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Almost-Periodic Response Determination for Models of the Basilar Membrane

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Electrical networks consisting of linear passive elements and many nonlinear resistors are often used to model the basilar membrane. The inputs to these networks are typically a sum of sinusoids switched on at $t = 0$, and the resulting quantities of interest because of their interpretation as analogs of experimental observables are the steady-state response components of a certain current and of certain voltages. In this paper, recently obtained mathematical results concerning the input-output representation of nonlinear systems are used to give, for the first time, a locally convergent expansion for all of the steady-state quantities of interest. Also given is a good deal of information concerning general properties of the expansion, and this establishes important properties of the nonlinear network's response. Of particular practical interest is a term in the expansion that contains a component whose frequency is $(2f_1 - f_2)$ when the network's input consists of a sum of two sinusoids, with frequencies f_1 and f_2. One of our main results is an explicit expression for this $(2f_1 - f_2)$ component.

I. INTRODUCTION

Electrical networks of the type shown in Fig. 1, together with sophisticated frequency-domain measurement techniques, play a central role in the modeling and analysis of the peripheral auditory system.[1-9] In the figure—which shows a one-dimensional lumped-element transmission-line model of the basilar membrane—the inductors and capacitors are linear, the box at the upper left contains

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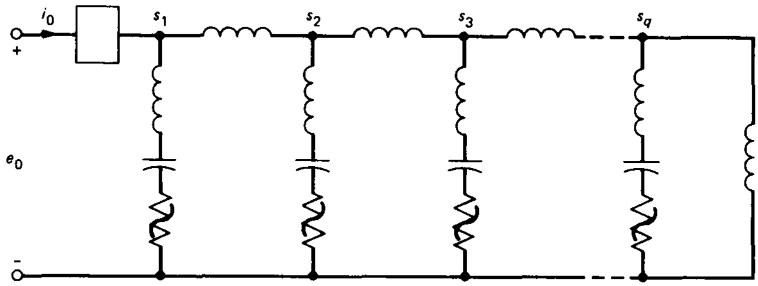

Fig. 1—Network model.

lumped elements, and, as indicated, the resistors are nonlinear. The voltage e_0 applied to the network is typically a finite sum of sinusoids (often a sum of just two sinusoids) switched on at some finite time that we take to be $t = 0$. The resulting quantities of interest, because of their interpretation as analogs of experimental observables, are the steady-state response components of the current i_0 and of one or more of the voltages $s_1, \cdots, s_q$.

In models of interest today the number q of nonlinear resistors is typically taken to be between 200 and 500. The resistors are assumed to be current controlled, with each current-voltage relationship often represented by the sum of linear and cubic terms.[1-3]

The purpose of this paper is to use recently obtained results[10] concerning the input-output representation of nonlinear systems to give, for the first time, an expansion for all of the steady-state quantities of interest in Fig. 1. The expansion is in terms of e_0 and is locally convergent. By this we mean that whenever the sum of the Fourier coefficients of e_0 is sufficiently small, and some reasonable additional conditions are met, the steady-state quantities exist and are given by the sum of the terms in the expansion, with each term dependent on the frequencies and Fourier coefficients of e_0. We emphasize that the expansion provides an exact representation of the response; it is not merely an approximation or a formal expansion whose convergence has not been proved. However, in this paper we do not give lower bounds on the *size* of the region of convergence. Questions of this type are the subject of ongoing studies.[11]

In Section II it will become clear that the terms in the expansion are defined by a certain recursive process. Of particular practical interest at the present time is the term we call the third-order term, which contains a component whose (radian) frequency is $(2\omega_1 - \omega_2)$ when e_0 consists of a sum of two sinusoids, one of frequency ω_1, and another of frequency ω_2. One of our main results is an explicit expression for this $(2\omega_1 - \omega_2)$ component, under some very reasonable assumptions.

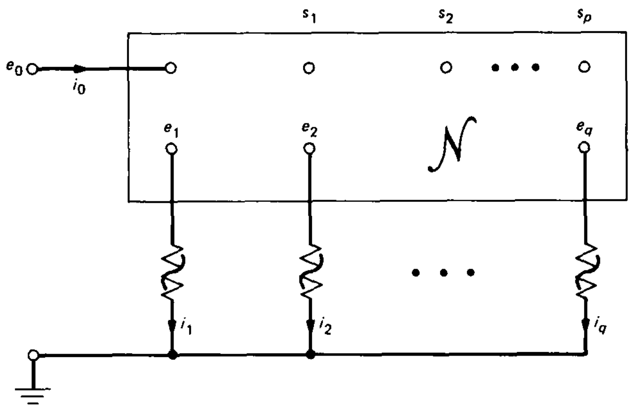

Fig. 2—More general network model.

II. EXISTENCE, PROPERTIES, AND EVALUATION OF THE STEADY-STATE QUANTITIES

2.1 Formulation

To enable attention to be more sharply focused on the concepts of importance to us, it is helpful to generalize our problem. Thus, we consider instead of Fig. 1 the network of Fig. 2, in which $\mathcal{N}$ is a linear time-invariant network and $s_1, \cdots, s_p$ are voltages in $\mathcal{N}$, measured with respect to the ground terminal, where p is any positive integer.

Let i and e, respectively, denote the transpose of the current and voltage row vectors $(i_1, \cdots, i_q)$ and $(e_1, \cdots, e_q)$. Assume that $\mathcal{N}$ has the representation

$$i(t) = \int_0^t h_a(t - \tau)e_0(\tau)d\tau + \int_0^t h_c(t - \tau)e(\tau)d\tau + u_1(t), \qquad t \geqslant 0, \quad (1)$$

in which h_a and h_c are $q \times 1$ and $q \times q$ matrix-valued impulse response functions and u_1 (which takes into account initial conditions) is a bounded continuous function that approaches zero as $t \to \infty$.* Similarly, let r stand for the transpose of the response $(s_1, \cdots, s_p, i_0)$ and suppose that there are $(p + 1) \times 1$ and $(p + 1) \times q$ matrix-valued impulse response functions h_d and h_b, respectively, for which

* We could have assumed that u_1 and the transient functions u_2 and u_3 to be introduced are all zero functions. However, we wish to establish that the steady-state responses are *robust* with respect to these functions in the strong sense that, under the conditions to be described, they are independent of them.

$$r(t) = \int_0^t h_d(t - \tau)e_0(\tau)d\tau + \int_0^t h_b(t - \tau)e(\tau)d\tau + u_2(t), \qquad t \geqslant 0, \quad (2)$$

where u_2 is also a bounded continuous function that approaches zero as $t \longrightarrow \infty$.

Each element of h_a, h_b, h_c, and h_d is assumed to be an absolutely integrable real-valued function on $[0, \infty)$ with possibly an impulse at $t = 0$. We use H_a to denote the Fourier transform of h_a, i.e.,

$$H_a(\omega) = \int_0^\infty h_a(t)e^{-j\omega t}dt, \qquad -\infty < \omega < \infty.$$

Similarly, H_b, H_c, and H_d stand for the Fourier transforms of h_b, h_c, and h_d, respectively. Of course $H_a(\omega)$, $H_b(\omega)$, $H_c(\omega)$, and $H_d(\omega)$ are also matrices. Notice that, from (1) and (2), each of these matrices has a natural transfer-function interpretation. For example, from (1) we see that the elements of H_a are the voltage-to-current transfer functions from the system input e_0 to the "inputs" i of the nonlinear resistors, when these resistors are replaced with short circuits.

The nonlinear resistors in Fig. 2 are assumed to be represented by

$$e_k(t) = R_k[i_k(t)], \qquad (k = 1, \cdots, q) \qquad (3)$$

with each R_k an analytic function in some neighborhood of the origin of the complex plane, such that $R_k(z)$ is real when z is real, $R_k(0) = 0$, and $dR_k(z)/dz = 0$ at $z = 0$. (In particular, the R_k can be polynomials with real coefficients.) In Fig. 1 the nonlinear resistors typically have a relatively large linear part. These linear parts can be taken into account in Fig. 2 in $\mathscr{N}$. Using known properties of networks with positive elements, it is not difficult to show that the assumptions made above concerning h_a, h_b, h_c, h_d, u_1, and u_2 are satisfied for the network of Fig. 1 when put in the form of Fig. 2, as long as the linear part of each resistor has positive resistance, all linear elements are passive, the impedance of the two-terminal box is not zero at zero frequency, and each s_k in Fig. 2 is an s_k in Fig. 1.

2.2 Steady-state responses: properties and evaluation

We now assume that e_0 is given by

$$e_0(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t} + u_3(t), \qquad t \geqslant 0,$$

in which the sum of the $|a_k|$ is finite; $j = (-1)^{1/2}$; the ω_k are real; and u_3, like u_1 and u_2, is bounded, continuous, and approaches zero as $t \rightarrow \infty$. We do not require that the ω_k are multiples of some fixed constant.

Thus, the input is assumed to be the sum, for $t \geqslant 0$, of a so-called "almost-periodic" signal

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t}, \qquad -\infty < t < \infty \tag{4}$$

and a transient part u_3. Although all almost-periodic signals have a generalized Fourier series of the form (4), the sum of the magnitudes of the Fourier coefficients need not be finite. We shall use AP to denote the subset of almost-periodic functions for which this sum *is* finite.

At this point we are able to state our main result, which is: Under the assumptions already discussed, and for $\sum_{k=-\infty}^{\infty} |a_k|$ as well as u_1, u_2, and u_3 sufficiently small,*

1. There are unique bounded functions i, e, and r that satisfy eqs. (1), (2), and (3), and (regarding uniqueness) a certain very reasonable neighborhood condition† concerning i,

2. There is a $(p + 1)$-vector-valued function r_{ss} defined on $(-\infty, \infty)$, with each of its $(p + 1)$ components belonging to (AP), such that

$$r(t) - r_{ss}(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

(i.e., the response r approaches the steady state r_{ss} as $t \rightarrow \infty$), and

3. r_{ss} is independent of u_1, u_2, and u_3. It is given by

$$r_{ss}(t) = \sum_{m=1}^{\infty} [r_{ss}(t)]_m, \qquad -\infty < t < \infty, \tag{5}$$

in which the $[r_{ss}(\cdot)]_m$ are $(p + 1)$-vector-valued functions, with components belonging to AP, defined by

$$[r_{ss}(t)]_1 = \sum_{k=-\infty}^{\infty} H_d(\omega_k) a_k e^{j\omega_k t}$$

and

$$[r_{ss}(t)]_m = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_m=-\infty}^{\infty} \mathscr{R}_m(\omega_{k_1}, \cdots, \omega_{k_m}) a_{k_1} \cdots a_{k_m} e^{j(\omega_{k_1} + \cdots + \omega_{k_m})t} \tag{6}$$

for $m \geqslant 2$, where the $\mathscr{R}_m(\omega_{k_1}, \cdots, \omega_{k_m})$, which depend on H_a, H_b, H_c, and the derivatives of the R_k at the origin, but not on the coefficients a_k, are defined by the recursive relations (10), (11), and (12) in the Appendix. The infinite sum in (5) converges uniformly in t.

Notice that a fundamental property of the class of network models

* By "small" for u_1, u_2, and u_3 is meant small in the reasonable sense of the $\mathscr{B}_0$ norm of Ref. 10, p. 692.

† The condition is simply that the function i must lie in a certain neighborhood of the origin. See the first of the two footnotes in the Appendix.

considered is that, with excitations as indicated, each component of any steady-state response r_{ss} belongs to AP. In particular, the r_{ss} are well behaved; any r_{ss} is continuous in t and has a Fourier series, and the Fourier series converges to $r_{ss}(t)$ for each t.

It is shown in the Appendix that the result described above follows from the main theorem in Ref. 10. In addition, bounds in Ref. 10, Section 2.4.3 show that the following can be added to 1 through 3.

4. There are positive constants α and β such that, with $([r_{ss}(t)]_m)_k$ the kth component of $[r_{ss}(t)]_m$,

$$\sum_{m=(M+1)}^{\infty} \max_k |([r_{ss}(t)]_m)_k| \leq \alpha \left(\beta \sum_{k=-\infty}^{\infty} |a_k|\right)^{(M+1)}, \quad -\infty < t < \infty,$$

for any positive integer M [which provides useful information concerning the error in discarding all terms in (5) beyond the Mth].

2.3 The $(2\omega_1 - \omega_2)$ component of $[r_{ss}(t)]_3$

Each $[r_{ss}(t)]_m$ in (5) is of order m in the sense that the effect of multiplying all of the Fourier coefficients of e_0 by a constant λ is to cause $[r_{ss}(t)]_m$ to be replaced by $\lambda^m[r_{ss}(t)]_m$. Of particular interest in applications is an explicit expression for $T(\omega_1, \omega_2, a_1, a_2)$, the component at the frequency $(2\omega_1 - \omega_2)$ of the *third* order term in (5), when

$$e_0 = a_1 e^{j\omega_1 t} + a_{-1} e^{-j\omega_1 t} + a_2 e^{j\omega_2 t} + a_{-2} e^{-j\omega_2 t},$$

where a_{-1} and a_{-2} are the complex conjugates of a_1 and a_2, respectively, $0 < \omega_1 < \omega_2 < 2\omega_1$,* and $\alpha_k(l) = 0$ $(k = 1, \cdots, q)$ for $l = 2$, and where here and in the Appendix $\alpha_k(l)$ denotes $d^l R_k(z)/dz^l|_{z=0}$.

Under the condition on the α_k (2) indicated, the expression (12) for the $\mathscr{R}_m$ yields

$$\mathscr{R}_3(\omega_{k_1}, \omega_{k_2}, \omega_{k_3}) = \frac{1}{6} H_b(\omega_{k_1} + \omega_{k_2} + \omega_{k_3})\text{diag}[\alpha_1(3), \cdots, \alpha_q(3)]$$

$$\cdot\hat{\chi}[H_a(\omega_{k_1}), H_a(\omega_{k_2}), H_a(\omega_{k_3})],$$

where "diag" indicates a diagonal matrix and $\hat{\chi}[H_a(\omega_{k_1}), H_a(\omega_{k_2}), H_a(\omega_{k_3})]$ denotes the q-vector whose kth element is the product $[H_a(\omega_{k_1})]_k[H_a(\omega_{k_2})]_k[H_a(\omega_{k_3})]_k$ of kth elements for each k. Thus, using (6) with $m = 3$, $a_0 = 0$, and $a_k = 0$ for $|k| > 2$, as well as the observation that $(\omega_{k_1} + \omega_{k_2} + \omega_{k_3}) = (2\omega_1 - \omega_2)$ only if one of the ω_{k_l} is $-\omega_2$ and

* For ω_1 and ω_2 that meet these conditions, $(2\omega_1 - \omega_2)$ is not equal to ω_1, ω_2, $3\omega_1$, $3\omega_2$ or $(2\omega_2 - \omega_1)$, which are the only other positive frequencies at which $[r_{ss}(t)]_3$ can have components. However, higher-order terms of odd index *can* possess components at $(2\omega_1 - \omega_2)$. For example, $(\omega_{k_1} + \cdots + \omega_{k_5}) = (2\omega_1 - \omega_2)$ if $\omega_{k_1} = \omega_{k_2} = \omega_1$, $\omega_{k_3} = -\omega_2$, and $\omega_{k_4} + \omega_{k_5} = 0$.

the other two are equal to ω_1, it easily follows that the sum of the coefficients of $\exp[j(2\omega_1 - \omega_2)t]$ in $[R_{ss}(t)]_3$ is

$$\frac{1}{2} H_b(2\omega_1 - \omega_2)\mathrm{diag}[\alpha_1(3), \cdots, \alpha_q(3)]\hat{\chi}[H_a(\omega_1), H_a(\omega_1), H_a(-\omega_2)]a_1^2 a_{-2}.$$

This shows that

$$T(\omega_1, \omega_2, a_1, a_2) = \mathrm{Re}\{H_b(2\omega_1 - \omega_2)\mathrm{diag}[\alpha_1(3), \cdots, \alpha_q(3)] \cdot\hat{\chi}[H_a(\omega_1), H_a(\omega_1), H_a(-\omega_2)]a_1^2 a_{-2}\exp[j(2\omega_1 - \omega_2)t]\},$$

where Re{ } stands for the real part of { }. Since H_a and H_b have a direct interpretation in terms of the structure of the network of Fig. 2, so does $T(\omega_1, \omega_2, a_1, a_2)$.

As a matter of convenience we have chosen to let $\mathcal{N}$ take into account the linear parts of the nonlinear resistors. We could have assumed instead that $\mathcal{N}$, without these linear parts, has sufficient damping that our conditions on h_a, h_b, h_c, h_d, u_1, and u_2 are satisfied. Under some very reasonable assumptions (see Ref. 10, comments on p. 694 concerning H.3), our expression for $T(\omega_1, \omega_2, a_1, a_2)$ would then explicitly exhibit its dependence on these linear parts, and this may be of interest in some cases. It can be shown, using a result in Ref. 10, Section 2.4.3, that the alternative expression for $T(\omega_1, \omega_2, a_1, a_2)$ that we would have obtained is

$$\mathrm{Re}\{H_b(2\omega_1 - \omega_2)F(2\omega_1 - \omega_2)\mathrm{diag}[\alpha_1(3), \cdots, \alpha_q(3)] \cdot\hat{\chi}[E(\omega_1)H_a(\omega_1), E(\omega_1)H_a(\omega_1), E(-\omega_2)H_a(-\omega_2)] \cdot a_1^2 a_{-2}\exp[j(2\omega_1 - \omega_2)t]\},$$

in which, with 1_q the identity matrix of order q,

$$F(\omega) = \{1_q - \mathrm{diag}[\alpha_1(1), \cdots, \alpha_q(1)]H_c(\omega)\}^{-1},$$

and

$$E(\omega) = \{1_q - H_c(\omega)\mathrm{diag}[\alpha_1(1), \cdots, \alpha_q(1)]\}^{-1}$$

(with both inverses existing for $-\infty < \omega < \infty$).

2.4 Discussion

In this paper we have derived and discussed a general expansion for the response of a cochlear model having a nonlinear membrane. The nonlinearities of the model take into account the membrane's nonlinear damping. Of particular interest is the third-order term in the expansion for the case described in Section 2.3, in which the input is a sum of sinusoids at frequencies ω_1 and ω_2. This term is the first term in the expansion that gives rise to a component at the frequency $(2\omega_1 - \omega_2)$.

The expression for the third-order term is seen to depend on two transfer-function matrices H_a and H_b, where H_b relates the output response vector r to the voltages across the resistors in Fig. 2 under the condition that e_0 is zero, and H_a relates the currents through the resistors to the input voltage e_0 under the condition that the resistors are replaced by short circuits.

In the expression for T, the transfer function H_b is evaluated only at $(2\omega_1 - \omega_2)$. The function H_b has the interpretation that it corresponds to a filter that alters the distortion products after their generation on the basilar membrane.

The terms $\alpha_1(3), \cdots, \alpha_q(3)$ are measures of the generator strength of the nonlinear distortion as a function of position, in the sense that each $\alpha_k(3)$ is proportional to the coefficient of the cubic term in the power series expansion of the resistor function R_k. Cubic nonlinearities have been used previously in basilar membrane models to model the generation of distortion products.

The transfer function H_a enters the expression for T in a particularly interesting way. Notice that any element of T, say the lth, is the real part of

$$\sum_{k=1}^{q} [H_b(2\omega_1 - \omega_2)]_{lk}\alpha_k(3)[H_a(\omega_1)]_k^2[H_a(-\omega_2)]_k a_1^2 a_{-2} e^{j(2\omega_1-\omega_2)t},$$

which is a linear combination of q terms with, so to speak, H_a appearing three times in each term, twice for ω_1 and once for ω_2.

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APPENDIX

Proof of the Main Steady-State Response Result; Recursive Relations for the $\mathcal{R}_m$

Theorem 3 of Ref. 10 would be directly applicable to the network governed by (1), (2), and (3) if $h_a(t - \tau)$, $h_b(t - \tau)$, $h_c(t - \tau)$, and $h_d(t - \tau)$ were *square* matrices of the *same* size. Since this condition is not met, we proceed to construct a suitable related set of system equations.

Let $n = (q + p + 2)$, and define K_a, K_b, K_c, and K_d to be the convolution operators associated with h_a, h_b, h_c, and h_d, respectively. Let v, x, y, and w be given by $v = (e_0, u_1, u_2)^{tr}$, $x = (i, x^{[p+2]})^{tr}$, $y = (e, y^{[p+2]})^{tr}$, and $w = (r, w^{[q+1]})^{tr}$, where "tr" denotes transpose, and $x^{[p+2]}$, $y^{[p+2]}$, and $w^{[q+1]}$ are unspecified vector-valued functions (on $t \geqslant 0$) of the indicated dimensions. Notice that v, x, y, and w are all n-vector valued. Finally, let $\eta_k (k = 1, \cdots, n)$ be the functions defined by $\eta_k = R_k (1 \leqslant k \leqslant q)$, with η_k equal to the zero function for $(q + 1) \leqslant k \leqslant n$.

Consider the equations

$$x = Av + Cy \tag{7}$$

$$w = Dv + By \tag{8}$$

$$y = Nx \tag{9}$$

in which by (9) we mean $y_k(t) = \eta_k[x_k(t)]$ for each k and t, and in which A, C, D, and B are given in partitioned form by

$$A = \begin{pmatrix} K_a & I(q) & Z(q, p + 1) \\ Z(p + 2, 1) & Z(p + 2, q) & Z(p + 2, p + 1) \end{pmatrix},$$

$$C = \begin{pmatrix} K_c & Z(q, p + 2) \\ Z(p + 2, q) & Z(p + 2, p + 2) \end{pmatrix},$$

$$D = \begin{pmatrix} K_d & Z(p + 1, q) & I(p + 1) \\ Z(q + 1, 1) & Z(q + 1, q) & Z(q + 1, p + 1) \end{pmatrix},$$

and

$$B = \begin{pmatrix} K_b & Z(p + 1, p + 2) \\ Z(q + 1, q) & Z(q + 1, p + 2) \end{pmatrix},$$

where, for any positive integers q and s, $I(q)$ denotes the identity operator on the space of q-vector valued functions on $t \geq 0$, and $Z(q, s)$ is the zero operator from the space of s-vector-valued functions on $t \geq 0$ into the corresponding space of functions whose values are of dimension q.

We see that if (7), (8), and (9) are satisfied, then (1), (2), and (3) are met, and that if the latter set of equations are satisfied and $x^{[p+2]}$, $y^{[p+2]}$, and $w^{[q+1]}$ are zero functions, then (7), (8), and (9) are satisfied. Using the fact that A, B, C, D, and N meet the conditions of Theorem 3 of Ref. 10, it follows from that theorem that statements 1 and 2 of Section 2.2 hold.* It also follows from the theorem that r_{ss} is independent of u_1, u_2, and u_3, and using the relation $w = (r, w^{[q+1]})^{tr}$, that $r_{ss}(t)$ can be written in the form (5) with the components of each $[r_{ss}(\cdot)]_m$ elements of AP, with

$$[r_{ss}(t)]_1 = \sum_{k=-\infty}^{\infty} H_d(\omega_k) a_k e^{j\omega_k t}$$

and each $[r_{ss}(t)]_m$ for $m \geq 2$ specified as follows (after some straightforward analysis involving partitioned matrices).

With $c_1, c_2, \cdots$ arbitrary n-vectors, and $\beta_1, \beta_2, \cdots$ arbitrary real numbers, let q-vector-valued functions $Q_1, Q_2, \cdots$ and $(p + 1)$-vector-valued functions $P_2, P_3, \cdots$ be defined by $Q_1(c_1, \beta_1) = H_a(\beta_1)(c_1)_1$,

$$Q_m(c_1, \cdots, c_m, \beta_1, \cdots, \beta_m) = H_c(\beta_1 + \cdots + \beta_m)S_m$$

for $m \geq 2$, and

$$P_m(c_1, \cdots, c_m, \beta_1, \cdots, \beta_m) = H_b(\beta_1 + \cdots + \beta_m)S_m$$

for $m \geq 2$, in which†

$$S_m = \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+\cdots+k_l=m \\ k_j>0}} \operatorname{diag}[\alpha_1(l), \cdots, \alpha_q(l)]$$

$$\cdot\hat{\chi}[Q_{k_1}(c_1, \cdots, c_{k_1}, \beta_1, \cdots, \beta_{k_1}), \cdots, Q_{k_l}(c_{(m-k_l+1)}, \cdots, c_m, \beta_{(m-k_l+1)}, \cdots, \beta_m)],$$

$(c_1)_1$ is the first component of c_1,

$$\alpha_k(l) = \left.\frac{d^l R_k(z)}{dz^l}\right|_{z=0} \quad (k = 1, \cdots, q)$$

for each l, "diag" indicates a diagonal matrix, and $\hat{\chi}$ is defined by the

* The "neighborhood condition" of statement 1 is inherited from Ref. 10, part (iib) of Theorem 3 via the relationship between (1), (2), and (3) and (7), (8), and (9).

† In the expression for S_m, $\sum_{\substack{k_1+\cdots+k_l=m \\ k_j>0}}$ denotes a sum over all positive integers k_1, $\cdots$, k_l that add to m.

condition that $\hat{\chi}[c_1, \cdots, c_l]$ is the q-vector with kth element $(c_1)_k \cdots (c_l)_k (1 \leqslant k \leqslant q)$. In terms of these P_m, we have

$$[r_{ss}(t)]_m = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_m=-\infty}^{\infty} P_m(d_{k_1}, \cdots, d_{k_m}, \omega_{k_1}, \cdots, \omega_{k_m}) e^{j(\omega_{k_1}+\cdots+\omega_{k_m})t},$$

where $d_k = (a_k, 0, \cdots, 0)^{tr}$ for each k.

Observe that for any m and k with $1 \leqslant k \leqslant m$, each Q_m and each P_m is linear in c_k and independent of $(c_k)_l$ for $l \geqslant 2$. Thus, each $Q_m(c_1, \cdots, c_m, \beta_1, \cdots, \beta_m)$ is equal to $Q_m(u, \cdots, u, \beta_1, \cdots, \beta_m)(c_1)_1 \cdots (c_m)_1$, where $u = (1, 0, \cdots, 0)^{tr}$, and similarly for the P_m. Therefore, if $\mathscr{S}_m$ and $\mathscr{R}_m$ are defined by

$$\mathscr{S}_1(\beta_1) = H_a(\beta_1), \tag{10}$$

$$\mathscr{S}_m(\beta_1, \cdots, \beta_m) = H_c(\beta_1 + \cdots + \beta_m) \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+\cdots+k_l=m \\ k_j>0}} \operatorname{diag}[\alpha_1(l), \cdots, \alpha_q(l)] \cdot \hat{\chi}[\mathscr{S}_{k_1}(\beta_1, \cdots, \beta_{k_1}), \cdots, \mathscr{S}_{k_l}(\beta_{(m-k_l+1)}, \cdots, \beta_m)] \tag{11}$$

for $m \geqslant 2$, and

$$\mathscr{R}_m(\beta_1, \cdots, \beta_m) = H_b(\beta_1 + \cdots + \beta_m) \cdot \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+\cdots+k_l=m \\ k_j>0}} \operatorname{diag}[\alpha_1(l), \cdots, \alpha_q(l)] \cdot \hat{\chi}[\mathscr{S}_{k_1}(\beta_1, \cdots, \beta_{k_1}), \cdots, \mathscr{S}_{k_l}(\beta_{(m-k_l+1)}, \cdots, \beta_m)] \tag{12}$$

for $m \geqslant 2$, we have

$$[r_{ss}(t)]_m = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_m=-\infty}^{\infty} \mathscr{R}_m(\omega_{k_1}, \cdots, \omega_{k_m}) a_{k_1} \cdots a_{k_m} e^{j(\omega_{k_1}+\cdots+\omega_{k_m})t}, \quad -\infty < t < \infty$$

for $m = 2, 3, \cdots$. This completes the Appendix.

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