AT&T Technical Journal
Vol. 64, No. 7, September 1985
Printed in U.S.A.

Laplace Transform Inequalities With Application to Queueing

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(Manuscript received April 11, 1985)

Inequalities satisfied by the Laplace transforms of convex and log-convex functions are obtained. Applications are made to the M/G/1 queue waiting time and to an important teletraffic congestion problem, arising in parcel blocking studies.

I. INTRODUCTION

The purpose of this paper is to establish certain inequalities satisfied by the Laplace transform of convex functions and to illustrate their use. The notion of α-convexity on which these results are based is fully discussed.[1] This notion had its origin in the author's investigations concerning the inversion of the Laplace transform;[2] subsequently, it has been applied to obtaining the results reported on here. The property of α-convexity forms a natural bridge between ordinary convexity ($\alpha = 0$) and the stronger property of log-convexity (all α). This enables the formulation of a criterion in terms of the transform of a function for ascertaining the log-convexity of the function, vd. Theorem 2 and (11).

An infinite set of inequalities satisfied by the Laplace transform of a log-convex function is obtained, vd. Theorem 3; these inequalities are illustrated by two applications. The first provides necessary conditions for the log-convexity of the complementary waiting time distribution in the First-In First-Out (FIFO) M/G/1 queue. The condi-

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tions are expressed in terms of the transform of the complementary service time distribution.

The second application concerns the important teletraffic problem of ascertaining the time congestion of a call that overflows a primary group and is offered to a secondary group. Simple upper and lower bounds are obtained for a function $[\delta_j(x, a)]$ arising in Brockmeyer's analysis of the problem and in terms of which the time congestion is obtained.[3] These results form an important part of "parcel blocking" analyses.[4]

II. α-CONVEXITY

A function $f(x)$ is said to be α-convex on an interval I if $e^{\alpha x}f(x)$ is convex on I. Clearly, ordinary convexity corresponds to $\alpha = 0$. A sufficient condition for convexity of $f(x)$ is[5]

$$f''(x) \geq 0, \quad x \in I. \tag{1}$$

Introducing the function $h(x)$ by

$$h(x) = e^{-\alpha x} \frac{d^2}{dx^2} [e^{\alpha x} f(x)], \tag{2}$$

$$= f''(x) + 2\alpha f'(x) + \alpha^2 f(x). \tag{3}$$

Then, in view of (1), the condition for α-convexity is

$$h(x) \geq 0, \quad x \in I. \tag{4}$$

It should be observed that α-convexity does not imply convexity ($\alpha = 0$). Thus consider, for example, $f(x) = x^3$, which is α-convex for $x \geq 0$, $\alpha \geq 0$. For $\alpha = 1$, however, x^3 is α-convex for $-3 - \sqrt{3} \leq x \leq -3 + \sqrt{3}$.

The α-convexity of a function may permit stronger bounds to be obtained on integrals of the function than ordinary convexity. For example, let $p(x) \geq 0$, and let $f(x)$ be convex on I, then Jensen's inequality states[5]

$$\int_I f(x)p(x)dx \geq f(\mu) \int_I p(x)dx,$$

$$\mu = \int_I xp(x)dx \Big/ \int_I p(x)dx. \tag{5}$$

If $f(x)$ is α-convex on I, then, since

$$\int_I f(x)p(x)dx = \int_I e^{\alpha x} f(x) e^{-\alpha x} p(x)dx, \tag{6}$$

one has

$$\int_I f(x)p(x)dx \geq e^{\alpha\mu}f(\mu) \int_I e^{-\alpha x}p(x)dx,$$

$$\mu = \int_I xe^{-\alpha x}p(x)dx \Big/ \int_I e^{-\alpha x}p(x)dx. \quad (7)$$

This result can be stronger than (5).

Let the Laplace transform, $\tilde{f}(s)$, of $f(x)$ be defined by

$$\tilde{f}(s) = \int_0^\infty e^{-sx}f(x)dx, \quad s > -\gamma, \quad (8)$$

and let $f(x)$ be α-convex for $x \geq 0$; then the following theorem may be stated.

Theorem 1:

$$\tilde{f}(s) \geq \frac{1}{s+\alpha} e^{\frac{\alpha}{s+\alpha}} f\left(\frac{1}{s+\alpha}\right), \quad \alpha > -s;$$

or, equivalently,

$$f(x) \leq \frac{1}{x} e^{-\alpha x} \tilde{f}\left(\frac{1}{x} - \alpha\right), \quad \alpha < \frac{1}{x} + \gamma.$$

Proof: Use of (7) with $p(x) = e^{-sx}$.

A function $f(x) > 0$ is said to be log-convex on I if $\ln f(x)$ is convex on I. Thus, the condition for log-convexity is

$$f''(x)f(x) - f'(x)^2 \geq 0, \quad x \in I. \quad (9)$$

In particular, log-convexity implies convexity, hence $e^{\alpha x}f(x)$ is convex; thus, a log-convex function is α-convex for all α. The following theorem asserts also the converse.

Theorem 2: A function $f(x) > 0$ is log-convex on an interval I if and only if it is α-convex on I for all α.

Proof: It is necessary to prove only that $f(x)$ is log-convex on I if it is α-convex on I for all α. This follows from (3) on observing that the discriminant of the quadratic in α is $f'(x)^2 - f''(x)f(x)$; hence, α-convexity for all α implies (9) and the consequent log-convexity of $f(x)$.

Corollary: The sum of log-convex functions is log-convex.

Proof: The sum of α-convex functions corresponding to the same α is clearly again α-convex for the same α; hence, the statement follows on applying the theorem.

A function $f(x)$ is said to be completely monotone on I if

$$(-1)^r f^{(r)}(x) \geq 0, \quad x \in I, \quad r = 0, 1, 2, \cdots. \tag{10}$$

The Bernstein theorem,[6] which states that $f(x) \geq 0$ if and only if $\tilde{f}(s)$ is completely monotone for s real and in the domain of convergence of (8), may be used to translate condition (4) in terms of $\tilde{f}(s)$. Accordingly, let $\tilde{f}(s)$ converge for $s > 0$ and let

$$\tilde{h}(s) = (s + \alpha)^2 \tilde{f}(s) - (s + 2\alpha) f(0+) - f'(0+), \quad s > 0. \tag{11}$$

Then $f(x)$ is α-convex if and only if $\tilde{h}(s)$ is completely monotone for $s > 0$. Thus, also, $f(x)$ is log-convex if and only if $\tilde{h}(s)$ is completely monotone for $s > 0$ and all α.

III. INEQUALITIES FOR $\tilde{f}(s)$ FROM LOG-CONVEXITY

It will now be assumed that $f(x)$ is log-convex for $x > 0$ and that $\tilde{f}(s)$ converges for $s > 0$. Thus, one has

$$\min_{\alpha} (-1)^n \tilde{h}^{(n)}(s) \geq 0, \quad s > 0, \quad n = 0, 1, 2, \cdots. \tag{12}$$

The following theorem will now be proved.

Theorem 3: If $f(x)$ is log-convex for $x > 0$, then for all $s > 0$, one has

$$\tilde{f}(s)^{-1} \leq \frac{sf(0+) - f'(0+)}{f(0+)^2}, \quad f(0+) \neq 0,$$

$$\frac{d}{ds} \tilde{f}(s)^{-1} \geq \frac{1}{f(0+)}, \quad f(0+) \neq 0,$$

$$(n - 1)\tilde{f}^{(m)}(s)\tilde{f}^{(m-2)}(s) \geq n\tilde{f}^{(m-1)}(s)^2, \quad n \geq 2.$$

The equality signs are achieved for $f(x) = e^{-\gamma x}$.

Proof: One has for

$$\alpha = \frac{f(0)}{\tilde{f}(s)} - s, \tag{13}$$

$$\min_{\alpha} \tilde{h}(s) = sf(0) - f'(0) - \frac{f(0)^2}{\tilde{f}(s)} \geq 0. \tag{14}$$

This establishes the first inequality. For

$$\alpha = \frac{\tilde{f}(s)}{\tilde{f}'(s)} - s, \tag{15}$$

one has

$$\min_{\alpha}[-\tilde{h}'(s)] = \frac{\tilde{f}(s)^2}{\tilde{f}'(s)} + f(0) \geq 0, \tag{16}$$

which, after a little manipulation, yields the second inequality. For the choice

$$\alpha = -n \frac{\tilde{f}^{(n-1)}(R)}{\tilde{f}^{(n)}(R)} - R, \tag{17}$$

direct calculation shows that

$$\min_{\alpha}[(-1)^n \tilde{h}^{(n)}(s)] = (-1)^n n \left[(n-1)\tilde{f}^{(n-2)}(s) - n \frac{\tilde{f}^{(n-1)}(s)^2}{\tilde{f}^{(n)}(s)} \right] \geq 0. \tag{18}$$

In view of the complete monotonicity of $\tilde{f}(s)$, the remaining inequalities are established.

IV. CONVEXITY IN M/G/1

An application of Theorem 3 will now be made to waiting time in the FIFO M/G/1 queue.[1] In the following, λ is the arrival rate, μ is the service rate, $\rho = \lambda/\mu < 1$, μ_2 is the second moment about the origin of service time, and $\tilde{\beta}(s)$, $\tilde{F}(s)$ are the Laplace transforms of the complementary service and waiting time distributions, respectively. Theorem 4 may now be stated.

Theorem 4: Necessary conditions for the complementary waiting time distribution, $F(x)$, to be log-convex are

$$\frac{2}{\mu^2\mu_2} \frac{1}{s + \dfrac{2}{\mu\mu_2}} \leq \tilde{\beta}(s) \leq \frac{1}{s+\mu}, \qquad \mu_2 \geq \frac{2}{\mu^2}.$$

Proof: The Pollaczek-Khintchine formula[1] may be written in the form

$$\tilde{F}(s) = \frac{1}{s} \frac{\rho - \lambda\tilde{\beta}(s)}{1 - \lambda\tilde{\beta}(s)}. \tag{19}$$

Clearly,

$$\tilde{\beta}(s) \sim \frac{1}{s}, \qquad s \to \infty; \tag{20}$$

hence,

$$\tilde{F}(s) \sim \frac{\rho}{s} - \frac{\lambda(1-\rho)}{s^2}, \qquad s \to \infty. \tag{21}$$

Thus,

$$F(0+) = \rho, \qquad F'(0+) = -\lambda(1-\rho). \tag{22}$$

Application of the first inequality of Theorem 3 yields

$$s\,\frac{1-\lambda\tilde{\beta}(s)}{\rho-\lambda\tilde{\beta}(s)} \le \frac{s}{\rho} + \mu\,\frac{1-\rho}{\rho}. \tag{23}$$

Observing that $\tilde{\beta}(s)$ is monotone decreasing, and that $\lambda\tilde{\beta}(0) = \rho < 1$, one has $(s > 0)$

$$1 - \lambda\tilde{\beta}(s) > 0, \qquad \rho - \lambda\tilde{\beta}(s) > 0. \tag{24}$$

Hence, multiplying (23) through by $\rho - \lambda\tilde{\beta}(s)$, and solving for $\tilde{\beta}(s)$, one obtains

$$\tilde{\beta}(s) \le \frac{1}{s+\mu}. \tag{25}$$

Using the second inequality of Theorem 3 in the form

$$\tilde{F}(s)^{-1} \ge \tilde{F}(0)^{-1} + \frac{s}{F(0+)} \tag{26}$$

with the evaluation

$$\tilde{F}(0) = \frac{1}{2}\,\frac{\lambda\mu_2}{1-\rho} \tag{27}$$

yields the required lower bound for $\tilde{\beta}(s)$. Finally, multiplying the upper and lower bounds by s and evaluating the limit $s \to \infty$ provides the last inequality of the theorem.

V. AN OVERFLOW MODEL

Consider the traffic model of Fig. 1, in which a is the offered load, assumed Poisson, to the primary trunk group of x trunks whose overflow of m erlangs and peakedness z is offered to the secondary trunk group of c trunks. Clearly,

$$m = aB(x, a), \tag{28}$$

in which $B(x, a)$ is the Erlang loss function.[7,8] The blocking (call congestion) experienced by a call overflowing the primary is given by the formula (equivalent random method)[8]

$$B_c = \frac{B(x+c, a)}{B(x, a)}. \tag{29}$$

An alternative expression for B_c may be obtained from the Brockmeyer analysis.[3] Let $P(k)$ be the probability k trunks are busy on the secondary group and let $m(k)$ be the corresponding load offered to the secondary; then

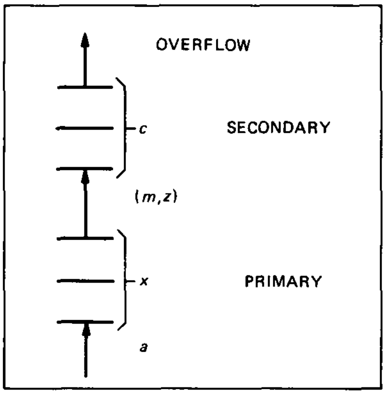

Fig. 1—Overflow system.

$$m = \sum_{k=0}^{c} m(k)P(k), \tag{30}$$

$$B_c = \frac{m(c)P(c)}{m}. \tag{31}$$

One has also that the time congestion, B_T, is given by

$$B_T = P(c). \tag{32}$$

In order to obtain $P(c)$, the function $\delta_j(x, a)$ is introduced by

$$\delta_j(x, a) = \frac{G_x(-j-1, a)}{G_x(-j, a)}, \tag{33}$$

in which $G_j(x, a)$ are Poisson-Charlier polynomials.[7] Then

$$B_T = \delta_c(x, a)B(x + c, a). \tag{34}$$

It has been found[4] that $\delta_j(x, a)$ satisfies the recursion

$$\delta_0(x, a) = B(x, a)^{-1},$$

$$\delta_j(x, a) = 1 + \frac{x-a}{j} + \frac{a}{j}\,\delta_{j-1}(x, a)^{-1}, \qquad j \geq 1. \tag{35}$$

Another recursion for $\delta_j(x, a)$ may be obtained by considering the integral

$$I_j(x, a) = \int_0^\infty e^{-ay}(1+y)^x y^j dy. \tag{36}$$

Since

$$G_x(-j, a) = (-1)^x \frac{a^j}{\Gamma(j)} \int_0^\infty e^{-ay}(1+y)^x y^{j-1} dy, \tag{37}$$

one has from (33)

$$\delta_j(0, a) = 1, \qquad j \ge 0,$$

$$\delta_j(x, a) = \frac{a}{j}\frac{I_j(x, a)}{I_{j-1}(x, a)}, \qquad j \ge 1. \tag{38}$$

One easily establishes

$$I_j(x + 1, a) = I_j(x, a) + I_{j+1}(x, a), \tag{39}$$

$$I_j(x, a) = \frac{x}{a} I_j(x - 1, a) + \frac{j}{a} I_{j-1}(x, a); \tag{40}$$

hence, from (39) and (40)

$$\delta_j(x, a) = \frac{x}{j}\frac{I_j(x - 1, a)}{I_{j-1}(x, a)} + 1. \tag{41}$$

Use of (39) finally yields

$$\delta_j(x, a) = \frac{x}{a + j\delta_j(x - 1, a)} \delta_j(x - 1, a) + 1, \quad x \ge 1. \tag{42}$$

The recursion of (42) with initial value of (38) provides a convenient and stable method for the exact computation of $\delta_j(x, a)$; however, in many practical investigations, it is useful to have upper and lower bounds showing simple and explicit dependence on the arguments. For this purpose, let $P_x(t)$ be the recovery function for a group of x trunks, that is, the probability that all trunks are busy at time t given they were all busy at time zero; then[9]

$$\tilde{P}_x(s) = \frac{G_x(-R, a)}{RG_x(-R - 1, a)}. \tag{43}$$

Comparison of (43) with (33) shows that

$$\tilde{P}_x(j) = \frac{1}{j\delta_j(x, a)}. \tag{44}$$

It is also known that $P_x(t)$ is log-convex in t for $t \ge 0$.[9] The following theorem may now be proved.

Theorem 5: For $j \ge 1$, $x \ge 0$, $a \ge 0$, the following inequalities hold

$$\frac{1}{2}\left[1 + \frac{x - a + 1}{j} + \sqrt{\left(1 + \frac{x - a + 1}{j}\right)^2 + 4\frac{j(a - 1) - x}{j^2}}\right]$$

$$\le \delta_j(x, a) \le \frac{1}{2}\left[1 + \frac{x - a}{j} + \sqrt{\left(1 + \frac{x - a}{j}\right)^2 + \frac{4a}{j}}\right].$$

Proof: The upper bound is due to A. A. Fredericks,[4] who shows that

$$\delta_j(x, a) \leq \delta_{j-1}(x, a), \qquad j \geq 1; \tag{45}$$

hence, from (35) one gets

$$\delta_j(x, a) \leq 1 + \frac{x - a}{j} + \frac{a}{j}\, \delta_j(x, a)^{-1}. \tag{46}$$

Solution of the quadratic provides the upper bound of the theorem.

Since $P_x(0) = 1$, the second inequality of Theorem 3 applied to $\tilde{P}_x(s)$ in the form

$$\tilde{P}_x(s + h)^{-1} - \tilde{P}_x(s)^{-1} \geq h, \qquad h \geq 0, \tag{47}$$

yields, from (44),

$$(j + h)\delta_{j+h}(x, a) - j\delta_j(x, a) \geq h, \qquad h \geq 0. \tag{48}$$

Setting $h = 1$ and writing (48) in the form

$$\delta_{j-1}(x, a)^{-1} \geq \frac{j - 1}{j\delta_j(x, a) - 1} \tag{49}$$

yields, after substitution into (35),

$$j\delta_j(x, a) \geq j + x - a + a\, \frac{j - 1}{j\delta_j(x, a) - 1}. \tag{50}$$

From (48) it follows that $j\delta_j(x, a) \geq 1$; hence, in (50), multiplying through by $j\delta_j(x, a) - 1$ yields the inequality

$$\delta_j(x, a)^2 - \left(1 + \frac{x - a + 1}{j}\right) \delta_j(x, a) - \frac{j(a - 1) - x}{j^2} \geq 0. \tag{51}$$

Solution of this quadratic finally yields the lower bound of the theorem.

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