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Trunk Implementation Plan for Hierarchical Networks

By A. N. KASHPER* and G. C. VARVALOUCAS*

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The Trunk Implementation Plan (TIP) is a multiyear schedule of planned trunk augments and disconnects that minimizes the impact of varying demand and forecast uncertainties on the cost of implementing a network meeting objective service criteria. This paper presents a theoretical development of the TIP algorithm for hierarchical networks that accounts for modularity, facility, and demand servicing constraints. First, we solve the only-route TIP problem analytically using stochastic dynamic programming techniques. We show that our analytical solution yields a numerically efficient algorithm for calculating a multiyear, minimum-cost policy. Second, we present the results of our analysis showing that, in the presence of forecast uncertainty, a near-optimal traffic network is obtained by introducing reserve capacity on the final trunk groups only. Based on this result, we construct the TIP algorithm for hierarchical networks by combining conventional network engineering principles with an optimal disconnect policy for high-usage trunk groups and the only-route TIP sizing procedure for final groups. Using this algorithm we obtain an economical multiyear schedule of trunk augments and disconnects for hierarchical networks.

I. INTRODUCTION

1.1 Background and motivation

In the Bell operating companies and AT&T Communications, the trunk forecasting process consists of: (1) traffic measurement and offered load estimation, (2) projection of future traffic demands, and

* AT&T Bell Laboratories.

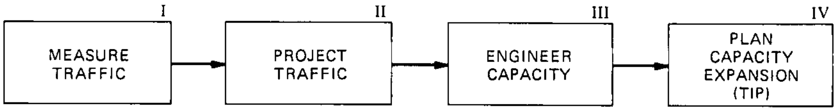

Fig. 1—Trunk network forecasting process.

(3) determination of the trunk group sizes for each of five or more future years. Currently, the engineering procedure utilized in (3) is based on independent, single-year network designs, each of which minimizes the cost of satisfying anticipated demands for a given future year.

Although substantial work has been done to improve the quality of the trunk forecasting process,[1-3] the existing methods do not account for several important implementation considerations, namely, the existing trunk network, the variation of trunk demand from year to year, uncertainty of demand forecasts, economics of maintaining or rearranging trunks, and facility constraints. Consequently, in practice, the output of (3) is modified by trunk forecasters to make the final multiyear schedule of planned trunk augments and disconnects feasible and economically sensible. The adjustments to the mechanized forecasting process are based on heuristic trunk disconnect guidelines and engineering judgment. However, no quantitative attempt is made to find an optimal multiyear trunk provisioning plan.

Accordingly, we identified the need for a mechanized system that would compute an economical capacity expansion plan for the trunk network. As Fig. 1 illustrates, the capacity expansion planning can be regarded as the fourth major function of the trunk forecasting process. The new mechanized system, called the Trunk Implementation Plan (TIP), will provide a multiyear schedule of trunk augments and disconnects that accounts for facility constraints while minimizing the impact of forecast uncertainty and demand dynamics on the cost of implementing a network meeting objecting service criteria.

In this work, we present a theoretical development of the TIP algorithm and generalize the mathematical model of Ref. 4 to reflect modularity and facility constraints. The problem formulation in Ref. 4 assumes a nonmodular engineering environment and no facility constraints; i.e., the multiyear schedule of trunk augments and disconnects is cost-effective under the assumption that the sizes of the trunk groups are nonnegative real numbers and that the cost of a trunk group is proportional to the number of trunks in the group.

However, the assumptions of nonmodular engineering do not hold, for example, for a new generation of digital terminals such as the Digital Carrier Trunk (DCT), used for the 1A *ESS*,* and the Digital

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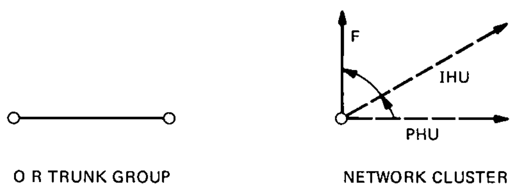
Interface Frame (DIF), used for *4ESS*. The DCT and DIF require that digital carriers [equivalent to 24 voice frequency (VF) circuits] terminate directly on the switch. The dedication of network facilities by destination implies that the cost of a trunk group has a per-module component in addition to the per-circuit component. Therefore, in Ref. 1, W. Elsner concluded that trunk groups terminating on such facilities should be modularly engineered. In particular, Ref. 1 shows that an engineering procedure that assumes only modular sizes (24 trunks for two-way groups) for high-usage and certain final trunk groups provides significant economic benefits; see Fig. 2 for definitions of trunking terminology.

In this paper, we replace the continuous TIP model of Ref. 4 by a discrete TIP formulation that incorporates modularity constraints. Then we derive an optimal modular network expansion policy for high-usage and final trunk groups. In addition, we show how to modify the TIP multiyear trunk-sizing procedures to reflect facility constraints.

1.2 Overview

Section II defines the notation and describes the mathematical model for the only-route TIP problem. In Section III we present a complete solution to the only-route TIP problem that accounts for modularity, facility, and demand servicing constraints. Section IV shows how to combine conventional network engineering principles



HU - HIGH-USAGE GROUP–DESIGNED TO OVERFLOW TO AN ALTERNATE ROUTE

PHU - PRIMARY HU GROUP–HU THAT DOES NOT RECEIVE OVERFLOW TRAFFIC

IHU - INTERMEDIATE HU GROUP–HU THAT RECEIVES OVERFLOW TRAFFIC

F - FINAL GROUP–LAST-CHOICE GROUP THAT IS ENGINEERED TO A SPECIFIC BLOCKING OBJECTIVE

O R - ONLY-ROUTE GROUP–F GROUP THAT RECEIVES ONLY FIRST ROUTE TRAFFIC

Fig. 2—Trunking terminology.

with the only-route TIP results to design an economical multiyear hierarchical network.

II. TIP MODEL FOR ONLY-ROUTE TRUNK GROUPS

We start our derivation of the TIP algorithm by considering the multiyear engineering problem for only-route trunk groups (see Fig. 2 for a definition of only-route trunk groups). As we discuss in Section 4.3, our solution of the only-route problem will be utilized to plan multiyear capacity expansion for hierarchical networks.

2.1 Mathematical model

2.1.1 Notation

First, we define the notation used in our mathematical model.

$T(k)$—number of trunks in service at the beginning of the kth year

$u(k)$—number of planned trunk augments/disconnects at the beginning of the kth year

$d(k)$—the maximum number of trunks (peak demand) in trunks during year k to guarantee the engineered blocking level

F_k—the distribution of the peak trunk demand during year k

$c_1^k(u(k))$—capital cost during year k

$c_2^k(u(k))$—labor cost during year k

$c_3^k(T(k), u(k))$—maintenance cost during year k

$c_4^k(d(k), T(k), u(k))$—underprovision cost during year k

N—number of years in the forecast horizon

We assume that the number of trunks in service, $T(k)$; the planned trunk level, $T(k) + u(k)$; and the peak demand, $d(k)$; are expressed in modules of m trunks. In the message network, m is equal to 1, 12, or 24. Consequently, F_k is a discrete distribution function, defined in accordance with the established rounding rules for engineering modular final groups.[1] That is, if F_k' is a continuous distribution of the peak demand, then F_k is obtained by

$$F_k(m\ell) = F_k'(m\ell + t_\ell),$$

where ℓ is a nonnegative integer, and t_ℓ is a rounding threshold, $0 < t_\ell < m$.

2.1.2 Trunk group dynamics

According to the AT&T practice, if the blocking objective on an only-route or alternate final trunk group is violated significantly, then the trunk group is augmented during the year on an emergency basis (demand servicing) to restore the engineered blocking level. Therefore, the number of trunks in service at the beginning of the $(k + 1)$th year is the sum of the planned trunk level for year k and the demand servicing augmentation, if any, during year k. Thus, the trunk group dynamics that reflect the planned and demand servicing components of the trunk provisioning process are modeled by

$$T(k+1) = [T(k) + u(k)] + \max[0, d(k) - (T(k) + u(k))]$$
$$= \max[y(k), d(k)], \qquad (1)$$

where $y(k) = T(k) + u(k)$ represents the planned trunk level at year k.

2.1.3 Objective function

The goal of the only-route TIP is to minimize the expected present worth of trunk provisioning costs. If we denote the present worth of the total cost for year k by $g_k(d(k), T(k), u(k))$, then the TIP objective function can be expressed as

$$\min_{\mathbf{u}} J(\mathbf{u}) = \min_{\mathbf{u}} E \left\{ \sum_{i=0}^{N-1} g_i[d(i), T(i), u(i)] \right\}, \qquad (2)$$

where $\mathbf{u} = (u(0), \cdots, u(N-1))$ and the expected value is taken over the demands $d(0), d(1), \cdots, d(N-1)$.

The present worth of trunk provisioning costs at year k is equal to

$$g_k(d(k), T(k), u(k)) = \rho^k[c_1^k(u(k)) + c_2^k(u(k))$$
$$+ c_3^k(T(k), u(k)) + c_4^k(d(k), T(k), u(k))], \qquad (3)$$

where ρ is the discount factor ($\rho < 1$) that measures the worth of the next year's dollars in terms of present dollars.

The capital, labor, maintenance, and underprovision costs are assumed to be piecewise linear with respect to modules of m trunks and are defined for $k = 0, \cdots, N-1$ by

$$c_1^k(u(k)) = \begin{cases} a_1^k u(k) & u(k) \geq 0 \\ b_1^k u(k) & u(k) < 0, \end{cases} \qquad (4)$$

$$c_2^k(u(k)) = \begin{cases} a_2^k u(k) & u(k) \geq 0 \\ -b_2^k u(k) & u(k) < 0 \end{cases} \qquad (5)$$

$$c_3^k(T(k), u(k)) = a_3^k y(k), \qquad (6)$$

$$c_4^k(d(k), T(k), u(k)) = a_4^k \max(0, d(k) - y(k)), \qquad (7)$$

where $y(k) = T(k) + u(k)$ is a planned trunk level at year k.

Equation (7) states that if the peak demand $d(k)$ exceeds the planned trunk level $y(k)$ during year k, then the cost of providing the additional trunks is proportional to the trunk shortage; if $d(k)$ does not exceed the planned level, then no cost is incurred. Thus, the underprovision cost reflects the demand servicing policy as described by the trunk-group dynamics equation (1). The assumption that the underprovision cost is linear with respect to the trunk shortage will be discussed in Section 2.1.4.

We assume that in (4) through (7) the per-trunk costs $a_1^k, \cdots, a_4^k$ and b_1^k, b_2^k are nonnegative and satisfy the conditions

$$a_1^k + a_2^k > b_1^k - b_2^k, \tag{8a}$$

$$b_1^k - b_2^k > 0, \tag{8b}$$

$$a_1^k + a_2^k > \rho(a_1^{k+1} + a_2^{k+1}), \tag{9a}$$

$$b_1^k - b_2^k > \rho(b_1^{k+1} - b_2^{k+1}), \tag{9b}$$

$$a_4^k > a_1^k + a_2^k + a_3^k. \tag{10}$$

Inequalities (8a) and (8b) state: first, that the cost of buying and installing a trunk module is always greater than its salvage value minus the disconnect expense; and second, that there is always an incentive for disconnecting a trunk module. Inequalities (9a) and (9b) show that it is uneconomical to augment a trunk group if not necessary and also uneconomical to delay the disconnect decision. Finally, (10) reflects the fact that it is always more costly to augment a trunk group on an emergency, rather than on a planned, basis.

2.1.4 Demand servicing constraints

In general, the underprovision cost or, equivalently, the unsatisfied demand penalty cost, involves all of the costs of planned servicing: capital, labor, and maintenance, plus a penalty due to the fact that demand servicing cannot be carried out with the normal planning intervals and orderly procedures associated with planned servicing. It is important to note that, in practice, the incremental underprovision cost depends on a variety of factors such as switch, trunk, and/or personnel availability. Typically, if the existing personnel and facilities can satisfy the emergency servicing need, then the cost of underprovisioning is marginally higher than the cost of planned trunk augmentation and can be computed by (7). However, if there is a shortage of personnel or facilities, demand servicing becomes much more expensive (than planned servicing) and highly undesirable.

Thus, to complete the problem formulation we need to specify that the feasible solutions in the TIP model correspond to a level of demand servicing not greater than an allowable threshold. In particular, we require that the expected amount of demand servicing at year k not exceed a specified level

$$E\{\max(0, d(k) - T(k) - u(k))\} \leq \beta_k E d(k),$$

where $k = 0, \cdots, N - 1$ and β_k is a given constant.

2.1.5 Mathematical model

The TIP problem can be viewed as a sequential stochastic decision process. The state of the system, the number of trunks in service, varies according to the trunk group dynamics given in eq. (1). At each state of the process the cost function $g_k(d(k), T(k), u(k))$ is defined via (4) through (7). The problem then is: given the initial trunk level $T(0)$ and future peak demand distributions $F_0, F_1, \cdots, F_{N-1}$, find a set of decisions (augments, disconnects), $\mathbf{u}^* = \{u^*(0), \cdots, u^*(N-1)\}$, the optimal policy, that minimizes the total expected trunk provisioning cost over N years

$$J(\mathbf{u}^*) = \min_{\mathbf{u}} E\left\{\sum_{k=0}^{N-1} g_k(d(k), T(k), u(k))\right\} \tag{11}$$

subject to possible capacity limitation conditions

$$0 \le T(k) + u(k) \le \gamma(k) \tag{12}$$

and demand servicing constraints

$$E\{\max(0, d(k) - T(k) - u(k))\} \le \beta_k Ed(k), \tag{13}$$

where $k = 0, 1, \cdots, N-1$ and $\gamma(k)$ are given modular thresholds that represent facility constraints.

III. SOLUTION FOR ONLY-ROUTE TRUNK GROUPS

Throughout the rest of this paper, we assume that the random variables $d(k)$ are statistically independent. We shall prove that under this assumption the optimal control law to the only-route TIP problem of Section 2.1.5 is defined by N pairs of scalars $(\underset{\sim}{S}(0), \tilde{S}(0)), \cdots, (\underset{\sim}{S}(N-1), \tilde{S}(N-1))$. The pair $(\underset{\sim}{S}(k), \tilde{S}(k))$ provides two critical levels for year k. Specifically, at year k, the optimal control law is to augment the number of trunks up to the level $\underset{\sim}{S}(k)$, to maintain the trunk level if $\underset{\sim}{S}(k) \le T(k) < \tilde{S}(k)$, or to disconnect down to the level $\tilde{S}(k)$; that is, for $k = 0, 1, \cdots, N-1$ the optimal decision $u_k^* = u_k^*(T(k))$ is given by

$$u_k^*(T(k)) = \begin{cases} \underset{\sim}{S}(k) - T(k) & \text{if} \quad T(k) < \underset{\sim}{S}(k) \\ 0 & \text{if} \quad \underset{\sim}{S}(k) \le T(k) < \tilde{S}(k) \\ \tilde{S}(k) - T(k) & \text{if} \quad T(k) \ge \tilde{S}(k). \end{cases} \tag{14}$$

We note that the independence assumption for future demands is critical for obtaining an analytical solution to the only-route TIP problem. However, our numerical experience shows that cost-effectiveness of the TIP solution as compared to currently used heuristic trunk provisioning strategies is not sensitive to this assumption.

We start our derivation by considering the TIP formulation that ignores demand servicing constraints. Specifically, Sections 3.1 and 3.2 present a complete solution to the modular only-route TIP problem described by (11) and (12). A similar nonmodular capacity management problem is considered in Ref. 5. In Section 3.3, we generalize this solution and obtain a numerically efficient algorithm that computes an optimal policy under the constraint on the level of demand servicing.

3.1 Optimality conditions

Since the demands are assumed to be independent, the TIP objective function (2) can be transformed to the form[6]

$$\min_{\mathbf{u}} J(\mathbf{u}) = \min_{u(0)} \mathop{E}_{d(0)}\{g_o(d(0), T(0), u(0)) + \min_{u(1)} \mathop{E}_{d(1)}\{\cdots + \min_{u(N-1)} \mathop{E}_{d(N-1)} \{g_{N-1}(d(N-1), T(N-1), u(N-1))\} \cdots\}\}.$$

To convert the right-hand side into a recursive form, we introduce the optimal cost-to-go function for the year k, $V_k(T(k))$, that is, the minimum expected cost over all possible strategies for years k, $k+1, \cdots, N-1$ that assume $T(k)$ trunks in service at the beginning of year k. Then, by Bellman's principle of optimality,[6] the optimal policy in year k is obtained by solving the backward dynamic programming recursion:

$$V_k(T(k)) = \min_{u(k)} \mathop{E}_{d(k)} \{g_k(d(k), T(k), u(k)) + \rho V_{k+1}(\max(d(k), T(k) + u(k)))\}, \quad (15)$$

where $V_N(T(N)) = 0$ and the minimum is taken over all modular $u(k)$ such that the planned trunk level at year k satisfies the condition

$$0 \le T(k) + u(k) \le \gamma(k). \quad (16)$$

The sum of trunk provisioning costs for year k is composed of two functions $g_{k,1}(\cdot)$ and $g_{k,2}(\cdot)$ defined by

$$g_k(d(k), T(k), u(k)) = \begin{cases} g_{k,1}(d(k), T(k), u(k)), & u(k) \ge 0 \\ g_{k,2}(d(k), T(k), u(k)), & u(k) < 0 \end{cases}$$

$$= \begin{cases} a_1^k u(k) + a_2^k u(k) + a_3^k(T(k) + u(k)) \\ \quad + a_4^k \max(d(k), T(k) + u(k)), & u(k) \ge 0 \\ b_1^k u(k) - b_2^k u(k) + a_3^k(T(k) + u(k)) \\ \quad + a_4^k \max(d(k), T(k) + u(k)), & u(k) < 0. \end{cases} \quad (17)$$

In what follows, we consider $g_k(\cdot)$ as a function of $d(k)$, $T(k)$, and

$y(k) = T(k) + u(k)$. Then, from the optimality principle and (17) the year k cost-to-go function is given by

$$J_k(T(k), y(k)) = \begin{cases} J_{k,1}(T(k), y(k)), & y(k) \geq T(k) \\ J_{k,2}(T(k), y(k)), & 0 \leq y(k) < T(k) \end{cases}$$

$$= \begin{cases} \underset{d(k)}{E}[g_{k,1}(d(k), T(k), y(k)) \\ \quad + \rho V_{k+1}(\max(d(k), y(k)))], \\ \qquad\qquad\qquad\qquad y(k) \geq T(k) \\ \underset{d(k)}{E}[g_{k,2}(d(k), T(k), y(k)) \\ \quad + \rho V_{k+1}(\max(d(k), y(k)))], \\ \qquad\qquad\qquad\qquad 0 \leq y(k) < T(k). \end{cases} \tag{18}$$

Note that from (17) $J_{k,1}(T, y)$ and $J_{k,2}(T, y)$ are well-defined functions for all $y = 0, m, 2m, \cdots$.

To prove that a solution of (18) is given by (14) we first develop sufficient conditions for the optimality of an $(\underset{\sim}{S}, \tilde{S})$ policy without capacity constraints ($\gamma = \infty$). We observe that $J_{k,1}(T, T) = J_{k,2}(T, T)$ for all $T \geq 0$. Accordingly, to prove the optimality of (14) it is sufficient to show that the solutions of the minimization problems

$$\min_{y \geq T \geq 0} J_1(T, y) \tag{19}$$

and

$$\min_{T \geq y \geq 0} J_2(T, y) \tag{20}$$

are defined by

$$y_1^* = \begin{cases} \underset{\sim}{S}, & T < \underset{\sim}{S} \\ T, & T \geq \underset{\sim}{S} \end{cases} \tag{21}$$

and

$$y_2^* = \begin{cases} \tilde{S}, & T > \tilde{S} \\ T, & T \leq \tilde{S}, \end{cases} \tag{22}$$

respectively, for some numbers $\underset{\sim}{S}$ and $\tilde{S}$ such that

$$0 \leq \underset{\sim}{S} \leq \tilde{S} < \infty.$$

To simplify the notation, we have dropped the index k in (19) through (22) and in the formulas in the rest of this section.

Consequently, the optimality of (21) and (22) is evident if there exist a pair of numbers $(\underset{\sim}{S}, \tilde{S})$, $\underset{\sim}{S} \leq \tilde{S}$ such that for any $T \geq 0$

$$\begin{cases} J_1(T, x) \geq J_1(T, y), & 0 \leq x \leq y < \underset{\sim}{S} \\ J_1(T, x) \leq J_1(T, y), & \underset{\sim}{S} \leq x \leq y \leq \infty \end{cases} \tag{23}$$

and

$$\begin{cases} J_2(T, x) \leq J_2(T, y), & \tilde{S} \leq x \leq y < \infty \\ J_2(T, x) \geq J_2(T, y), & 0 \leq x \leq y \leq \tilde{S}. \end{cases} \tag{24}$$

To demonstrate (23) and (24) and to construct $\underset{\sim}{S}$ and $\tilde{S}$ we shall prove an even stronger statement. Specifically, we consider the first differences of J_1 and J_2, defined by

$$\Delta J_1(T, y) = J_1(T, y + m) - J_1(T, y)$$

and

$$\Delta J_2(T, y) = J_2(T, y + m) - J_2(T, y),$$

and show that ΔJ_1 and ΔJ_2 satisfy the conditions:

1. $\Delta J_1(T, y) = \Delta J_1(y)$ and $\Delta J_2(T, y) = \Delta J_2(y)$,
2. $\Delta J_1(y)$ and $\Delta J_2(y)$ are nondecreasing, and
3. $0 \leq \underset{\sim}{S} \leq \tilde{S} < \infty$,

where

$$\underset{\sim}{S} = \min\{y \mid \Delta J_1(y) \geq 0,\ y = 0,\ m,\ \cdots\}$$

and

$$\tilde{S} = \min\{y \mid \Delta J_2(y) \geq 0,\ y = 0,\ m,\ \cdots\}. \tag{25}$$

Note that (25) defines the minimum points of J_1 and J_2, respectively. Furthermore, conditions (1) through (3) not only imply the optimality of (14) but also suggest that we can find the critical thresholds efficiently by applying a modular version of the bisection methods to the first differences of J_1 and J_2. The proof that $J_{k,1}$ and $J_{k,2}$ [defined by (18)] satisfy these conditions is given in Appendix A.

The generalization to the constrained case follows immediately. Because of the monotonicity of $\Delta J_1(y)$ and $\Delta J_2(y)$ the optimal solution under capacity constraints is given by

$$\underset{\sim}{S}^* = \min[\underset{\sim}{S}, \gamma]$$

and

$$\tilde{S}^* = \min[\tilde{S}, \gamma]. \tag{26}$$

3.2 Computational procedure

As we prove in Appendix A, an optimal modular TIP policy is described by N pairs of critical threshold levels $\{(\underset{\sim}{S}(0), \tilde{S}(0)), \cdots,$

$(\underset{\sim}{S}(N-1), \tilde{S}(N-1))\}$ and each pair $(\underset{\sim}{S}(k), \tilde{S}(k))$ can be computed by (25).

Consequently, to obtain an algorithm for calculating the optimal policy we shall derive backward recursions for $\Delta J_{k,1}(T(k) + u(k))$ and $\Delta J_{k,2}(T(k) + u(k))$. In Appendix A we show that

$$\Delta J_{k,1}(y(k)) = [a_1^k + a_2^k + a_3^k - a_4^k(1 - F_k(y(k)))]m + \rho \Delta \underset{d(k)}{E} V_{k+1}(\max(d(k), y(k))) \quad (27)$$

and

$$\Delta J_{k,2}(y(k)) = [b_1^k - b_2^k + a_3^k - a_4^k(1 - F_k(y(k)))]m + \rho \Delta \underset{d(k)}{E} V_{k+1}(\max(d(k), y(k))), \quad (28)$$

where $y(k) = T(k) + u(k)$.

Thus, we can confine our effort to the derivation of the backward recursion for the first difference of the expected optimal cost-to-go function defined by

$$\Delta \underset{d(k)}{E} V_{k+1}[\max(d(k), y(k))] = \underset{d(k)}{E} V_{k+1}[\max(d(k), y(k) + m)] - \underset{d(k)}{E} V_{k+1}[\max(d(k), y(k))]. \quad (29)$$

The details of our derivation are presented in Appendix B; we demonstrate

$$\Delta \underset{d(k)}{E} V_{k+1}[\max(d(k), y(k)] = F_k(y) \cdot \begin{cases} -(a_1^{k+1} + a_2^{k+1})m & \text{if} \quad y(k) < \underset{\sim}{S}(k+1) \\ [a_3^{k+1} - a_4^{k+1}(1 - F_{k+1}(y(k))]m \\ \quad + \rho \Delta \underset{d(k+1)}{E} V_{k+2}\{\max[d(k+1), y(k)]\} \\ \quad \text{if} \quad \underset{\sim}{S}(k+1) \le y(k) < \tilde{S}(k+1) \\ -(b_1^{k+1} - b_2^{k+1})m & \text{if} \quad y(k) \ge \tilde{S}(k+1). \end{cases} \quad (30)$$

Consequently, the backward recursions (27), (28), (30), and formulas (25) define an algorithm that yields the complete set of optimal threshold levels $\{\underset{\sim}{S}(N-1, \tilde{S}(N-1), \cdots (\tilde{S}(0), \underset{\sim}{S}(0)\}$ for the problem described by (15) and (16).

3.3 Final solution of the only-route TIP problem

Now we apply the Lagrangean relaxation approach to show that the computational procedure of Section 3.2 can be used to find an optimal solution to the original TIP formulation (11) through (13) that includes demand servicing constraints. Indeed, let us consider the functional

$$H(\mathbf{u}, \lambda) = J(\mathbf{u}) + \sum_{k=0}^{N=1} \lambda_k \rho^k E \max(0, d(k) - T(k) - u(k)), \quad (31)$$

where $\lambda = (\lambda_0, \cdots, \lambda_{N-1})$.

Clearly, for any given $\lambda \geq \mathbf{0}$, minimizing $H(\mathbf{u}, \lambda)$ with respect to $\mathbf{u}$ subject to (16) is equivalent to solving the problem described by (11) and (12) with the incremental underprovision cost a_4^k replaced by

$$\hat{a}_4^k = a_4^k + \lambda_k.$$

To demonstrate that a solution to the problem (12) and (31) (with appropriately fixed λ) is, in fact, a solution to the original TIP problem, (11) through (13), we need to prove the following general proposition:

Let $\mathbf{u}^*$ be a minimum of the functional

$$J(\mathbf{u}) + \sum_{k=0}^{N-1} \lambda_k E_k(\mathbf{u})$$

for all $\mathbf{u} \in U$, where $J(\mathbf{u})$, $E_k(\mathbf{u})$ are arbitrary real-valued functions, U is some set of admissible controls, and $\lambda \geq \mathbf{0}$. Then, $\mathbf{u}^*$ is a solution to the problem:

$$\min_{\mathbf{u} \in U} J(\mathbf{u})$$

subject to

$$E_k(\mathbf{u}) \leq E_k(\mathbf{u}^*) \quad (32)$$

for all k such that $\lambda_k > 0$.

The proof of this proposition is simple. By our hypothesis, for all $\mathbf{u} \in U$ we have

$$J(\mathbf{u}^*) + \sum_{k=0}^{N-1} \lambda_k E_k(\mathbf{u}^*) \leq J(\mathbf{u}) + \sum_{k=0}^{N-1} \lambda_k E_k(\mathbf{u}),$$

or

$$J(\mathbf{u}^*) \leq J(\mathbf{u}) + \sum_{k=0}^{N-1} \lambda_k [E_k(\mathbf{u}) - E_k(\mathbf{u}^*)]. \quad (33)$$

Since λ is a nonnegative vector the second term of the right-hand side of (33) is nonpositive for any $\mathbf{u} \in U$ such that conditions (32) are satisfied. Therefore,

$$J(\mathbf{u}^*) \leq J(\mathbf{u})$$

for all admissible controls $\mathbf{u}$ such that $E_k(\mathbf{u}) \leq E_k(\mathbf{u}^*)$ when $\lambda_k > 0$. Q.E.D.

This result implies that the optimal solution of the original TIP problem is also of the $(\underset{\sim}{S}, \tilde{S})$ type. Moreover, the proposition shows that if we can find an optimal solution to the TIP problem (11) and (12) with some incremental underprovision costs $\hat{a}_4^k = a_4^k + \lambda_k$, $\lambda_k > 0$, and if this solution results in expected demand servicing equal to $100\beta_k$ percent, then it is an optimal solution to the problem described by (11) through (13).

To utilize this result we need to derive formulas for computing the expected demand servicing level for a given policy $\pi = \{(\underset{\sim}{S}(0), \tilde{S}(0)), \cdots, (\underset{\sim}{S}(N-1), \tilde{S}(N-1))\}$. The derivation is given in Appendix C.

Now we can describe an algorithm to solve the TIP problem described by (11) through (13). A numerical procedure for obtaining the TIP solution can be outlined as follows:

Step 1—Set the initial vector of the Lagrange multipliers:

$$\lambda = \mathbf{0} \tag{34}$$

and identify the set K of years k for which the demand servicing constraint level would be exceeded unless a positive value were set for λ_k.

Step 2—Using the computational procedure described in Section 3.2, determine

$$\pi = \{(\underset{\sim}{S}(0), \tilde{S}(0)), \cdots, (\underset{\sim}{S}(N-1), \tilde{S}(N-1))\}$$

that minimizes

$$H(\mathbf{u}, \lambda). \tag{35}$$

Step 3—Using the Step 2 solution π, for all years k, $k \in$ K, calculate

$$\eta_k(\lambda) = \{E \max(0, d(k) - T(k))\} - \beta_k E d(k) \tag{36}$$

and determine whether $|\eta_k| < \epsilon_k$, where $\epsilon_k > 0$ is some tolerance level. If for every k the tolerance level ϵ_k is satisfied, stop; otherwise go to Step 4.

Step 4—To reduce $|\eta_k|$, go back to Step 2 with λ replaced by

$$\lambda + \omega\eta(\lambda), \tag{37}$$

where ω is an appropriate positive constant. (The constant ω can be adjusted by trial and error to speed up the iteration procedure.)

In general, the Lagrange multipliers vary from year to year depending on the demand characteristics, constraint levels, and cost coefficients. Our numerical experience revealed, however, that under reasonable assumptions for TIP application [growing (declining) demand pattern, $\beta_k = \beta$, and $a_i^k = a_i$] the vector of Lagrange multipliers λ^* can be approximated by $\lambda' = (\lambda', \cdots, \lambda')$, where the constant λ' depends only on a given level of demand servicing β and the coefficient of variation of the demand.

Using the approach outlined in Appendix C, we verified that the approximation of λ^* by λ' does not result in a significant cost penalty, i.e.,

$$\min_{\mathbf{u}} H(\mathbf{u}, \lambda^*) \approx \min_{\mathbf{u}} H(\mathbf{u}, \lambda').$$

Consequently, to obtain a numerically efficient TIP algorithm and to facilitate TIP implementation, we developed a conversion table (as illustrated on Fig. 3) that defines λ' as a function of the demand servicing constraint level β.

IV. NETWORK TIP

This section extends the only-route TIP to determine a multiyear schedule of trunk augments and disconnects for a hierarchical network that minimizes the present worth of the expected cost of planned and demand servicing subject to capacity and demand servicing constraints.

We note that there is a fundamental difference between the only-route and the hierarchical network TIP problems. The only-route TIP problem answers two basic questions: first, how much extra capacity is needed on a trunk group to hedge against forecast uncertainty and/or to satisfy the constraint on the amount of demand servicing; second,

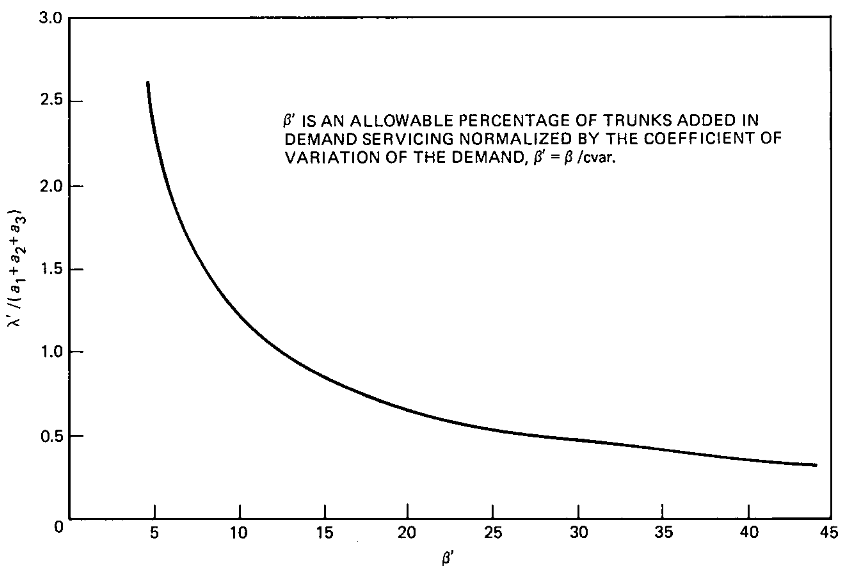


Fig. 3—Approximation of Lagrange multipliers.

how should the year-by-year trunk requirements be smoothed to obtain the optimal balance between the costs of maintaining and rearranging trunks over a planning horizon. In the network case the major additional question is to determine *where* to provide extra capacity in the network, that is, should this additional capacity be provided on all of the trunk groups in the network or on specific trunk groups only?

4.1 Overview of network TIP solution

To develop a network TIP algorithm, we exploit the heuristic principles used in conventional trunk group sizing procedures. In particular, we decouple the network TIP problem into individual cluster TIP problems, where a cluster is defined by a final trunk group and all subtending high-usage groups that overflow to that final group (Fig. 2 illustrates trunking terminology). To simplify the analysis we shall assume that the demand servicing policy is to augment only the final group when the blocking objective is violated. Our numerical experience shows that if an unbiased traffic load forecasting algorithm is used (such as in Ref. 3), then our demand servicing policy assumption is not critical to the optimality of the final TIP solution. Consequently, in this section we present a cluster-sizing procedure that minimizes the expected present worth of planned servicing expenditures on each high-usage (HU) trunk group plus the planned and demand servicing expenditures on the corresponding final trunk group subject to facility and demand servicing constraints.

The key idea of our solution is based on a heuristic argument that suggests that a near optimal solution can be obtained by accounting for forecast uncertainty and demand servicing constraints on the final trunk group only. We draw this conclusion from our analysis in Section 4.2, where we consider a single-year TIP problem. Extending this to a multiyear case we then assume that Truitt's engineering procedure, the ECCS rule,[7] can be used to find single-year, initial HU trunk requirements that then will be adjusted to eliminate uneconomical trunk group rearrangements.

Accordingly, in Section 4.3 we derive an optimal disconnect policy for HU trunk groups, that is, we show how to satisfy the ECCS trunk requirements for primary HU trunk groups while minimizing the present worth of the planned servicing cost over a planning horizon. As Fig. 4 shows, when the initial five-year TIP solution on primary HU groups is obtained, the TIP algorithm proceeds by calculating overflow traffic and sizing the intermediate HU trunk groups using, again, the Truitt's engineering procedure and an optimal disconnect policy. After all subtending HU groups have been sized, the final trunk group becomes the last and only choice for the remaining traffic to reach its destination. Consequently, the capacity expansion planning

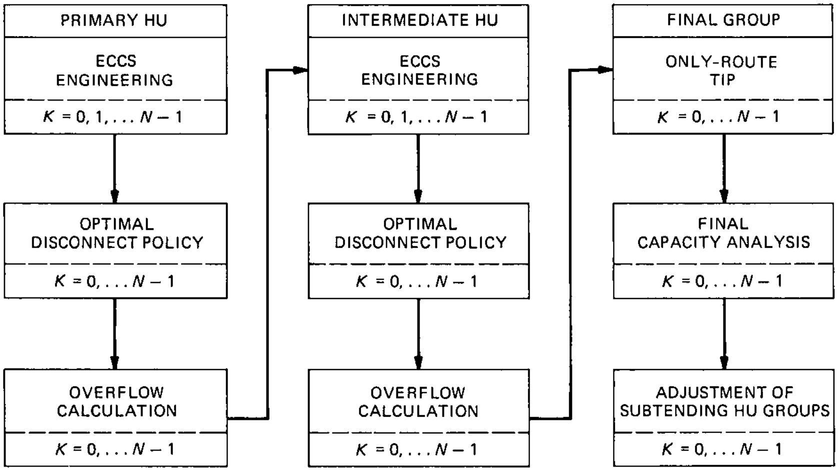


Fig. 4—Network TIP algorithm.

problem for the final trunk group reduces to an only-route TIP problem (11) through (13).

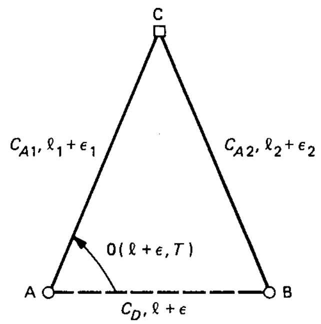
In Section 4.4 we show that under certain circumstances the initial cluster TIP solution may provide less (more) trunk capacity on the final trunk group than that necessary to satisfy the blocking objective and the demand servicing constraint. In that case we show how to improve the initial TIP solution by increasing (reducing) the sizes of the subtending HU groups in an economical fashion.

4.2 Alternate routing under uncertainty

To analyze the impact of forecast error on the optimal design of hierarchical networks we first formulate a single-year TIP problem for a Truitt alternate routing triangle.[7] Referring to Fig. 5, we assume that load $(\ell + \epsilon)$ is offered to the direct (HU) group, and background loads $(\ell_1 + \epsilon_1)$ and $(\ell_2 + \epsilon_2)$ are offered to the first and second legs of the alternate route, respectively, where ϵ, ϵ_1, ϵ_2 are the errors in the load forecast. The trunk group sizes on the direct and alternate routes are denoted by T, T_1, and T_2, respectively. Then the problem of determining T, T_1, T_2 to minimize the expected cost of trunk provisioning activities during the year is given by

$$\min_{T,T_1,T_2} E[C_D T + C_{A1} T_1 + C_{A2} T_2 + C_{S1} \max(0, d_1 - T_1) + C_{S2} \max(0, d_2 - T_2)], \quad (38)$$

where d_1 and d_2 are the number of trunks required on the alternate route to satisfy the network service objective; C_D is the incremental



ROUTE	COST	OFFERED LOAD	TRUNKS IN SERVICE	TRUNKS REQUIRED
AB	C_D	$\ell+\epsilon$	T	–
AC	C_{A1}	$\ell_1+\epsilon_1+O(\ell+\epsilon,T)$	T_1	d_1
CB	C_{A2}	$\ell_2+\epsilon_2+O(\ell+\epsilon,T)$	T_2	d_2

Fig. 5—Alternate routing under uncertainty.

cost of adding a trunk to the direct route on a planned basis; C_{A1}, C_{A2} and C_{S1}, C_{S2} are the incremental costs of planned and demand servicing on the alternate route legs. Also, in (38) we assume that when the calculated number of trunks required exceeds the number of trunks in service the demand servicing augmentation is performed on the final group only.

Our goal is to minimize (38) under the demand servicing constraints given by

$$E \max(0, d_1 - T_1) \leq \beta E d_1$$

and

$$E \max(0, d_2 - T_2) \leq \beta E d_2, \tag{39}$$

where the expected value is taken with respect to ϵ, ϵ_1 and ϵ, ϵ_1, ϵ_2, respectively.

In our numerical studies we investigated the cases when 10 to 100 erlangs of traffic are offered to the direct group and 10 to 500 erlangs are offered to the alternate route. We also assumed that the coefficient of variation of the demand forecast on the HU group, CV_D, and each leg of the alternate route, CV_{A1} and CV_{A2}, vary between 0.0 to 0.25, and that the demand servicing threshold, β, is in the range from 5 to 30 percent. Finally, we assumed that the forecast errors ϵ, ϵ_1, and ϵ_2 are statistically independent.

4.2.1 Major conclusions

We compared the optimal single-year TIP solution with the solution that sizes the direct route by Truitt's formula and then sizes the final to satisfy the blocking objective and demand servicing constraint at minimum cost. Our sensitivity analysis revealed that the optimal trunk requirement on the HU trunk group does not change significantly with changes in the coefficient of variation of the forecast, i.e., the optimal solution accounts for uncertainty by providing extra capacity, mainly on the final trunk group. More importantly, the cost difference between the optimal and the ECCS-based solutions is less than 1 percent. Thus, we conclude that the expected trunk provisioning cost function is very flat in the neighborhood of a solution point and the Truitt's HU solution is relatively close to the optimal HU trunk size.

To exploit the single-year ECCS design as a basis for a five-year trunk plan on HU groups, we next address the question of how to adjust the single-year trunk requirements to obtain an economical trunk implementation schedule for a given planning horizon.

4.3 Optimal disconnect policy for high-usage groups

We shall use the notions and notation of Section 2.1, except that $d(k)$ now represents the deterministic (rather than random) ECCS trunk requirement at year k.

As explained in Section 3.3, we omit the demand servicing constraint while sizing HU trunk groups. Consequently, for HU trunk groups the objective is to fulfill the ECCS trunk requirements at minimum cost, i.e., to minimize

$$\sum_{k=0}^{N-1} \rho^k[c_1^k(u(k)) + c_3^k(u(k)) + c_3^k(T(k), u(k))] \tag{40}$$

subject to the ECCS constraints

$$T(k) + u(k) \geq d(k),$$

where $T(0)$ is given and $T(k+1)$ is defined by

$$T(k+1) = T(k) + u(k). \tag{41}$$

Under conditions (8) and (9) we will show that the optimal decision, $u^*(k)$, has the following form:

$$u^*(k) = \begin{cases} d(k) - T(k), & \text{if } d(k) \geq T(k) \\ \min\left\{ \max\limits_{i=0,\cdots j^*} d(k+i) - T(k), 0 \right\}, & \text{if } d(k) < T(k), \end{cases} \tag{42}$$

where j^* is the largest integer j such that

$$b_1^k - b_2^k + \sum_{i=0}^{j^*-1} \rho^i a_3^{k+i} < \rho^{j^*}(a_1^{k+j^*} + a_2^{k+j^*}). \tag{43}$$

Note that in (43) the cost coefficients have superscripts, while only ρ is raised to a power.

Condition (43) states that the present worth of money recovered by disconnecting a trunk module in year k and maintaining $T(k + i) - m$ trunks is less than the present worth of a trunk module purchase and installation in year $k + j^*$ but is greater than this cost in year $k + j^* + 1$.

The second half of the policy (42) dictates that if $T(k)$ or more trunks are required in some year $k, k + 1, \cdots, k + j^*$, no trunks are disconnected. Otherwise, trunks are disconnected to the lowest possible level not requiring any reconnections prior to year $k + j^* + 1$.

To demonstrate that $u^*(k) = d(k) - T(k)$ if $d(k) \geq T(k)$, we need to show only that if $u(k) > d(k) - T(k)$, then the corresponding control strategy $\mathbf{u}^1$ and $(u(0), \cdots, u(N - 1))$ can be improved by the strategy $\mathbf{u}^2 = (u(0), \cdots, u(k) - m, u(k + 1) + m, \cdots, u(N - 1))$, where $\mathbf{u}^2$ is necessarily feasible. To show this we consider the two possible scenarios:

$$u(k + 1) > 0 \quad \text{and} \quad u(k + 1) \leq 0.$$

For $u(k + 1) > 0$, the cost difference, $L(\mathbf{u}^1) - L(\mathbf{u}^2)$, of the two strategies $\mathbf{u}^1$, $\mathbf{u}^2$ is

$$L(\mathbf{u}^1) - L(\mathbf{u}^2) = \rho^k m(a_1^k + a_2^k + a_3^k) - \rho^{k+1} m(a_1^{k+1} + a_2^{k+1}).$$

Then from (27) and the positivity of a_3^k, $L(\mathbf{u}^1) > L(\mathbf{u}^2)$. Similarly, when $u(k + 1) \leq 0$ we obtain

$$L(\mathbf{u}^1) - L(\mathbf{u}^2) = \rho^k m(a_1^k + a_2^k + a_3^k) - \rho^{k+1} m(b_1^{k+1} - b_2^{k+1}), \tag{44}$$

or, from (8) and (9), $L(\mathbf{u}^1) > L(\mathbf{u}^2)$.

To prove the second part of statement (42) we consider two cases. First, let us show that if we have a control strategy $\mathbf{u}^1 = (u(0), \cdots, u(N - 1))$ that disconnects fewer trunk modules in year k than suggested by (42), i.e.,

$$\max_{i=0,\cdots,j^*} \{d(k + i)\} - T(k) < u(k) \leq 0, \tag{45}$$

then $\mathbf{u}^1$ can be improved.

First assume that $u(k + j) < 0$ for some j such that $1 \leq j \leq j^*$. Let $k + j$ be the first such year. Then a better feasible policy is given by $\mathbf{u}^2 = (u(0), \cdots, u(k) - m, \cdots, u(k + j), \cdots)$. Indeed, for the difference in planned servicing cost we get

$$L(\mathbf{u}^1) = L(\mathbf{u}^2) = \rho^k m \left(b_1^k - b_2^k + \sum_{i=0}^{j-1} \rho^i a_3^{k+i} \right) - \rho^{k+j} m(b_1^{k+j} - b_2^{k+j}), \tag{46}$$

and from (9),

$$L(\mathbf{u}^1) - L(\mathbf{u}^2) > 0.$$

If there is no year $k + j$ $(1 \le j \le j^*)$ for which $u(k + j) < 0$, then from (43) a less expensive feasible solution is presented by $\mathbf{u}^2 = (u(0), \cdots, u(k) - m, \cdots, u(k + j^* + 1) - m, \cdots)$.

Now, we consider the second case and demonstrate that if we disconnect more trunks than specified by (42), i.e., $u^1(k) < u^*(k) = \max\{d(k + 1)\} - T(k)$, then the solution $\mathbf{u}^1$ can be improved by $\mathbf{u}^2 = (u(0), \cdots, u(k) + m, \cdots, u(k + j) - m, \cdots)$, where j is such that $d(k + j) = \max\{d(k + i)\}$. By the first part of (42) we add only as many trunks as needed. Therefore, if $u^1(k) < u^*(k)$ then we can assume that $u^1(k + j) \ge 0$. Consequently, we get

$$L(\mathbf{u}^1) - L(\mathbf{u}^2) = -m\rho^k \left[b_1^k - b_2^k + \sum_{i=0}^{j-1} \rho^i a_3^{k+i} - \rho^j (a_1^{k+j} + a_2^{k+j}) \right]$$

and from (43) we conclude

$$L(\mathbf{u}^1) - L(\mathbf{u}^2) > 0.$$

The proof is complete.

Using the intuitively appealing solution described by (42) we can construct a simple, numerically efficient scheme that evaluates the optimal policy $\mathbf{u}^*$:

Step 1—If $d(k) - T(k)$ is positive, set $u^*(k) = d(k) - T(k)$ and go to Step 3.

Step 2—If $d(k) - T(k) \le 0$, find maximum j for which (43) is satisfied; that is, find

$$j^* = \max \left\{ j \mid 1 \le j \le N - 1 - k, \quad b_1^k - b_2^k + \sum_{i=0}^{j-1} \rho^i a_3^{k+i} < \rho^j (a_1^{k+j} + a_2^{k+j}) \right\}.$$

Then, compute $d^* = \max_{0 \le i \le j^*} d(k + i)$. If $d^* \ge T(k)$, set $u^*(k) = 0$ and go to Step 3; if $d^* < T(k)$, set $u^*(k) = d^* - T(k)$ and go to Step 3.

Step 3—If $k = N - 1$, stop. Otherwise, set $T(k + 1) = T(k) + u^*(k)$, replace k by $k + 1$, and go to Step 1.

If $d(k) > T(k)$, the first step of algorithm simply augments the trunk group up to the ECCS requirement at year k. The second step determines how many trunks to disconnect if the current requirement is less than the number of trunks in service.

4.4 Final solution for HU trunk groups

The TIP algorithm described in Section 4.1 constructs a near optimal schedule of trunk augments and disconnects by smoothing the

ECCS high-usage trunk group requirements and by accounting for forecast uncertainty on final trunk groups only. As we stated in Section 3.1, the TIP solution on final groups is defined by N pairs of critical thresholds $(\underset{\sim}{S}^*(k), \tilde{S}^*(k))$, $k = 0, \cdots, N-1$. In practice, it is quite possible that because of the condition

$$T(k) + u(k) \leq \gamma(k),$$

the final group cannot be augmented to satisfy the blocking objective and the demand servicing constraint at year k. In that case we show how to adjust the sizes of subtending HU groups to reduce the load on the final.

Thus, in the development to follow we assume that $\underset{\sim}{S}(k)$ found by (25) is greater than the constraint level at year k,

$$\underset{\sim}{S}(k) > \gamma(k),$$

and, therefore, from eq. (26)

$$\underset{\sim}{S}^*(k) = \tilde{S}^*(k) = \gamma(k).$$

We note that the lower optimal threshold $\underset{\sim}{S}(k)$ represents the minimum number of trunks required to satisfy the blocking objective and demand servicing constraint. Consequently, the difference $\underset{\sim}{S}(k) - \underset{\sim}{S}^*(k)$ indicates the deficit in final trunks due to the facility constraints. To account for this deficit economically, we formulate the problem of augmenting high-usage groups to relieve the final. First, let us introduce the notation:

- $z(k)$ is the deficit in final trunks at year k, $z(k) = \underset{\sim}{S}(k) - \underset{\sim}{S}^*(k)$;
- $z_j(k)$ is the portion of the deficit (number of final trunks) covered by augmenting subtending high-usage group j at year k;
- $d_j(k)$ is the trunk requirement on the subtending group j at year k;
- $\beta_j(k)$ is the additional number of trunks on the subtending group j at year k that compensates for one final trunk;
- $\delta_j(k)$ is the maximum reduction in the final trunk requirement that can be obtained by augmenting trunk group j at year k;
- M is the total number of subtending trunk groups.

Also, we shall use the notation introduced in Section II for the cost functions, $c_1^{kj}(\cdot)$, $c_2^{kj}(\cdot)$, $c_3^{kj}(\cdot)$, controls, $u_j(k)$, and number of trunks in service at the beginning of the year, $T_j(k)$, where the index j identifies the subtending trunk groups.

Given the initial trunk levels for high-usage groups, $T_1(0), \cdots, T_M(0)$, we wish to find a multiyear schedule of trunk augments and disconnects that minimizes the present worth of the planned servicing costs,

$$\min L = \min_{\mathbf{u}_1, \cdots, \mathbf{u}_M} \sum_{k=0}^{N-1} \sum_{j=1}^{M} \rho^k [c_1^{kj}(u_j(k)) + c_2^{kj}(u_j(k)) + c_3^{kj}(T_j(k), u_j(k))] \quad (47)$$

subject to the following conditions:

1. The number of trunks in service on high-usage group j at year k must be greater than the new, inflated trunk requirement, that is,

$$T_j(k) \geq d_j(k) + \beta_j(k) z_j(k). \quad (48)$$

2. The total trunk requirements on the subtending high-usage groups at year k [given by (48)] must be sufficient to cover the deficit in final trunks, that is,

$$\sum_{j=1}^{M} z_j(k) \geq z(k). \quad (49)$$

3. Since we assumed that the demand servicing augmentation is performed on the final group only, the trunk group dynamics equation for the subtending high-usage groups is described by

$$T_j(k+1) = T_j(k) + u_j(k). \quad (50)$$

4. The unknown variables $z_j(k)$ must satisfy feasibility constraints

$$0 \leq z_j(k) \leq \delta_j(k). \quad (51)$$

To solve the nonlinear optimization problem (47) through (51) note that if all the nonnegative variables $z_j(k)$ are fixed, then the minimization problem can be decomposed as follows:

$$\min_{\mathbf{u}_1, \cdots, \mathbf{u}_M} L = \sum_{j=1}^{M} \min_{\mathbf{u}_j} L_j(z_j(0), \cdots, z_j(N-1))$$
$$= \sum_{j=1}^{M} \min_{\mathbf{u}_j} \sum_{k=0}^{N-1} \rho^k [c_1^{kj}(u_j(k)) + c_2^{kj}(u_j(k)) + c_3^{kj}(T_j(k), u_j(k))].$$

Also, when $z_j(k)$ are fixed, the minimum of the cost functional L_j, $L_j^*(z_j(0), \cdots, z_j(N-1))$ can be determined by the algorithm presented in Section 4.3. Therefore, the trunk capacity allocation problem is described by

$$\min_{z_j(k)} \sum_{j=1}^{M} L_j^*(z_j(0), \cdots, z_j(N-1)), \quad (52)$$

subject to

$$\sum_{j=1}^{M} z_j(k) \geq z(k), \qquad k = 0, \cdots, N-1,$$

where $z_j(k)$ satisfy feasibility constraints and $L_j^*(\cdot)$ is computed by the algorithm of Section 4.3.

We start by considering the case in which there is only one year, $k = k'$, such that $z(k) > 0$. Then since the unknown trunk quantities, $z_j(k')$, are integers we can use a forward dynamic programming recursion for solving (47), i.e.,

$$f_j(y_j) = \min_{z_j(k')}[L_j^*(z_j(k')) + f_{j-1}(y_j - z_j(k'))], \qquad j = 1, \cdots, M \quad (53)$$

where $f_o(y) = 0$, $0 \leq z_j(k') \leq y_j$; $y_M = z(k')$, and $L_j^*(z_j(k')) = L_j^*(0, \cdots, z_j(k'), \cdots, 0)$ is calculated by the algorithm of Section 4.3.

Note that modularity constraints on HU groups can be easily incorporated into the discrete dynamic programming formulation (53) to reduce the computational burden. Furthermore, the method outlined by (53) can be used sequentially for each year for which $z(k) > 0$. There is no guarantee, however, that this "one-year-at-a-time" procedure will terminate at a global optimum. Various refinements of the "one-year-at-a-time" method are considered in Ref. 8. In general, these refinements increase a chance to reach an optimum but require significantly more computation.

Finally, we add that the same mathemathical approach can be used to decrease the number of trunks on subtending high-usage groups economically when there is an extra capacity on the final. Recall that $\underset{\sim}{S}^*(k)$ represents the minimum trunk requirement at year k to satisfy the blocking and demand servicing constraints. Consequently, if $T(k) > \underset{\sim}{S}^*(k)$, then the difference between the planned trunk level and the $\underset{\sim}{S}^*(k)$ defines the amount of extra capacity on the final group at year k that can be used to reduce the planned servicing cost on subtending high-usage groups.

V. FINAL REMARKS

We have described a theoretical development of a new capacity expansion planning process, TIP, that provides a cost-effective multiyear schedule for trunk augments and disconnects for hierarchical networks. In contrast to the existing traffic engineering procedures, our solution accounts for forecast uncertainty, demand dynamics, trunk implementation costs, facility constraints, and demand servicing constraints. As we have shown in Sections III and IV, the dynamic programming approach to the stochastic capacity expansion problem yields a numerically efficient TIP algorithm that is easy to implement.

New, mechanized forecasting systems based on the TIP algorithm have been recommended for implementation in the operating companies and AT&T Communications.

VI. ACKNOWLEDGMENTS

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APPENDIX A

Proof of ($\underset{\sim}{S}$, $\tilde{S}$) Optimality

We start our inductive proof by showing that an ($\underset{\sim}{S}$, $\tilde{S}$)-type policy is optimal and by constructing the critical thresholds if N is equal to one, i.e., the last year, $N - 1$, is, in fact, the only year of the planning horizon.

A.1 The single-stage problem

For economy of notation, we shall drop the index, $N - 1$, from our equations. From (15) we wish to find an optimal policy, $u^* = u^*(T)$, that satisfies

$$\underset{d}{E}\{g(d, T, u^*)\} = \min_u \underset{d}{E}\{g(d, T, u)\}. \tag{54}$$

Equivalently, from (4) through (7), assuming T trunks in service at the beginning of the year, we seek a planned trunk level $y^*(T) = T + u^*(T)$ that minimizes the single-year cost functional:

$$J(T, y) = \begin{cases} a_1(y - T) + a_2(y - T) + a_3 y \\ \quad + a_4 E \max(0, d - y), & y \geq T \\ b_1(y - T) - b_2(y - T) + a_3 y \\ \quad + a_4 E \max(0, d - y), & y \leq T. \end{cases} \tag{55}$$

Note that when $u = 0$, the two branches of (55) are identical.

As we stressed in Section 3.1, to find an optimal solution of the single-stage problem described by (55), we shall show that the first differences of $J_1(T, T + u)$ and $J_2(T, T + u)$ satisfy conditions (1) through (3) of Section 3.1. Thus,

$$\Delta J_1(y) = (a_1 + a_2 + a_3 - a_4)m + a_4 E[\max(d, y + m) - \max(d, y)]$$

$$\Delta J_2(y) = (b_1 - b_2 + a_3 - a_4)m + a_4 E[\max(d, y + m) - \max(d, y)]. \tag{56}$$

The second term in (56) represents the expected savings in demand servicing if one additional trunk module is planned. This savings will occur with probability $1 - F(y)$. Thus,

$$\Delta J_1(y) = (a_1 + a_2 + a_3)m - a_4(1 - F(y))m,$$

$$\Delta J_2(y) = (b_1 - b_2 + a_3)m - a_4(1 - F(y))m. \tag{57}$$

From (57), condition (1) and (2) of Section 3.1 are satisfied. Therefore, we obtain the minimum points of $J_1(T, x)$ and $J_2(T, x)$ for all modular x by applying (25). From (57) $\underset{\sim}{S}$ and $\tilde{S}$ are the smallest values of x on the discrete set M such that

$$F(x) \geq 1 - \frac{a_1 + a_2 + a_3}{a_4}$$

and

$$F(x) \geq 1 - \frac{b_1 - b_2 + a_3}{a_4}. \tag{58}$$

Then, from inequalities (8) through (10), $\underset{\sim}{S}$ and $\tilde{S}$ satisfy (3). Consequently, the optimal single-stage decision for augmenting or disconnecting trunks in the unconstrained case is given by (3.1) and $\underset{\sim}{S}$ and $\tilde{S}$ are defined by (58). In the presence of capacity constraints, we need to modify $\underset{\sim}{S}$ and $\tilde{S}$ only by (26).

A.2 The multistage problem

In this section, we prove by induction the optimality of an $(\underset{\sim}{S}, \tilde{S})$-type policy for stage k by showing that the two branches of the cost-to-go function $J_k(T(k), y(k))$ satisfy conditions (1) to (3) of Section 3.1.

To simplify the recursion for $J_k(T(k), y(k))$ we introduce the auxiliary function $W_k(y)$:

$$W_k(y) = a_4^k y + \rho V_{k+1}(y). \tag{59}$$

From the optimality principle and (59), the cost-to-go function from state $T(k)$ can be expressed by

$$J(T(k), y(k)) = \begin{cases} (a_1^k + a_2^k + a_3^k - a_4^k)y(k) - (a_1^k + a_2^k)T(k) \\ \quad + \underset{d(k)}{E} W_k(\max(d(k), y(k))), \\ \quad y(k) \geq T(k) \\ (b_1^k - b_2^k + a_3^k - a_4^k)y(k) - (b_1^k - b_2^k)T(k) \\ \quad + \underset{d(k)}{E} W_k(\max(d(k), y(k))), \\ \quad y(k) \leq T(k). \end{cases} \tag{60}$$

From (15) and (59), $W_{k-1}(y)$ satisfies the recursion

$$W_{k-1}(y) = \min_{u(k)} \underset{d(k)}{E} \{a_4^{k-1}y + \rho[c_1^k(u(k)) + c_2^k(u(k))$$
$$+ a_3^k(y + u(k)) - a_4^k(y + u(k)) + W_k(\max(d(k), y + u(k)))]\}, \tag{61}$$

where $k = N - 1, \cdots, 0$.

As in Section 3.1, we consider the two branches of $J_k(T(k), y(k))$, that is, $J_{k,1}(T(k), y(k))$ and $J_{k,2}(T(k), y(k))$. Then, we need to show that $J_{k,1}$ and $J_{k,2}$ satisfy conditions (1) to (3).

From the definition of $J_{k,1}$ and $J_{k,2}$ and (60), condition (1) is trivial. To demonstrate (2) we have to prove that for any k

$$\Delta H_k(x) = \underset{d(k)}{E} W_k(\max(d(k), x + m)) - \underset{d(k)}{E} W_k(\max(d(k), x))$$

is a nondecreasing function of x. First, we consider the case $k = N - 1$.

We shall approach (2) by studying the properties of $W_{N-1}(x)$ and applying standard convexity results. In particular, since $W_{N-1}(x)$ and $\max(d, x)$ are monotonically increasing functions in x with nondecreasing first differences, the composite function $W_{N-1}(\max(d, x))$ must also be an increasing function in x with nondecreasing first differences.[9] In addition, the monotonicity of the functions $W_{N-1}(\max(d, x))$ and $\Delta W_{N-1}(\max(d, x))$ is preserved by the expected value operation. Thus, condition (2) is satisfied for $k = N - 1$.

To prove condition (2) inductively for an arbitrary k we have to show that recursion (61) preserves monotonicity of $W_k(y)$ and $\Delta W_k(y)$.

First, assuming the monotonicity of $W_k(y)$ and $\Delta W_k(y)$ we obtain (in a fashion similar to that for the case $k = N - 1$) that the composite function $W_k[\max(d(k), y + u(k))]$ has the same properties in y. Because of linearity in y of the remaining part of the right-hand side of (61) and the cost-of-money assumption, $\rho < 1$, the expression under the expected value sign in (61) is the sum of increasing functions in y with nondecreasing first differences. Thus

$$W_{k-1}(y) = \min_{u(k)} \underset{d(k)}{E} f(y, d(k), u(k)), \tag{62}$$

where $f(y, d(k), u(k))$ is increasing in y and $\Delta f(y, d(k), u(k))$ is nondecreasing in y.

Second, the monotonicity of $f(\cdot)$ and $\Delta f(\cdot)$ in y implies the monotonicity of the expected value. Thus, from D. Dantizig's convexity preservation result, it follows that the minimum of the expected value of $f(\cdot)$ is also a convex sequence in y, that is, the first differences are nondecreasing in y. Also, the monotonicity of $f(\cdot)$ is preserved by the minimum (infimum) operation. Indeed, if an arbitrary function $f(y, z)$ is increasing in y for each z, then for any $y_1 < y_2$, and z, we have

$$\inf_z f(y_1, z) \le f(y_1, z) \le f(y_2, z)$$

and, therefore,

$$\inf_z f(y_1, z) \le \inf_z f(y_2, z).$$

For completion of the proof, we need to demonstrate (3), i.e., that in the case with no capacity constraints, $\gamma(k) = \infty$, the minimum points of $J_{k,1}$ and $J_{k,2}$, $\underset{\sim}{S}(k)$ and $\tilde{S}(k)$, respectively, are finite and satisfy the condition

$$0 \le \underset{\sim}{S}(k) \le \tilde{S}(k) < \infty. \tag{63}$$

Calculating the first differences of $J_{k,1}$ and $J_{k,2}$, we obtain

$$\begin{aligned} \Delta J_{k,1}(y(k)) &= [a_1^k + a_2^k + a_3^k - a_4^k(1 - F_k(y(k))]m \\ &\quad + \rho\Delta \underset{d(k)}{E}\, V_{k+1}(\max(d(k), y(k))), \\ \Delta J_{k,2}(y(k)) &= [b_1^k - b_2^k + a_3^k - a_4^k(1 - F_k(y(k)))]m \\ &\quad + \rho\Delta \underset{d(k)}{E}\, V_{k+1}(\max(d(k), y(k))), \end{aligned} \tag{64}$$

where the first differences $\Delta E V_{k+1}$ are taken with respect to $y(k)$. Since $F_k(x) \longrightarrow 1$ as $x \longrightarrow \infty$, it follows from (64) that

$$\begin{aligned} \lim_{x\to\infty} \Delta J_{k,1}(x) &= [a_1^k + a_2^k + a_3^k]m - \rho[b_1^{k+1} - b_2^{k+1}]m, \\ \lim_{x\to\infty} \Delta J_{k,2}(x) &= [b_1^k - b_2^k + a_3^k]m - \rho[b_1^{k+1} - b_2^{k+1}]m. \end{aligned} \tag{65}$$

Relationship (65) shows that for sufficiently large x, both expressions are necessarily positive. Therefore, $\underset{\sim}{S}(k)$ and $\tilde{S}(k)$, which denote the smallest elements of the discrete set $M = \{0, m, 2m, \cdots\}$, such that

$$\Delta J_{k,1}(x) \ge 0 \quad \text{and} \quad \Delta J_{k,2}(x) \ge 0 \tag{66}$$

for (64), are finite. In addition, since $a_1^k + a_2^k > b_1^k - b_2^k$, it follows that $\Delta J_{k,1}(x) \geq J_{k,2}(x)$ for all x; hence,

$$\underset{\sim}{S}(k) \leq \widetilde{S}(k).$$

Thus, if $\gamma(k)$ is equal to infinity, then an $(\underset{\sim}{S}, \widetilde{S})$-type policy is optimal for the year k. We note that convexity preservation arguments hold for any region on which our sequences are defined. Consequently, in the constrained case, an optimal solution is given by (26).

The proof for the multistage problem is complete.

APPENDIX B

Derivation of the Recursion

To obtain explicit formulas for calculating $\Delta J_{k,1}$ and $\Delta J_{k,2}$, we shall use several recursions derived in Appendix A. In particular, since the optimal policy for stage $k+1$ is described by $(\underset{\sim}{S}(k+1), \widetilde{S}(k+1))$, applying $u^*(k+1)$ in (60) we can rewrite eq. (15) as

$$V_{k+1}(T(k+1)) = \begin{cases} (a_1^{k+1} + a_2^{k+1} + a_3^{k+1} - a_4^{k+1})\underset{\sim}{S}(k+1) \\ \quad - (a_1^{k+1} + a_2^{k+1})T(k+1) \\ \quad + W_{k+1}(\max(d(k+1), \underset{\sim}{S}(k+1))) \\ \quad \text{if} \quad T(k+1) < \underset{\sim}{S}(k+1) \\ a_3^{k+1}T(k+1) - a_4^{k+1}T(k+1) \\ \quad + W_{k+1}(\max(d(k+1), T(k+1))) \\ \quad \text{if} \quad \underset{\sim}{S}(k+1) \leq T(k+1) < \widetilde{S}(k+1) \\ (b_1^{k+1} + b_2^{k+1} + a_3^{k+1} - a_4^{k+1})\widetilde{S}(k+1) \\ \quad - (b_1^{k+1} - b_2^{k+1})T(k+1) \\ \quad + W_{k+1}(\max(d(k+1), \widetilde{S}(k+1))) \\ \quad \text{if} \quad T(k+1) \geq \widetilde{S}(k+1), \end{cases} \tag{67}$$

where the expected value is taken with respect to F_{k+1}, the demand distribution at year $k+1$. Note that at the boundary ($\underset{\sim}{S}(k+1)$ and $\widetilde{S}(k+1)$) the two corresponding branches of (67) are identical. Now, we shall carry (67) one step backward. Thus, we replace $T(k+1)$ by the trunk group dynamics equation and consider that $T(k)$ [rather than $T(k+1)$] is fixed. To simplify the notation we replace $T(k) + u(k)$ by y. In the rest of the section our objective is to obtain a recursion for

$$\Delta \underset{d(k)}{E}\, V_{k+1}(\max(d(k), y))$$

$$= \underset{d(k)}{E}\, V_{k+1}[\max(d(k), y+m)] - \underset{d(k)}{E}\, V_{k+1}[\max(d(k), y)].$$

From (67) we obtain

$$\Delta \underset{d(k)}{E} V_{k+1}(\max(d(k),y)) = \begin{cases} -(a_1^{k+1}+a_2^{k+1})mF_k(y) \\ \quad \text{if} \quad y<\underset{\sim}{S}(k+1) \\ [a_3^{k+1}-a_4^{k+1}+a_4^{k+1}F_{k+1}(y)]mF_k(y) \\ \quad +\Delta \underset{d(k)}{E}\Big\{\rho \underset{d(k+1)}{E} V_{k+2}\{\max \\ \quad \cdot[d(k+1),\max(d(k),y)]\}\Big\} \\ \quad \text{if} \quad \underset{\sim}{S}(k+1)\le y\le \widetilde{S}(k+1) \\ -(b_1^{k+1}-b_2^{k+1})mF_k(y) \\ \quad \text{if} \quad y\ge \underset{\sim}{S}(k+1). \end{cases} \tag{68}$$

First, we note that

$$\Delta V_{k+1}(\max(d(k),\, y)) = \begin{cases} \Delta V_{k+1}(y), & d(k) \le y \\ 0, & d(k) > y. \end{cases} \tag{69}$$

From (69) it follows that

$$\Delta \underset{d(k)}{E} V_{k+1}(\max(d(k),\, y)) = F_k(y)\Delta V_{k+1}(y). \tag{70}$$

Second, we set $R = \max[d(k),\, d(k + 1)]$. Then

$$\Delta V_{k+2}\{\max[d(k + 1),\, \max(d(k),\, y)]\}$$

$$= \Delta V_{k+2}\{\max(y,\, R)\} = \begin{cases} \Delta V_{k+2}(y), & R \le y \\ 0, & R > y. \end{cases} \tag{71}$$

Taking expectation with respect to the demand distribution F_{k+1}, we obtain

$$\Delta \underset{d(k+1)}{E} V_{k+2}\{\max(y,\, R)\} = \begin{cases} F_{k+1}(y)\Delta V_{k+2}(y), & d(k) \le y \\ 0, & d(k) > y. \end{cases} \tag{72}$$

Consequently,

$$\Delta \underset{d(k)}{E} \left\{ \underset{d(k+1)}{E} V_{k+2}\{\max(y,\, R)\} \right\} = F_k(y)F_{k+1}(y)\Delta V_{k+2}(y). \tag{73}$$

Applying (70) and (73), we arrive at

$$\Delta V_{k+1}(y) = \begin{cases} -(a_1^{k+1} + a_2^{k+1})m \quad \text{if} \quad y < \underset{\sim}{S}(k + 1) \\ [a_3^{k+1} - a_4^{k+1}(1 - F_{k+1}(y))]m \\ \quad + \rho F_{k+1}(y)\Delta V_{k+2}(y) \\ \quad \text{if} \quad \underset{\sim}{S}(k + 1) \le y < \widetilde{S}(k + 1) \\ -(b_1^{k+1} - b_2^{k+1})m \quad \text{if} \quad y \ge \underset{\sim}{S}(k + 1). \end{cases} \tag{74}$$

Finally, we note that because of (70), (74) gives the desired recursion:

$$\Delta \underset{d(k)}{E}\, V_{k+1}(\max(d(k), y))$$

$$= F_k(y)\cdot \begin{cases} -(a_1^{k+1} + a_2^{k+1})m \\ \quad \text{if} \quad y < \underset{\sim}{S}(k+1) \\ [a_3^{k+1} - a_4^{k+1}(1 - F_{k+1}(y))]m \\ \quad + \rho\Delta \underset{d(k+1)}{E}\, V_{k+2}\{\max[d(k+1), y]\} \\ \quad \text{if} \quad \underset{\sim}{S}(k+1) \le y < \tilde{S}(k+1) \\ -(b_1^{k+1} - b_2^{k+1})m \\ \quad \text{if} \quad y \ge \tilde{S}(k+1). \end{cases}$$

APPENDIX C

Computing Demand Servicing Level

As we discussed in Section 3.3, to solve the TIP problem described by (11) through (13) we need to learn how to compute the expected level of demand servicing for a given $(\underset{\sim}{S}, \tilde{S})$-type policy π.

For a given π, the planned trunk level $y(k) = T(k) + u(k)$ is a random variable that depends only on the previous demands $d(0), \cdots, d(k-1)$ and is independent of the future demands $d(k), \cdots, d(N-1)$. Accordingly, for the expected level of demand servicing corresponding to π, we get

$$\begin{aligned} &E\{\max(0, d(k) - y(k))\} \\ &\quad = E_{y(k)}\{E\{\max(0, d(k) - y(k)) \mid y(k)\}\} \\ &\quad = \int_0^\infty \left[\int_x^\infty (y - x) dF_k(y) \right] dG_k(x), \end{aligned} \tag{75}$$

where $G_k(x)$ is the distribution function of the planned trunk level $y(k)$.

By the definition of the optimal policy u^*, the distribution function G_k of $T(k) + u^*(k)$ is

$$G_k(x) = P(y(k) \le x) = P(T(k) + u^*(k) \le x)$$

$$= \begin{cases} 0, & \text{if} \quad x\, \underset{\sim}{S} < (k) \\ P(T(k) \le x), & \text{if} \quad \underset{\sim}{S}(k) \le x < \tilde{S}(k) \\ 1, & \text{if} \quad x \ge \tilde{S}(k) \end{cases} \tag{76}$$

for $k = 0, 1, \cdots, N-1$.

Using the fact that, for a given policy π, the planned trunk level $y(k)$ is independent of the demand at that year, $d(k)$, and our assumption that $d(k-1)$ is independent of $d(0), d(1), \cdots, d(k-2)$, we can derive a simple recursive formula for $G_k(x)$ by calculating $P(T(k) \le x)$):

$$\begin{aligned} H_k(x) &= P(T(k) \le x) \\ &= P\{\max(d(k-1), y(k-1)) \le x\} \\ &= P(d(k-1) \le x) \cdot P(y(k-1) \le x) \\ &= F_{k-1}(x) \cdot G_{k-1}(x) \\ &= F_{k-1}(x) \cdot \begin{cases} 0, & \text{if } x < \underset{\sim}{S}(k-1) \\ H_{k-1}(x), & \text{if } \underset{\sim}{S}(k-1) \le x < \tilde{S}(k-1) \\ 1, & \text{if } x \ge \tilde{S}(k-1), \end{cases} \end{aligned} \tag{77}$$

for $k = 1, 2, \cdots, N-1$.

Recalling that, at the beginning of the first year, the number of trunks in service, $T(0)$, is specified, we have

$$H_0(x) = \begin{cases} 0, & x < T(0) \\ 1, & x \ge T(0). \end{cases} \tag{78}$$

Formulas (76) to (78) together with expression (27) specify the forward recursion for calculating the expected demand servicing level at year k ($k = 0, \cdots, N-1$) associated with the policy π.

Note that using similar independence arguments we can derive forward recursions for calculating other quantities of practical interest such as the total expected cost of trunk provisioning over the planning horizon, the difference in capital cost for two competing ($\underset{\sim}{S}$, $\tilde{S}$)-type policies, or the probability of demand servicing. The last quantity is another important measure of demand servicing activity, since it allows us to predict the portion of only-route groups that will require emergency servicing in a given year.

To calculate the probability of demand servicing, for example, we observe that for a given policy π the planned trunk level $y(k)$ depends only on the previous demands $d(0), \cdots, d(k-1)$. Thus we obtain

$$P\{d(k) > y(k)\} = \int_0^\infty [1 - F_k(x)]dG_k(x),$$

where $G_k(\cdot)$ is the distribution function of $y(k)$. Integrating by parts and replacing G_k by H_k, we arrive at

$$P(d(k) > y(k)) = \int_{\underset{\sim}{S}(k)}^{\tilde{S}(k)} H_k(x)dF_k(x) + 1 - F_k(\tilde{S}(k)),$$

$$k = 0, 1, \cdots, N-1.$$

AUTHORS

Arik N. Kashper, M.Sc. (Mathematics), 1969, Leningrad University, USSR; Ph.D (Systems Engineering), 1979, University of Arizona, Tucson; AT&T Bell Laboratories, 1979—. At the University of Arizona, Mr. Kashper worked in the area of system identification and parameter estimation. At AT&T Bell Laboratories, he is concerned with various aspects of traffic network planning. Member, SIAM.

G. C. Varvaloucas, B.S. (Electrical Engineering), 1972, National Technical University of Athens; M.S., 1975, and Ph.D., 1978 (Systems Engineering), Case Western Reserve University; AT&T Bell Laboratories, 1977—. Mr. Varvaloucas has done work in the general area of telephone network planning. He is presently Supervisor of the Interexchange Architecture Planning Group, with responsibility for defining certain aspects of the interexchange network planning process in the post-divestiture environment. Member, IEEE.