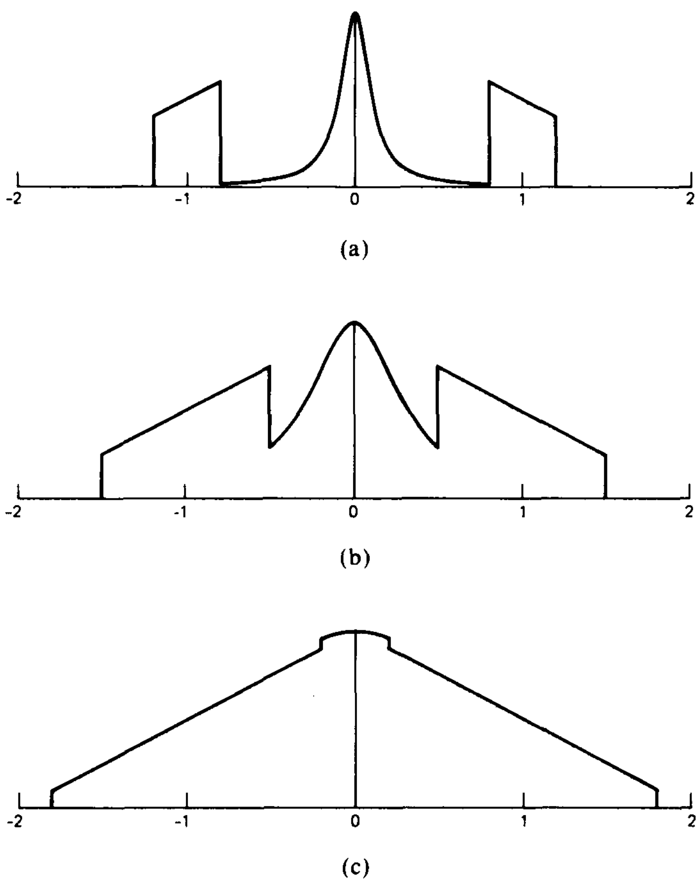

Fig. 1—The Fourier transform $F_0(\omega/\pi; \Omega, \alpha)$ of the extremal function for various values of Ω and α. (a) $\Omega = 1.2\pi$, $\alpha^2 = 10$. (b) $\Omega = 1.5\pi$, $\alpha^2 = 1$. (c) $\Omega = 1.8\pi$, $\alpha^2 = 0.1$.

extent, the adoption of the mean-square error criterion), and one is faced with the dilemma of uncertainty in using such channels. In the model here, the "expected value of α^2" might be regarded from a pragmatical viewpoint as a mathematical non sequitur which might better be replaced by a design hypothesis, $|x(t)| \leq \alpha$. Then the use of $f_0(t;\ \Omega,\ \alpha)$ guarantees that the mean-square error will not exceed $\epsilon_0^2(\Omega;\ \alpha)$ if the hypothesis is true. The design problem is much like that of deciding how much insurance to buy.

On the other hand, one might adopt a hedging strategy, where extra bandwidth is used to *guard* against fading, while zero error is obtained in the absence of fading by using a truly interpolatory f. That is, in the problem of minimizing

$$\sum_{k\neq 0} |f(k)|^2 + \alpha^2 \sum_{-\infty}^{\infty} |f'(k)|^2 \tag{33}$$

over f in $B_2(\Omega)$, $\pi < \Omega \leq 2\pi$, $f(0) = 1$, one decides to make the first sum zero and minimize the second sum under the constraints. The extremal function for this problem, then, depends only on Ω, and is found to be†

$$f_1(t;\ \Omega) = \frac{\sin \pi t}{\pi t}\left\{\frac{\sin(\Omega - \pi)t}{\pi t} + \left(2 - \frac{\Omega}{\pi}\right)\cos(\Omega - \pi)t\right\},$$

$$(\pi < \Omega \leq 2\pi), \tag{34}$$

the minimum value of the second sum in (33) being, under the constraints,

$$\sum_{-\infty}^{\infty} |f_1'(k;\ \Omega)|^2 = \frac{\pi^2}{3}\left(2 - \frac{\Omega}{\pi}\right)^3, \qquad (\pi < \Omega \leq 2\pi). \tag{35}$$

The Fourier transform of f_1 is

$$\begin{aligned} F_1(\omega;\ \Omega) &= 1 \text{ for } |\omega| < 2\pi - \Omega, \\ &= 1 - \frac{|\omega|}{2\pi} \text{ for } 2\pi - \Omega < |\omega| < \Omega, \\ &= 0 \text{ for } |\omega| > \Omega. \end{aligned} \tag{36}$$

The graph of $F_1(\omega;\ \Omega)$ (see Fig. 2) suggests a log cabin; so it will be called the log-cabin characteristic, and $f_1(t;\ \Omega)$ will be called the log-cabin kernel.

One may choose an optimum scaling of the log-cabin kernel as a substitute for $f_0(t;\ \Omega,\ \alpha)$ in the original problem of minimizing, over f in $B_2(\Omega)$, $\pi < \Omega \leq 2\pi$,

† It is no surprise that $f_1(t;\ \Omega) = \lim_{\alpha\to 0} f_0(t;\ \Omega,\ \alpha)$.

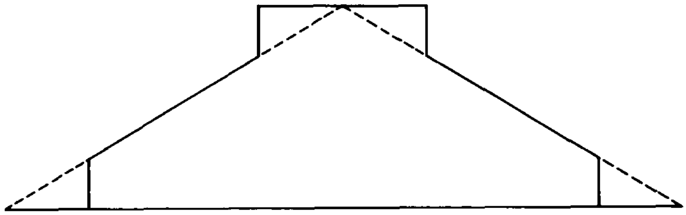

Fig. 2—The log-cabin characteristic.

$$\epsilon^2(f) = |1 - f(0)|^2 + \sum_{k\neq 0} |f(k)|^2 + \alpha^2 \sum_{-\infty}^{\infty} |f'(k)|^2. \qquad (37)$$

The resulting function is

$$f_{10}(t;\, \Omega,\, \alpha) = \gamma f_1(t;\, \Omega), \qquad (38)$$

where

$$\gamma = \left[1 + \frac{\alpha^2\pi^2}{3}\left(2 - \frac{\Omega}{\pi}\right)^3\right]^{-1}.$$

(In case α is small, the optimum scaling may be ignored, as it only amounts to a second-order correction.) The optimally scaled log-cabin kernel gives for the mean-square error in the original problem,

$$\epsilon_{10}^2(\Omega;\, \alpha) = \frac{\dfrac{\alpha^2\pi^2}{3}\left(2 - \dfrac{\Omega}{\pi}\right)^3}{1 + \dfrac{\alpha^2\pi^2}{3}\left(2 - \dfrac{\Omega}{\pi}\right)^3}, \qquad \pi < \Omega \leq 2\pi. \qquad (39)$$

In Fig. 3, 10 $\log_{10}\{\epsilon_0^2(\Omega;\, \alpha)\}$ and 10 $\log_{10}\{\epsilon_{10}^2(\Omega;\, \alpha)\}$ are plotted, for various values of α^2, versus the normalized angular frequency, Ω/π, which corresponds to $2WT$ for a top frequency W and a sampling interval T. To evaluate the effectiveness of $f_0(t;\, \Omega,\, \alpha)$, some reference value of mean-square error must be adopted. It seems natural that $\epsilon_0^2(\Omega;\, \alpha)$ should be compared with $\epsilon_{10}^2(\pi;\, \alpha)$, the minimum mean-square error obtainable (at Nyquist rate) with an optimal scaling of $(\sin \pi t)/\pi t$. Thus, for $\alpha^2 = 1$, a 50-percent increase in bandwidth gives an improvement of approximately 6.3 dB. For small values of α^2, the improvement in decibels is approximately $-30 \log_{10}[2 - (\Omega/\pi)]$, which for $\Omega = 1.5\pi$ is approximately 9 dB; i.e., the mean-square error is reduced by a factor of 8 for a 50-percent increase in bandwidth. The difference between $\epsilon_0^2(\Omega;\, \alpha)$ and $\epsilon_{10}^2(\Omega;\, \alpha)$ is insignificant for moderate to small values of α^2. Of course, the extremely small errors given for Ω near 2π have to be discounted in practice as being purely theoretical values.

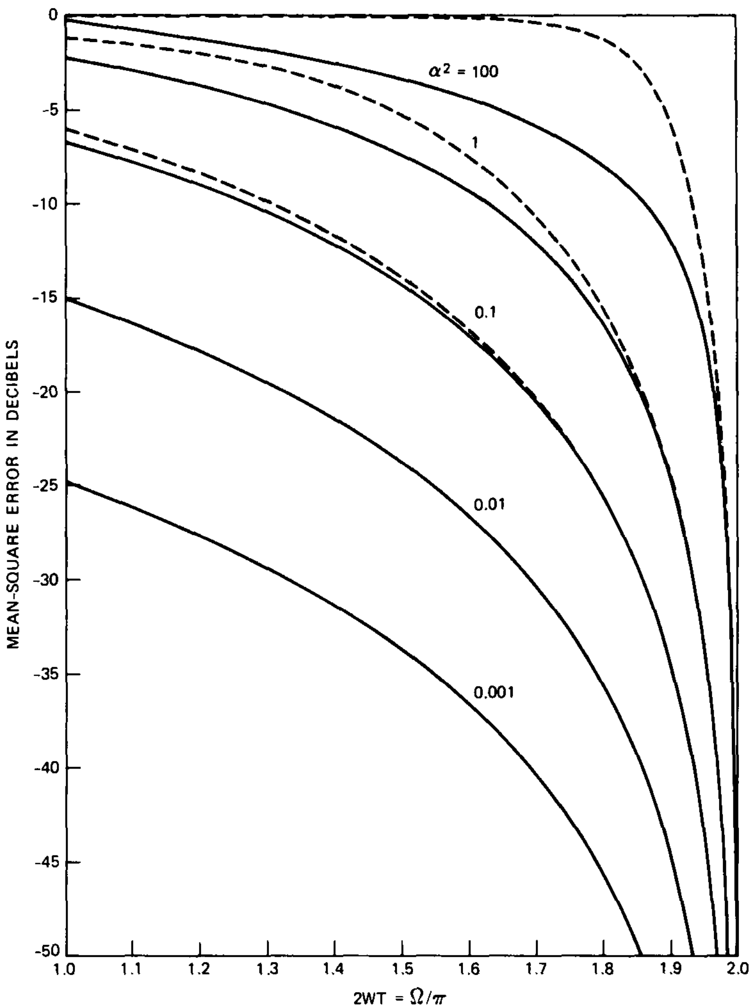

Fig. 3—Bandwidth-error exchange for various values of α^2. The solid line is for the optimal kernel; the dotted line is for the log-cabin kernel.

Whether or not the simple fading channel model is wholly acceptable, there is another reason for recommending or rationalizing the use of $f_0(t;\ \Omega,\ \alpha)$, or the log-cabin kernel; this reason being a reduced sensitivity to sampling jitter. We outline the justification of this reason.

In the problem of sampling jitter, it is supposed that

$$s(t) = \sum_{-\infty}^{\infty} a_k f(t - k), \tag{40}$$

where f is a real-valued function in $B_2(\Omega)$ and the a_k are independent random variables of zero mean and unit variance. In the absence of sampling jitter, f is taken, in case $\Omega = \pi$, to be $(\sin \pi t)/\pi t$, so that $s(t)$ is sampled at time(s) $t = n$ to obtain $a_n = s(n)$. In the presence of sampling jitter, $s(t)$ is inadvertently sampled at time(s) $t = n + \tau$, where τ is assumed to be a random variable with density $p(\tau)$, usually symmetric about $\tau = 0$. This jitter results in an error

$$\epsilon_n = a_n\{1 - f(\tau)\} - \sum_{k \neq n} a_k f(\tau + n - k). \tag{41}$$

The expected mean-square error over the $\{a_k\}$ is

$$\epsilon^2(\tau) = \{1 - f(\tau)\}^2 + \sum_{k \neq 0} f^2(k + \tau)$$

$$= 1 - 2f(\tau) + \sum_{-\infty}^{\infty} f^2(k + \tau), \tag{42}$$

which, when averaged over τ, gives

$$\epsilon^2 = 1 - 2\int_{-\infty}^{\infty} f(\tau)p(\tau)d\tau + \int_{-\infty}^{\infty} \left\{ \sum_{-\infty}^{\infty} f^2(k + \tau) \right\} p(\tau)d\tau. \tag{43}$$

In case $0 < \Omega \leqslant \pi$, the sum in the last integral is $\int_{-\infty}^{\infty} f^2(t)dt$, independent of τ. In this case,

$$\epsilon^2 = 1 - 2\int_{-\infty}^{\infty} f(\tau)p(\tau)d\tau + \int_{-\infty}^{\infty} f^2(t)dt. \tag{44}$$

It is easy to show that ϵ^2 in (44), which is the same as ϵ^2 in (43) only for $0 < \Omega \leqslant \pi$, is minimized over f in $B_2(\Omega)$ by taking

$$f(t) = \int_{-\infty}^{\infty} p(\tau) \frac{\sin \Omega(t - \tau)}{\pi(t - \tau)} d\tau. \tag{45}$$

That is, $f(t)$ is obtained by bandlimiting $p(t)$ [projecting $p(t)$ on $B_2(\Omega)$].

Recall (26), where

$$f_0(t;\, \Omega,\, \alpha) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{\cos \omega t}{1 + \alpha^2\omega^2} d\omega, \qquad 0 < \Omega \leqslant \pi.$$

From this, it is seen that

$$f_0(t;\, \Omega,\, \alpha) = \int_{-\infty}^{\infty} \frac{e^{-|\tau|/\alpha}}{2\alpha} \frac{\sin \Omega(t - \tau)}{\pi(t - \tau)} d\tau, \qquad 0 < \Omega \leqslant \pi. \tag{46}$$

Thus, for $0 < \Omega \leqslant \pi$, $f_0(t;\, \Omega,\, \alpha)$ may be interpreted as the function in $B_2(\Omega)$ which minimizes the mean-square error due to (reduced bandwidth *and*) a sampling jitter τ with density $(2\alpha)^{-1}e^{-|\tau|/\alpha}$, $-\infty < \tau < \infty$.

The fact that τ is unbounded requires the assumption of an ensemble of sampling mechanisms. However, this is not important in the case of small α, when all that really matters is the second moment of $p(\tau)$.

The mean-square error due to sampling jitter can be decreased at the expense of extra bandwidth. For $\Omega > \pi$, the sum $\sum_{-\infty}^{\infty} f^2(k + \tau)$ is no longer, because of aliasing, the same as $\int_{-\infty}^{\infty} f^2(t)dt$. The case $\pi < \Omega \leq 2\pi$ can be treated, as in the proof of the Theorem, by decomposing f as

$$f(t) = g(t) + h(t),$$

where g is the low-frequency component, bandlimited to $[-(2\pi - \Omega), (2\pi - \Omega)]$, and h is the bandpass component with center frequency π and upper frequency Ω. It turns out that the low-frequency component of the optimal f is obtained by bandlimiting the density $p(\tau)$. So in case of the two-sided exponential density, $g(t) = f_0(t; 2\pi - \Omega, \alpha)$ is optimal, as in $f_0(t; \Omega, \alpha)$. The optimal bandpass component $h(t)$ is not quite the same as $h_0(t; \Omega)$ in (28), but is very close to $h_0(t; \Omega)$ in case the density is symmetric with small support $[-a, a]$. In fact, as $a \to 0$, the optimal f in $B_2(\Omega)$, $\pi < \Omega \leq 2\pi$, for the (symmetric) sampling-jitter problem tends to $f_1(t; \Omega)$, the log-cabin kernel. So the log-cabin kernel is near optimal for reducing, at the expense of bandwidth, the mean-square error due to small symmetrically distributed sampling jitter.

An obvious generalization of the problem considered here is the minimization over f in $B_2(\Omega)$ of

$$\epsilon^2(f) = |1 - f(0)|^2 + \sum_{k \neq 0} |f(k)|^2 + \sum_{n=1}^{N} \alpha_n^2 \left\{ \sum_{k=-\infty}^{\infty} |f^{(n)}(k)|^2 \right\}. \quad (47)$$

In response to a question raised by the Referee, a numerical approach to this problem is not recommended, at least for moderate N, because, first, the analytical solution is tractable, simpler, and exact; and second, certain anomalies can be anticipated.

For $0 < \Omega \leq \pi$, the solution is almost trivial, the extremal function being

$$f_0(t; \Omega, \bar{\alpha}_N) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{\cos \omega t}{P(\omega^2; \bar{\alpha}_N)} d\omega, \qquad (0 < \Omega \leq \pi), \quad (48)$$

where

$$P(\omega^2; \bar{\alpha}_N) = 1 + \sum_{n=1}^{N} \alpha_n^2 \omega^{2n}.$$

Always (for any Ω), the minimum mean-square error is

$$\epsilon_0^2(\Omega; \bar{\alpha}_N) = 1 - f_0(0; \Omega, \bar{\alpha}_N). \quad (49)$$

Clearly, $\epsilon^2(f)$ can be made zero for $\Omega \geqslant (N+1)\pi$ by taking $f(t) = (\sin \pi t)^{N+1}/(\pi t)^{N+1}$. For $\pi < \Omega < (N+1)\pi$, aliasing complicates the problem. However, the aliasing can be handled in a systematic way, and the solution, if tedious, is straightforward, except in anomalous cases where only even-order derivatives appear in the last sum in (47). In the anomalous cases, the minimum, or properly speaking, the infimum of $\epsilon^2(f)$ over f in $B_2(\Omega)$ is not attainable for $\pi < \Omega < (N+1)\pi$. For example,

$$\epsilon^2(f) = |1 - f(0)|^2 + \sum_{k \neq 0} |f(k)|^2 + \alpha_2^2 \sum_{-\infty}^{\infty} |f''(k)|^2$$

can be made arbitrarily small for f in $B_2(2\pi)$, but only at the expense of making the norm (in L_2) of f very large.

There will be near-anomalous cases where the infimum *is* attained but for an f of large norm, i.e., cases where the solution is not satisfactory for reasons not taken into account. One consideration is the average power of the transmitted signal, which is directly proportional to $\int_{-\infty}^{\infty} f^2(t)dt$. Another consideration is the sensitivity to sampling jitter.

It is obvious, then, in the general problem, that $\epsilon^2(f)$ *should* be minimized with a constraint on the norm of f. This severely complicates the problem. The practical alternative is to solve the problem and then appraise the solution. What is needed is a better understanding of how the alphas affect the norm of the solution for $\Omega > \pi$. It turns out for the simple problem considered here that the norm of $f_0(t; \Omega, \alpha)$ decreases with Ω for $\pi < \Omega < 2\pi$. It would be nice to have, at least, sufficient conditions on the alphas for this decrease to obtain in the general problem. On the basis of the solution (48) to the general problem for $\Omega = \pi$, and the analogous sampling-jitter problem, it is conjectured that a sufficient condition on the alphas is that the reciprocal of the polynomial $P(\omega^2; \overline{\alpha}_N)$ in (48) be the Fourier transform of a positive function.

IV. PROOF OF THE THEOREM

The minimization problem is obviously equivalent to the problem of minimizing over f in $B_2(\Omega)$,

$$Q(f; \alpha) = \sum_{-\infty}^{\infty} |f(k)|^2 + \alpha^2 \sum_{-\infty}^{\infty} |f'(k)|^2, \tag{50}$$

subject to $f(0) = 1$.

Now suppose $f = f_1 + if_2$, where f_1 and f_2 are real-valued functions in $B_2(\Omega)$ with $f_1(0) = 1$, $f_2(0) = 0$. Then, obviously,

$$Q(f; \alpha) = Q(f_1; \alpha) + Q(f_2, \alpha). \tag{51}$$

So we may restrict our attention to real-valued f in $B_2(\Omega)$. Next suppose that $f = f_1 + f_2$, where f_1 and f_2 are real-valued functions in $B_2(\Omega)$, f_1 being even, and f_2 odd. Once again we obtain (51), since f_1 and f_2 (f_1' and f_2') are orthogonal over the integers. So we may restrict our attention to real-valued even functions f in $B_2(\Omega)$, $f(0) = 1$, in seeking the minimum.

Now define

$$Q^*(f; \alpha) = \int_{-\infty}^{\infty} |f(t)|^2 dt + \alpha^2 \int_{-\infty}^{\infty} |f'(t)|^2 dt. \tag{52}$$

Then we have

Lemma 1. For f in $B_2(\Omega)$, $0 < \Omega \leqslant \pi$, we have

$$Q(f; \alpha) = Q^*(f; \alpha). \tag{53}$$

This follows from the well-known result,

$$\tau \sum_{-\infty}^{\infty} |f(k\tau + \theta)|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt,$$

$$f \text{ in } B_2(\Omega), \quad \theta \text{ real}, \quad 0 < \tau \leqslant \pi/\Omega, \tag{54}$$

and the fact that if f belongs to $B_2(\Omega)$ then so does f'. □

Next define

$$\rho(\Omega; \alpha) = \inf_{\substack{f \in B_2(\Omega) \\ f(0)=1}} \{Q^*(f; \alpha)\}, \qquad 0 < \Omega < \infty,$$

$$0 \leqslant \alpha < \infty. \tag{55}$$

Then it is a simple matter to establish:

Lemma 2. We have

$$\rho(\Omega; \alpha) = \frac{\pi}{\Omega} \cdot \frac{(\alpha\Omega)}{\arctan(\alpha\Omega)}, \tag{56}$$

and the infimum in (55) is attained for, and only for,

$$f(t) = f(t; \Omega, \alpha) = \rho(\Omega; \alpha) \cdot \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{\cos \omega t}{1 + \alpha^2\omega^2} d\omega. \tag{57}$$

It follows from Lemmas 1 and 2, and the definition (14) of $\mu(\Omega; \alpha)$, that

$$\mu(\Omega; \alpha) = \rho(\Omega; \alpha) - 1, \qquad 0 < \Omega \leqslant \pi. \tag{58}$$

So Lemmas 1 and 2 establish the first part of the theorem.

Proof of Lemma 2. From the previous argument [which applies as well to the integrals in (52)], the extremal f will be real and even; i.e.,

$$f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) \cos \omega t \, d\omega, \qquad f(0) = 1, \tag{59}$$

where $F(\omega)$ is real and even. Then

$$Q^*(f;\,\alpha) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F^2(\omega)(1 + \alpha^2\omega^2)d\omega. \tag{60}$$

Now suppose g is any real even function in $B_2(\Omega)$ satisfying $g(0) = 0$; i.e.,

$$g(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} G(\omega)\cos\,\omega t\;dt, \qquad \int_{-\Omega}^{\Omega} G(\omega)d\omega = 0. \tag{61}$$

Then

$$\begin{aligned} Q^*(f + g;\,\alpha) &= \frac{1}{2\pi}\int_{-\Omega}^{\Omega} [f(\omega) + G(\omega)]^2(1 + \alpha^2\omega^2)d\omega \\ &= Q^*(f;\,\alpha) + Q^*(g;\,\alpha) \\ &\quad + \frac{1}{\pi}\int_{-\Omega}^{\Omega} F(\omega)G(\omega)(1 + \alpha^2\omega^2)d\omega. \end{aligned} \tag{62}$$

The last integral vanishes, in accord with (61), if

$$F(\omega) = \frac{c}{1 + \alpha^2\omega^2}. \tag{63}$$

Then setting $f + g = f_1$, we have f_1 in $B_2(\Omega)$, $f_1(0) = 1$, and

$$Q(f_1) = Q(f;\,\alpha) + Q(f_1 - f;\,\alpha) \geqslant Q(f;\,\alpha) \tag{64}$$

with equality throughout if, and only if, $f_1 - f \equiv 0$.

So the restriction to $[-\Omega,\,\Omega]$ of $F(\omega)$ given by (63) is the Fourier transform of the extremal function, provided

$$\frac{c}{\pi}\int_{0}^{\Omega} \frac{d\omega}{1 + \alpha^2\omega^2} = 1, \tag{65}$$

requiring

$$c = \frac{\pi}{\Omega}\cdot\frac{(\alpha\Omega)}{\arctan(\alpha\Omega)}. \tag{66}$$

Then

$$\begin{aligned} \rho(\Omega;\,\alpha) &= \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F^2(\omega)(1 + \alpha^2\omega^2)d\omega \\ &= \frac{c^2}{2\pi}\int_{-\Omega}^{\Omega} \frac{d\omega}{1 + \alpha^2\omega^2} = c. \qquad \square \end{aligned} \tag{67}$$

Now to find the extremal f in $B_2(\Omega)$, $\pi < \Omega \leqslant 2\pi$, we first introduce $B_2(a,\,b)$, the class of square-integrable bandlimited functions whose Fourier transforms vanish outside the intervals $[-b,\,-a]$, $[a,\,b]$, where

$0 \leqslant a < b < \infty$. (In case $a > 0$, the functions are called bandpass functions.) For f in $B_2(\Omega)$, $\pi < \Omega \leqslant 2\pi$, we may make the decomposition,

$$f(t) = g(t) + h(t), \qquad g \text{ in } B_2(\beta\pi), \quad h \text{ in } B_2(\beta\pi, \Omega), \tag{68}$$

where $\beta = 2 - (\Omega/\pi)$, $0 \leqslant \beta < 1$. Next we use the general representation for h in $B_2(\beta\pi, \Omega)$,

$$h(t) = p(t)\cos \pi t + q(t) \sin \pi t, \qquad p, q \text{ in } B_2(\beta'\pi), \tag{69}$$

where $\beta' = 1 - \beta > 0$.

In $Q(f;\,\alpha)$, we need the values of $f(k)$ and $f'(k)$, which with the decomposition (68) become

$$f(k) = g(k) + (-1)^k p(k), \tag{70}$$

$$f'(k) = g'(k) + (-1)^k[p'(k) + \pi q(k)]. \tag{71}$$

Now it is convenient at this point to use the fact that the extremal f will be real, allowing us to write

$$\begin{aligned} Q(f;\,\alpha) &= \sum_{-\infty}^{\infty} [g(k) + (-1)^k p(k)]^2 \\ &\quad + \alpha^2 \sum_{-\infty}^{\infty} \{g'(k) + (-1)^k[p'(k) + \pi q(k)]\}^2 \\ &= \sum_{-\infty}^{\infty} g^2(k) + \alpha^2 \sum_{-\infty}^{\infty} [g'(k)]^2 + \sum_{-\infty}^{\infty} p^2(k) \\ &\quad + \alpha^2 \sum_{-\infty}^{\infty} [p'(k) + \pi q(k)]^2 + 2 \sum_{-\infty}^{\infty} (-1)^k g(k)p(k) \\ &\quad + 2\alpha^2 \sum_{-\infty}^{\infty} (-1)^k g'(k)[p'(k) + \pi q(k)]. \end{aligned} \tag{72}$$

Now recall that g and g' belong to $B_2(\beta\pi)$; p and $[p' + \pi q]$ belong to $B_2(\beta'\pi)$, where $0 < \beta \leqslant 1$ and $\beta' = 1 - \beta$. Therefore, gp and $g'[p' + \pi q]$ belong to $B_1(\pi)$, the class of absolutely integrable (L_1) functions whose Fourier transforms vanish outside $[-\pi, \pi]$. Hence, their Fourier transforms vanish at the endpoints, $\pm\pi$, since the Fourier transform of a function of L_1 is continuous. If follows, for example, from the Poisson sum formula, that the last two sums in (72) vanish, and the other sums may be written as integrals, in accord with (54). Thus

$$\begin{aligned} Q(f;\,\alpha) &= \int_{-\infty}^{\infty} g^2(t)dt + \alpha^2 \int_{-\infty}^{\infty} [g'(t)]^2 dt \\ &\quad + \int_{-\infty}^{\infty} p^2(t)dt + \alpha^2 \int_{-\infty}^{\infty} [p'(t) + \pi q(t)]^2 dt, \end{aligned} \tag{73}$$

g in $B_2(\beta\pi)$, p, q in $B_2(\beta'\pi)$.

We wish to minimize this quantity subject to

$$f(0) = p(0) + g(0) = 1.$$

First we make the last integral vanish by taking

$$q(t) = -\frac{1}{\pi} p'(t), \tag{74}$$

noting that this is not necessary in case $\alpha = 0$. Then

$$Q(f; \alpha) = Q^*(g; \alpha) + \int_{-\infty}^{\infty} p^2(t)dt, \quad g \text{ in } B_2(\beta\pi), \quad p \text{ in } B_2(\beta'\pi). \tag{75}$$

Now suppose

$$p(0) = \lambda \quad \text{and} \quad g(0) = 1 - \lambda. \tag{76}$$

Since

$$p(0) = \int_{-\infty}^{\infty} p(t) \frac{\sin \beta'\pi t}{\pi t} dt, \tag{77}$$

we have, from Schwarz's inequality,

$$\int_{-\infty}^{\infty} p^2(t)dt \geqslant p^2(0)/\beta', \tag{78}$$

with equality holding if, and only if,

$$p(t) = p(0) \frac{\sin \beta'\pi t}{\beta'\pi t}, \qquad (\beta'\pi = \Omega - \pi). \tag{79}$$

From Lemma 2, we have

$$Q^*(g; \alpha) \geqslant g^2(0) \cdot \rho(\beta\pi; \alpha), \tag{80}$$

with equality holding if, and only if,

$$g(t) = g(0) \cdot f(t; \beta\pi, \alpha). \tag{81}$$

Therefore

$$Q(f; \alpha) \geqslant \min_{\lambda} \{(1 - \lambda)^2 \rho(\beta\pi; \alpha) + \lambda^2/\beta'\} \tag{82}$$

where $\beta' = 1 - \beta > 0$.

The minimum occurs for

$$\lambda = \frac{1 - \beta}{(1 - \beta) + [\rho(\beta\pi; \alpha)]^{-1}}, \tag{83}$$

giving

$$Q(f;\,\alpha) \geq \{[\rho(\beta\pi;\,\alpha)]^{-1} + 1 - \beta\}^{-1} = m(\Omega,\,\alpha), \qquad \pi < \Omega \leq 2\pi, \quad (84)$$

with equality holding (for $\alpha > 0$) if, and only if,

$$f(t) = (1 - \lambda)f(t;\,\beta\pi,\,\alpha) + \lambda\left\{\phi(t)\cos \pi t - \frac{1}{\pi}\,\phi'(t)\sin \pi t\right\}, \quad (85)$$

where $f(t;\,\Omega,\,\alpha)$ is defined in (57), λ is given by (83), and

$$\phi(t) = \frac{\sin(\Omega - \pi)t}{(\Omega - \pi)t}. \quad (86)$$

The uniqueness qualification, "for $\alpha > 0$", owes to the fact [cf. (74)] that we need not have $q(t) = -p'(t)/\pi$, in case $\alpha = 0$, in order to minimize $Q(f;\,\alpha)$ with $f(0) = 1$. Obviously, any function in $B_2(\Omega)$ satisfying $f(0) = 1$ and having $(\sin \pi t)/\pi t$ as a factor will minimize $Q(f;\,0)$. Since equality may attain in (84) for f given by (85), we have

$$\mu(\Omega;\,\alpha) = m(\Omega;\,\alpha) - 1, \qquad \pi < \Omega \leq 2\pi, \quad (87)$$

and the theorem is proved. □

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AUTHOR

Benjamin F. Logan, Jr., B.S. (Electrical Engineering), 1946, Texas Technological College; M.S., 1951, Massachusetts Institute of Technology; Eng.D.Sc. (Electrical Engineering), 1965, Columbia University; AT&T Bell Laboratories, 1956—. While at MIT, Mr. Logan was a research assistant in the Research Laboratory of Electronics, investigating characteristics of high-power electrical discharge lamps. Also at MIT he engaged in analog computer development at the Dynamic Analysis and Control Laboratory. From 1955 to 1956 he worked for Hycon-Eastern, Inc., where he was concerned with the design of airborne power supplies. He joined AT&T Bell Laboratories as a member of the Visual and Acoustics Research Department, where he was concerned with the processing of speech signals. Currently, he is a member of the Mathematical Research Department. Member, Sigma Xi, Tau Beta Pi.