REPORT

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OPTIMALLY REHOMING LOCAL SERVING OFFICES TO A NEW POINT OF PRESENCE

Introduction

The agreement that governed AT&T's divestiture of the Bell Operating Companies separated the ownership of the local and long-haul portions of the public telephone network and required AT&T to designate specific nodes of its network as the sole points of interface with the local network. AT&T must also choose the assignment of the local serving offices to these points of interface, otherwise known as *points of presence* (POPs).

The local telephone companies bill AT&T when long-distance calls use the local networks to gain access to the AT&T network. One component of these "access" charges is linear in the traffic volume, and the unit charge depends on the distance between the local office and the point of presence. Thus in choosing the locations of its points of presence, AT&T must consider the trade-off between access charges and its own network costs.

In this paper we develop an approach to quantifying the benefits of building a new point of presence in a specific location. To this end, several simplifying assumptions are made about the way traffic that is reassigned or "rehomed" to the new point of presence is routed. We then formulate the problem of finding the optimal set of local serving offices to rehome to a new point of presence and give a computationally efficient algorithm to solve it. Although restrictive, these assumptions are reasonable approximations for a screening tool whose purpose is to identify promising point-of-presence locations for detailed study. They allow the costs and benefits attributable specifically to the new location's role as a point of presence to be identified. If other goals are important, such as capacity relief for existing switches or facilities, then a more complex fundamental traffic planning problem must be solved using planning tools designed for that purpose.

Assumptions and Notation

To begin with, if a local serving office is rehomed to the new POP, we assume that its traffic is routed from the new point of presence to the point of presence to which it was formerly assigned, and then out to the long-haul network exactly as before (Figure 1). Therefore the only change in the original network is a new facility connecting the old POP and the new POP.

Second, the new point of presence is assumed to be a *facility* POP that can cross-connect trunks but cannot switch individual calls, with the number of trunks terminating on the long-haul side of the POP equal to the number of trunks terminating on the local side. (The latter assumption is strictly true only if the traffic from the LSOs is not concentrated further at a local "tandem" switch before arriving at the POP.) Alternatively, we may assume that the new POP is a switch, provided all local serving offices that are candidates for rehoming have the same busy hour. The more general problem in which the busy hours may differ is much harder and is not considered here.

Let E represent the set of local offices that are candidates for rehoming to the proposed point of presence, and let $x_i > 0$

represent the net present value of the access charge savings realized by rehoming office $i \in E$ to the proposed POP. If the new POP is merely a cross-connect point, let d_i represent the number of trunks needed to carry the busy hour load offered by office i; if the new POP is a switch, then d_i represents the busy hour load, in, say, erlangs. Define

$$w(S) \equiv \sum_{i \in S} d_i, S \subseteq E$$

The function $f(D)$ gives the cost of building a facility of capacity D between the existing POP and the proposed POP. Since economies of scale are characteristic of telecommunications facilities, f is assumed to be nondecreasing and concave for positive capacities, while we take $f(0) = 0$.

If the new POP is a cross-connect node, then the capacity D of the new facility is measured in trunks. In this case, f is piecewise linear when a finite set of technologies is available and each technology is characterized by a fixed cost and a constant incremental cost per trunk. If the new POP is a switch, then D is expressed in erlangs or some other measure of offered load. In this case, f captures not only the economies of scale attributable to telecommunications facilities, but also the fact that the marginal capacity of a trunk group increases as offered load increases (given a fixed blocking probability).[1]

With the above notation and assumptions, the total cost associated with rehoming the set of serving offices $S \subseteq E$ to the new point of presence can be expressed as

$$Z(S) \equiv f(w(S)) - \sum_{i \in S} x_i \qquad (1)$$

AT&T's objective in choosing the new POP's service region S is to minimize $Z(S)$.

An Optimal Rehoming Algorithm

The optimal service region is found by the following algorithm. Its complexity is determined by the complexity of the sorting algorithm used in the first step, and hence is $O(|E| \log |E|)$. (See Reference 2 for an explanation of the $O(\cdot)$ notation).

Rehoming Algorithm.

1. Order the serving offices so that $x_{i1} / d_{i1} \geqslant x_{i2} / d_{i2} \geqslant \cdots \geqslant x_{i|E|} / d_{i|E|}$. Define $S_0 \equiv \varnothing$ and $S_k \equiv \{i_1, \ldots, i_k\}$, $1 \leqslant k \leqslant |E|$. Set $n = 0$ and $k = 1$.
2. If $Z(S_k) \leqslant Z(S_n)$, set $n = k$.
3. Set $k \leftarrow k + 1$. If $k > |E|$, stop. Else go to (2).

We claim that when the algorithm terminates, S_n is an optimal service region. This is an immediate consequence of the following lemma.

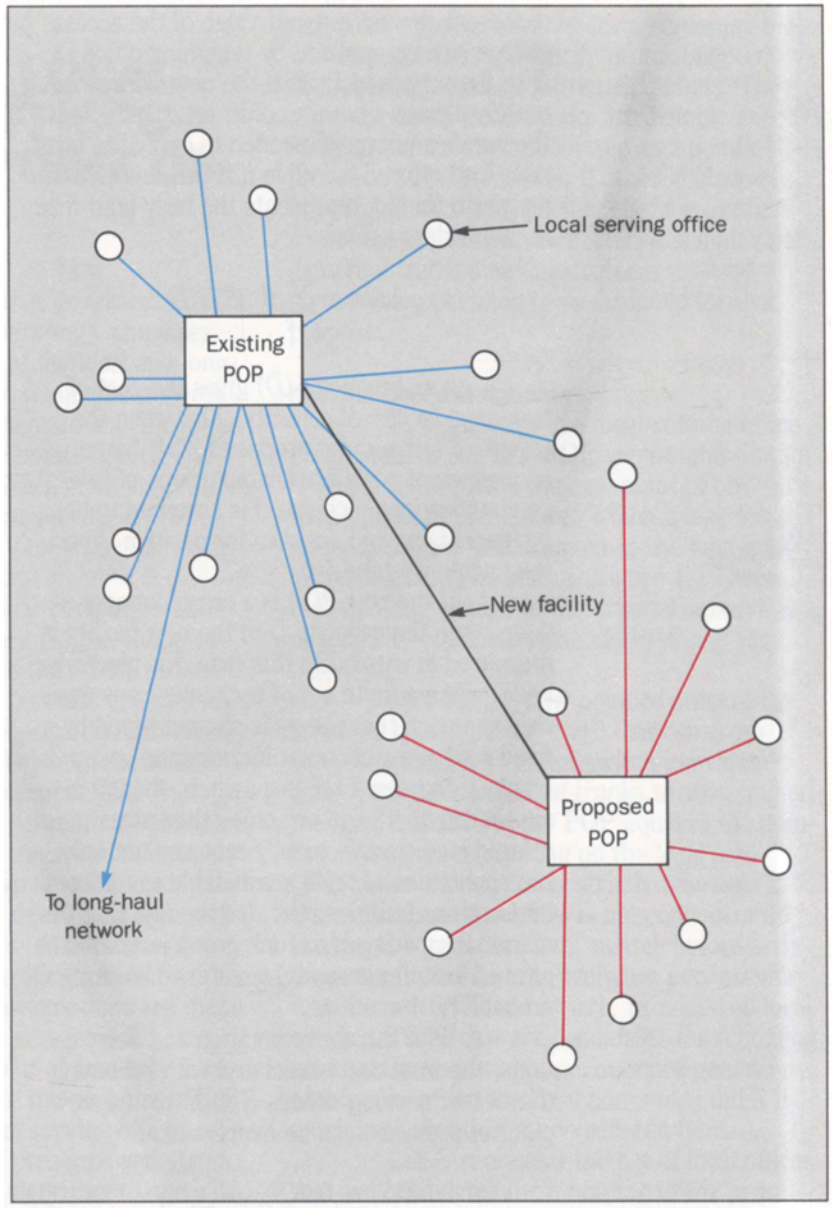
Lemma. Let $x_{i1} / d_{i1} \geqslant x_{i2} / d_{i2} \geqslant \cdots \geqslant x_{i|E|} / d_{i|E|}$, and suppose there exists a nonempty optimal set. Then for some n, $1 \leqslant n \leqslant |E|$, $S_n \equiv \{i_1, \ldots, i_n\}$ is optimal.

Proof. First, note that by the concavity of f,

(a) $x \leqslant y$ if and only if $f(x + z) - f(x) \geqslant f(y + z) - f(y)$, $x, y \geqslant 0$, $z > 0$.

(b) If $0 \leqslant x < z < y$, then $(f(y) - f(z))/(y - z) \leqslant (f(y) - f(x)) / (y - x) \leqslant (f(z) - f(x)) / (z - x)$.

Figure 1. Adding a new point of presence near an existing one.



Now define

$$T_1 \equiv \{i \in E: f(y + d_i) - f(y) < x_i \text{ for all } y \geqslant 0\}$$
$$T_2 \equiv \{i \in E: f(y + d_i) - f(y) = x_i \text{ for some } y \geqslant 0\}$$
$$D_i \equiv 0,\ i \in T_1$$
$$D_i \equiv \min\{y: f(y + d_i) - f(y) = x_i\},\ i \in T_2$$

Let S be optimal. Then it is easily verified using item (a) in the list above and the definition of D_i that

(c) $i \in S$ implies $D_i \leqslant w(S) - d_i$ (since we must have $Z(S) - Z(S - \{i\}) = f(w(S)) - f(w(S) - d_i) - x_i \leqslant 0$).

(d) $D_i < w(S)$ implies $i \in S$ or that $S \cup \{i\}$ is optimal. (For if $i \notin S$ and $S \cup \{i\}$ is not optimal, we obtain the contradiction $Z(S \cup \{i\}) - Z(S) = f(w(S) + d_i) - f(w(S)) - x_i > 0$).

Now suppose $x_i / d_i \geqslant x_j / d_j$ and $j \in S$. First note that $i \in T_1 \cup T_2$, for otherwise we have $(f(y + d_i) - f(y)) / d_i > x_i / d_i \geqslant x_j / d_j \geqslant (f(D_j + d_j) - f(D_j)) / d_j$, and choosing $y > D_j + d_j$ contradicts (b). We must show that $i \in S$, or that $S \cup \{i\}$ is optimal. If $D_i \leqslant D_j$, the result follows from (c) and (d) because $D_i \leqslant D_j \leqslant w(S) - d_j < w(S)$. Similarly, if $D_j < D_i$ but $D_i + d_i \leqslant D_j + d_j$, the result again follows from (c) and (d) since $D_i \leqslant D_j + d_j - d_i \leqslant w(S) - d_i < w(S)$. Finally, consider the case in which $D_j < D_i$ and $D_j + d_j < D_i + d_i$. Then by (b) and the definition of D_i, $x_i / d_i = (f(D_i + d_i) - f(D_i)) / d_i \leqslant (f(D_j + d_j) - f(D_j)) / d_j \leqslant x_j / d_j$, implying that these inequalities actually hold with equality. It follows that $f(y)$ has constant slope equal to x_i / d_i for $D_j \leqslant y \leqslant D_i + d_i$. Now if $w(S) > D_i$, the desired result holds by item (d). Otherwise, $Z(S \cup \{i\}) - Z(S) = f(w(S) + d_i) - f(w(S)) - x_i = d_i (x_i / d_i) - x_i = 0$, so that $S \cup \{i\}$ is also optimal.

Combinatorial Theory

The set function $Z(\cdot)$ in Equation (1) possesses the special property called submodularity.[3,4] That is, $Z(S) + Z(T) \geqslant Z(S \cup T) + Z(S \cap T)$; $S, T \subset E$. Submodularity in some sense is a generalization of the property of concavity. It arises in the POP location model discussed here because the efficiency of telecommunications facilities increases with capacity.

The set function $V(S) \equiv f(w(S))$ is a rank function because it is nondecreasing (that is, $V(S) \leqslant V(T)$ for every $\subset T \subseteq E$), submodular, and $V(\varnothing) = 0$. In their recent work on the greedy procedure for resource allocation problems, Federgruen and Groenevelt[5] consider the problem of testing whether a specific point x^0 is a member of a polymatroid

$$F \equiv \{x \in R^N : \sum_{i \in S} x_i \leqslant V(S), S \subseteq \{1, \dots, N\}\}$$

where by the definition of a polymatroid, $V(S)$ is a rank function. They refer to the special case where $V(S) \equiv f(w(S))$, and $f(\cdot)$ is a nondecreasing concave function with $f(0) = 0$, as the generalized symmetric case. With S_k defined as in the preceding section, they prove that in the generalized symmetric case, $x^0 \in F$ if and only if

$$\sum_{i \in S_k} x_i^0 \leqslant V(S_k)$$

for $k = 1, \ldots, N$. This result in turn can be used to prove the validity of the rehoming algorithm presented here.

Returning to the rehoming application, suppose that the proposed POP is to contain a switch, and let d_i^t denote the ith serving office's offered load in the tth hour, $t \in T$. Again the problem of interest is to choose a set $S \subseteq E$ to minimize

$$V(S) - \sum_{i \in S} x_i$$

but $V(S)$ here is defined as

$$f(\max\{\sum_{i \in S} d_i^t : t \in T\})$$

Not only has this problem lost the simple generalized symmetric structure described above, it is easily verified that $V(S)$ is not even submodular.

Conclusions and Acknowledgments

The savings achievable by establishing a new point of presence depends on the set of end offices that is reassigned to it. Therefore, any POP planning tool must contain a rehoming scheme. The model and a special case of the algorithm described here have been integrated into AT&T Communications' regional planning tool PET (POP Evaluation Tool), replacing naive strategies that reassign an end office to the proposed POP if it is closer to the new POP than to the old, or if rehoming the end office produces any access savings at all. The quality of POP planning methods has thus been improved.

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References

1. R. F. Rey, ed., *Engineering and Operations in the Bell System*, second edition, AT&T Bell Laboratories, Murray Hill, New Jersey, 1983.
2. Christos H. Papadimitriou and Kenneth Steiglitz, *Combinatorial Optimization: Algorithms and Complexity*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1982.
3. M. Grötschel and L. Lovász, "The Ellipsoid Method and Its Consequences in Combinatorial Optimization," *Combinatorica*, Vol. 1, No. 2, 1981, pp. 169-197.
4. G. L. Nemhauser and L. A. Wolsey, "Best Algorithms for Approximating the Maximum of a Submodular Set Function," *Mathematics of Operations Research*, Vol. 3, No. 3, August 1978, pp. 177-188.
5. A. Federgruen and H. Groenevelt, "The Greedy Procedure for Resource Allocation Problems: Necessary and Sufficient Conditions for Optimality," *Operations Research*, Vol. 34, No. 6, November-December 1986.

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