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Signals Designed for Recovery After Clipping—I. Localization of Infinite Products

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It is shown for signals of the form $s(t) = g(t) + \cos ct$, where g is bandlimited to $[-b, b]$, $0 < b < c < \infty$, and $(-1)^k s(k\pi/c) > 0$ (e.g., if $|g(t)| < 1$), that the zeros of s in the interval $(t - T, t + T)$ allow reconstruction of $s(t)$ with a relative error less than $2e^{-\lambda T}(1 - e^{-\lambda T})^{-2}$, where $\lambda = b - c$. The reconstruction algorithm gives an effective way of localizing the infinite product representation for $s(t)$. Some practical aspects of the algorithm are discussed. The scheme offers advantages for the recovery of signals after nonlinear distortion, such as occurs, for example, in magnetic recording.

I. INTRODUCTION

Let us assume that a signal $s(t)$ undergoes some nonlinear distortion, generally of unknown characteristic, during transmission (or perhaps in recording), which does not alter the zero crossings. We may suppose that the reliable information we receive is just sgn $s(t)$, the clipped signal. We are led to ask for what class of signals $s(t)$ can we make a reliable estimate of $As(t)$ from sgn $s(t)$, i.e., from the zero crossings of $s(t)$.

If we assume that $s(t)$ is a (real-valued) bandlimited signal with real simple zeros t_k, we have the formula,† assuming $s(0) \neq 0$,

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† This follows from a result of Titchmarsh,[1] who showed that (1) holds for $s(t)$ that have ordinary Fourier transforms that vanish outside an interval $[-\alpha, \alpha]$ but not outside any translate of the interval; i.e., for bandlimited functions whose spectral endpoints are centered about the origin. Then, (1) follows for bounded $s(t)$ by considering $s(t)(t - t_j)^{-1}(t - t_k)^{-1}$, where t_j and t_k are either real or complex conjugates.

$$s(t) = s(0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right), \tag{1}$$

where the product converges (conditionally) if the zeros are ordered such that $|t_{k+1}| > |t_k|$. Thus, from a strictly mathematical viewpoint, we have a "formula" for recovering $s(t)$ within a constant multiplier from sgn $s(t)$. However, for practical purposes the formula is worthless without additional restrictions on $s(t)$.

In the first place, the formula is valid only for strictly bandlimited $s(t)$. If the t_k are the zeros of an almost bandlimited $s(t)$, then the product may not even converge, or if it does, it may not even represent a bounded function, and in general cannot be expected to give a reasonable replica of $s(t)$.

In the second place, even if $s(t)$ is strictly bandlimited, the infinite product representation is, in general, impractical; it is very sensitive to (correlated) perturbations of the t_k and, more important, knowledge of all the t_k is crucial to estimating $s(t)$, relative to $s(0)$, for large t. In practice, one would like to make a good estimate, $s_T(t)$, of $s(t)$ just from the knowledge of the zeros of s in the interval $(t - T, t + T)$ for some fixed T and arbitrary t, $(-\infty < t < \infty)$. That is, in practice we require a *localized* reconstruction formula.

For a certain class of bandlimited $s(t)$, we can obtain an *exponentially localized* reconstruction formula; i.e., from a knowledge of the zeros of s in the interval $(t - T, t + T)$, we can make an estimate $s_T(t)$ such that

$$|s(t) - s_T(t)| \leq |s(t)| \cdot \epsilon_T, \tag{2}$$

where

$$\epsilon_T = 0\{e^{-\lambda T}\} \qquad T \rightarrow \infty$$

and $\lambda > 0$ is an appropriate parameter in the representation of $s(t)$.

This class of signals, which we denote by $S(b, c)$, consists of signals of the form

$$s(t) = g(t) + \cos ct, \tag{3}$$

where g (real valued) is a signal whose spectrum is confined to $[-b, b]$, $0 < b < c < \infty$, and such that

$$(-1)^k s(k\pi/c) > 0, \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{3a}$$

The alternation condition (3a) ensures that $s(t)$ has only real simple zeros,* one between each extremal of cos ct. Sufficient additional

* We ask the readers to accept this fact, or provide their own proof. A more general result will be given in a future paper.

conditions on g for (3a) are

$$|g(k\pi/c)| < 1 \qquad k = 0, \pm 1, \pm 2, \cdots \tag{3b}$$

and

$$|g(t)| < 1 \qquad -\infty < t < \infty, \tag{3c}$$

the latter condition being the more practical constraint. However, (3a) allows special functions g of the form

$$g(t) = (1 + x(t))\cos ct + \hat{x}(t)\sin ct, \tag{4}$$

where $x(t)$ is a bandpass signal (real valued) whose spectrum is confined to the disjoint intervals, $[\lambda, \mu]$ and $[-\mu, -\lambda]$, $0 < \lambda < \mu$,

$$\lambda + \mu \leq 2c, \tag{4a}$$

$$x(t) > -1, \tag{4b}$$

and

$$\hat{x}(t) \text{ is the Hilbert transform of } x(t). \tag{4c}$$

The spectrum of g, defined in (4), is confined to the union of the two intervals, $[-c + \lambda, -c + \mu]$ and $[-\mu + c, -\lambda + c]$. In case $c > \mu$, then g, defined in (4), is bandpass with spectrum confined to the disjoint intervals $[-b, -a]$, and $[a, b]$, where

$$0 < a = c - \mu < b = c - \lambda. \tag{4d}$$

In this case ($c > \mu$), we may consider $s(t)$ to be a full carrier lower-sideband signal associated with the bandpass baseband signal $x(t)$. Also, if g in (3) is bandpass with spectrum confined to the disjoint intervals $[-b, -a]$, $[a, b]$, and $0 < a < b < c$, we can always put g in the form (4).

The important parameter associated with the class $S(b, c)$ is

$$\lambda = c - b. \tag{5}$$

In words, λ is the width of the guard band between the top frequency of $g(t)$ and the frequency of $\cos ct$. We will refer to λ as the gap frequency (rad/s). It is the presence of the gap that allows the exponential localization of the reconstruction of $s(t)$, giving an error of the order of $e^{-\lambda T}$ from the knowledge of the zeros of s in the interval $(t - T, t + T)$. We can obtain a good approximation to $s(t)$ for λT, say, in the range 2π to 3π; i.e., T needs to be only 2 or 3 half periods (Nyquist intervals) of the gap frequency to obtain 0.1-percent error.

We should note that the signals in $S(b, c)$ are, in effect, normalized by taking the coefficient of $\cos ct$ to be $+1$. With the signals thus normalized, their zeros give a complete description of the signals.

In some applications we may think of s as a baseband signal g with added bias ($\cos ct$), and in other applications we may think of s as a lower-sideband signal with full carrier ($\cos ct$).

We first present the reconstruction formulas. Subsequent sections are devoted to the derivation of the formulas. Finally, some practical aspects of the reconstruction formulas are discussed in the concluding section.

II. THE RECONSTRUCTION FORMULAS

Here we describe an approximation, $s_T(t)$, determined only by the zeros of s in the interval $(t - T, t + T)$ such that for s in $S(b, c)$, we have

$$|s(t) - s_T(t)| \le \frac{2e^{-\lambda T}}{(1 - e^{-\lambda T})^2} |s(t)|, \qquad -\infty < t < \infty. \tag{6}$$

The first form of the approximation is

$$s_T(t) = (1/2)\{\operatorname{sgn} s(t)\}\cdot \exp\{\hat{h}_T(t)\}, \tag{7}$$

where

$$\hat{h}_T(t) = \sum_{-\infty}^{\infty} L_T(t - t_k) + \frac{c}{\lambda}\,\mu(\lambda T), \tag{7a}$$

$$L_T(t) = L_T(-t), \tag{7b}$$

$$L_T(t) = 0, \qquad |t| > T, \tag{7c}$$

$$L_T(t) = -\int_t^T \frac{f(x)}{x}\,dx, \qquad 0 < t < T, \tag{7d}$$

$$f(t) = \frac{T}{\sinh \lambda T}\,\frac{\sinh \lambda\sqrt{T^2 - t^2}}{\sqrt{T^2 - t^2}}, \tag{7e}$$

and

$$\mu(x) = \frac{xI_0(x)}{\sinh x} - \frac{x}{\sinh x}\cdot\frac{2}{\pi}\int_0^{\pi/2} e^{-x\sin\theta} d\theta \sim \sqrt{\frac{2x}{\pi}}, \quad x \to \infty. \tag{7f}$$

(Here I_0 is the modified Bessel function.) The function $L_T(t)$ is a sort of truncated $\log |t/T|$, depending both on λ and T, although in the notation we have suppressed the dependence on λ. Using the fact that $f(0) = 1$, we have

$$\begin{aligned} L_T(t) &= \log\left|\frac{t}{T}\right| + \int_t^T \frac{1 - f(x)}{x}\,dx \\ &= \log\left|\frac{t}{T}\right| + F\left(\frac{t}{T}; \lambda T\right), \qquad |t| < T \\ &= 0, \qquad |t| \ge T, \end{aligned} \tag{8}$$

where $F(x; \lambda T)$ is analytic everywhere. Graphs of $F(x; \beta)$ and $L_T(xT)$, $\beta = \lambda T = k\pi/2$ are shown in Fig. 1 for $-1 \leq x \leq 1$, $k = 1, 2, 3, \cdots, 6$. For $\lambda T = \pi/2$, in which case the reconstruction interval $(t - T, t + T)$ has a length equal to only a half-period (one Nyquist interval) of the gap frequency, the function F is not significantly different from zero over the interval $(-1, 1)$; so $L_T(t) \doteq \log |t/T|$ for the case $\lambda T = \pi/2$. For larger λT, the function F becomes significant.

In order to express $s_T(t)$ in product form we define

$$V_T(t) = \exp L_T(t). \tag{9}$$

Here, $V_T(t)$ is a sort of V-shaped function that we may write, using (7),

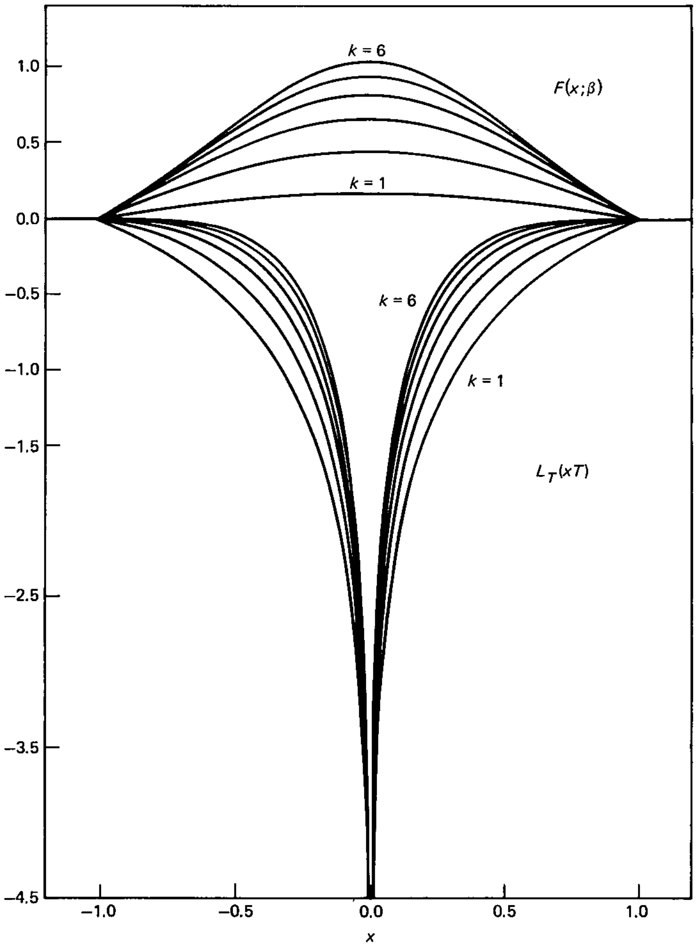


Fig. 1—The functions $L_T(xT)$ and $F(x; \beta)$ for $\beta = k\pi/2$, $k = 1, 2, 3, \cdots, 6$.

$$V_T(t) = \left|\frac{t}{T}\right| \cdot W\left(\frac{t}{T}; \lambda T\right), \qquad |t| < T$$

$$= 1, \qquad |t| \geq T, \tag{9a}$$

where

$$W(x; \lambda T) = \exp\{F(x; \lambda T)\}, \qquad |x| < 1. \tag{9b}$$

We have then,

$$s_T(t) = A\{\text{sgn } s(t)\} \prod_{-\infty}^{\infty} V_T(t - t_k), \tag{10}$$

where

$$A = \frac{1}{2} \exp\left\{\frac{c}{\lambda} \mu(\lambda T)\right\}. \tag{10a}$$

Graphs of the two functions, $V_T(xT) = |x| W(x; \lambda T)$ and $W(x; \lambda T)$, are shown in Fig. 2 for $-1 \leq x \leq 1$, $\lambda T = \beta = k\pi/2$, and $k = 1, 2, 3, \cdots, 6$.

The normalization function $\mu(\lambda T)$, defined in (7f) and appearing in (7a) and (10a), is graphed in Fig. 3.

It should be noted in formulas (7) and (10), recalling that $L_T(t) = 0$ for $|t| \geq T$ and $V_T(t) = 1$ for $|t| \geq T$, that the infinite sum and the infinite product involve for each t only those t_k that satisfy $|t - t_k| < T$; i.e., only those t_k in the interval $(t - T, t + T)$. So, (10) gives the desired localization of the infinite product (1) for signals in the class $S(b, c)$.

The error estimate in (6) is conservative, with equality possible only at the zeros of s. At least, we have the exponential decrease of the error with λT; e.g., for $\lambda T = 2\pi$,

$$\frac{2e^{-2\pi}}{(1 - e^{-2\pi})^2} \doteq 0.00375,$$

and for $\lambda T = 3\pi$,

$$\frac{2e^{-3\pi}}{(1 - e^{-3\pi})^2} \doteq 0.00016.$$

So, for T somewhere between 2 and 3 half periods of the gap frequency λ, we get a *relative* error, $(s_T(t) - s(t)) \div s(t)$, amounting to no more than 0.1 percent.

Figure 4 illustrates the simplest sort of test of the algorithm in reconstructing $s(t) = \cos t$. Here $g(t) = 0$, and $\lambda = c = 1$. Four cases of $s_T(t)$ are shown corresponding to $T = \pi/2$, π, $3\pi/2$, and 2π. The

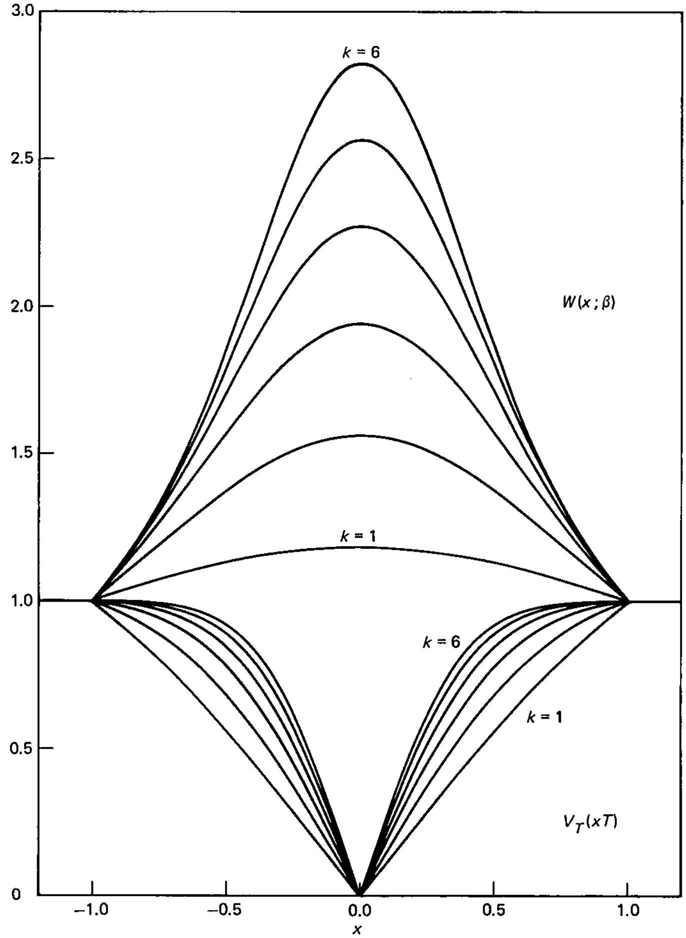


Fig. 2—The functions $V_T(xT)$ and $W(x; \beta)$ for $\beta = k\pi/2$, $k = 1, 2, 3, \cdots, 6$.

corresponding errors and relative errors, suitably magnified, are also shown.

In the first case ($T = \pi/2$), shown in Fig. 4a, only one zero of $\cos t$ appears in the observation interval $(t - T, t + T)$; so, we have in this case

$$s_T(t) = A \cdot V_T(t - \pi/2) \quad \text{for} \quad 0 \leq t \leq \pi/2.$$

Here $V_T(t) \equiv V_T(t; \lambda T)$ with $T = \pi/2$, $\lambda = 1$. For the relatively small product, $\lambda T = \pi/2$, the function $V_T(t)$ is essentially $|t/T|$ for $|t| < T$ (cf. Fig. 2), so that $s_T(t)$ has a quite triangular waveform. The magnitude of the error $(s_T(t) - s(t))$ in the reconstruction is maximum at $t = 0$, where it is approximately 0.22. The magnitude of the relative

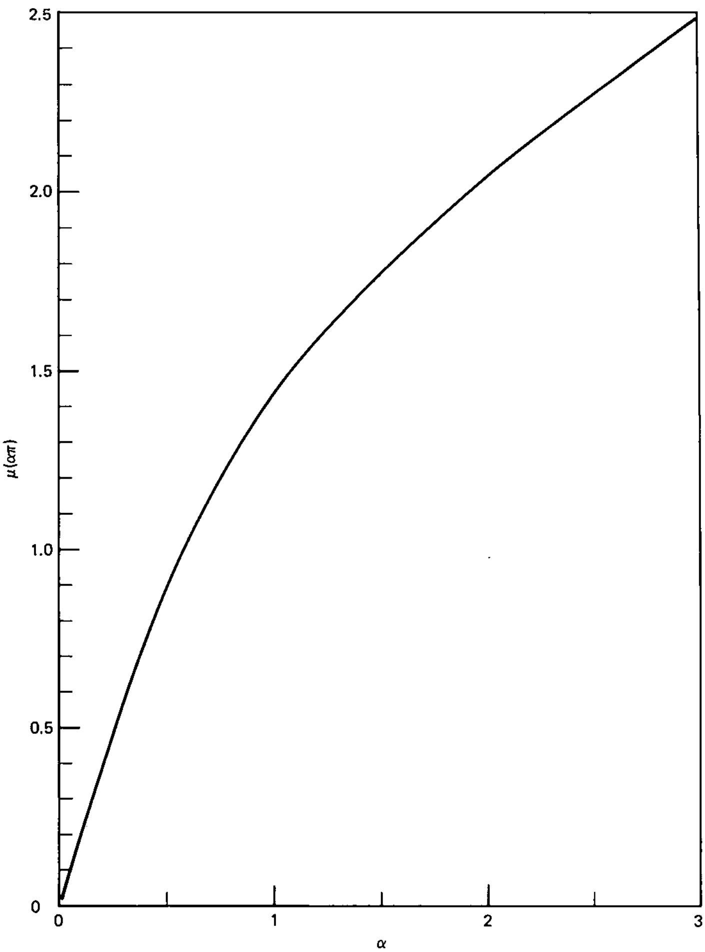


Fig. 3—Normalization function $\mu(\lambda T)$.

error, $(s_T(t) - s(t)) \div s(t)$, is also maximum at $t = 0$ with the same value, approximately 0.22. The estimate (6) gives an upper bound for the absolute relative error in this case of approximately 0.66. The reconstruction is rather crude in this case.

The approximations become progressively better in the other cases. For example, the maximum absolute relative error in the fourth case, $\lambda T = 2\pi$ (see Fig. 4d), is approximately 0.001, the upper bound (6) for this case being approximately 0.0037. In all cases the maximum absolute relative error is approximately one third of the upper bound, given by (6). Note that in the second and fourth cases the absolute relative error is maximum at the zeros of cos t where the error is zero.

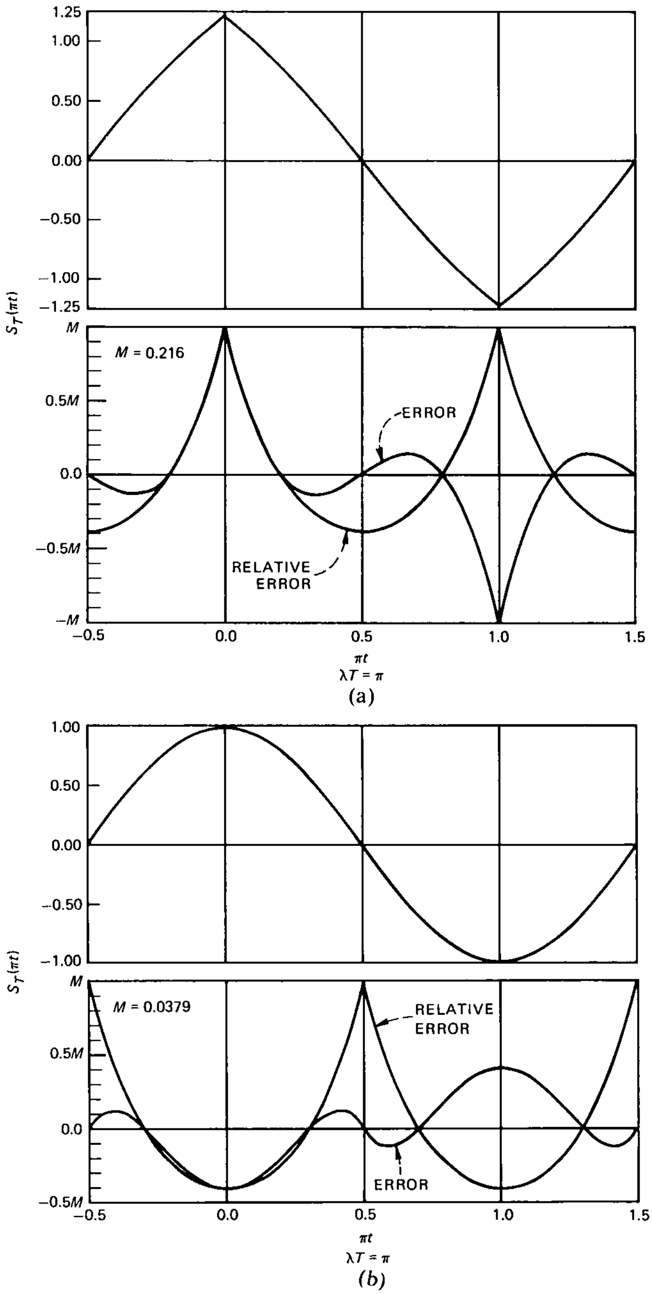
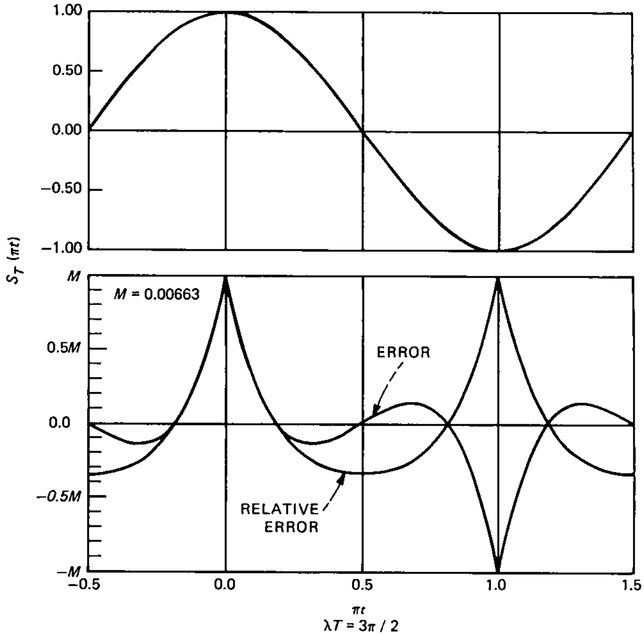
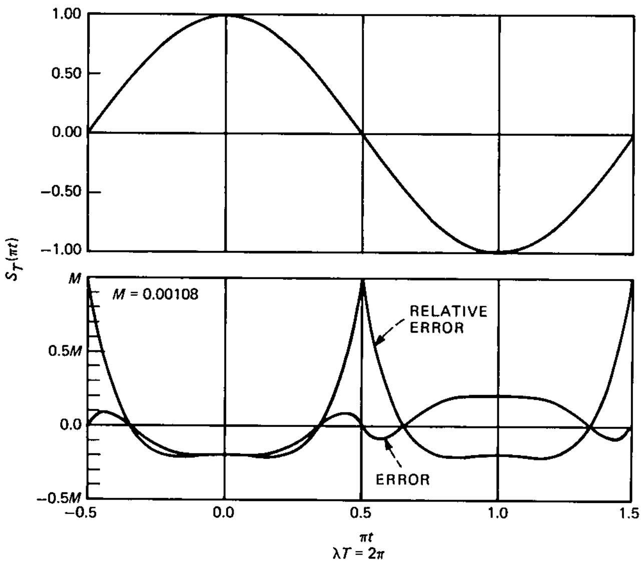


Fig. 4(a) and (b)—Reconstruction of cosine.



(c)



(d)

Fig. 4(c) and (d)—Reconstruction of cosine.

Note the similar character of the error waveforms in the first and third cases, where $2T$ is an odd multiple of π, which is distinctly different from that of the second and fourth cases, where $2T$ is an even multiple of π.

III. THE BASIC FORMULA

Given the zeros $\{t_k\}$ of a signal $s(t)$ in $S(b, c)$, we define the following important function:

$$h(t) = J(t) - ct, \tag{11}$$

where $J(t)$ is a jump function increasing by π at each zero t_k of $s(t)$, and

$$J(0) = 0. \tag{11a}$$

The function $\pi^{-1}J(t)$ is just the zero-counting function of $s(t)$. Since there is one, and only one, zero t_k between each extremal of cos ct, we may label the zeros such that

$$k\pi/c < t_k < (k + 1)\pi/c, \qquad k = 0, \pm 1, \pm 2, \cdots . \tag{12}$$

It is clear then that

$$h(k\pi/c) = 0, \qquad k = 0, \pm 1, \pm 2, \cdots \tag{13}$$

and

$$-\pi < h(t) < \pi, \qquad -\infty < t < \infty. \tag{14}$$

The function $h(t)$, which we will refer to as the *fundamental function* associated with the zeros $\{t_k\}$ of $s(t)$, may be described as a sort of meandering sawtooth function bounded between $-\pi$ and π.

In this section we want to show that $h(t)$ is a high-pass function with no spectrum in the frequency interval $(-\lambda, \lambda)$, where λ is the gap frequency ($\lambda = c - b$), and therefore has a Hilbert transform $\hat{h}(t)$. Furthermore, we will show that

$$\log|2s(t)| = \hat{h}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(x)}{t - x}\, dx, \tag{15}$$

and thus obtain the *basic formula*

$$s(t) = (1/2)\{\text{sgn } s(t)\}\exp\{\hat{h}(t)\}. \tag{16}$$

The fact that $h(t)$ is a high-pass function allows us to make a good estimate $\hat{h}_T(t)$ of $\hat{h}(t)$ just from the knowledge of h, or equivalently the zeros of s, in the interval $(t - T, t + T)$. Later, we detail the localization of the Hilbert transform to obtain (7) from (16).

To obtain these results, we introduce the complex variable $\tau = t + iu$, and consider for $s(t)$ in $S(b, c)$ the function

$$H(\tau) = i \log\{2e^{ic\tau}s(\tau)\}, \qquad u > 0. \tag{17}$$

We have

$$\begin{aligned} 2e^{ic\tau}s(\tau) &= 2e^{ic\tau}(g(\tau) + \cos c\tau) \\ &= 1 + 2e^{ic\tau}g(\tau) + e^{2ic\tau}. \end{aligned} \tag{18}$$

Now $g(\tau)$ is an entire function of exponential type $b < c$, real and bounded on the real line; so (cf. Ref. 2, p. 83),

$$|g(t + iu)| \leq (\cosh bu)\sup_t |g(t)|. \tag{19}$$

Hence,

$$2e^{ic(t+iu)}s(t + iu) = 1 + 0(e^{-\lambda u}), \qquad u \to \infty,$$

$$\text{where} \quad \lambda = c - b > 0, \tag{20}$$

and therefore,

$$H(t + iu) = 0(e^{-\lambda u}), \qquad u \to \infty. \tag{21}$$

Also, since $s(\tau)$ has only real zeros, $H(\tau)$ is analytic in the upper half plane. The real part of $H(\tau)$ is also bounded. It follows (Ref. 3, see Theorems 6.3.1 and 6.4.3) that

$$h(t) = \lim_{u \to 0+} \mathrm{Re}\{H(t + iu)\} \tag{22}$$

and

$$\hat{h}(t) = \lim_{u \to 0+} \mathrm{Im}\{H(t + iu)\} \tag{23}$$

are high-pass functions having no spectrum in the interval $(-\lambda, \lambda)$, and $\hat{h}(t)$ is the Hilbert transform of $h(t)$.

We have

$$h(t) = -\lim_{u \to 0+} \arg\{e^{ic\tau}(g(\tau) + \cos c\tau)\}. \tag{24}$$

Now, for $t = k\pi/c$ we have

$$e^{ict}(g(t) + \cos ct) = e^{ik\pi}s(k\pi/c) > 0. \tag{25}$$

Hence, $h(k\pi/c)$ is some multiple of 2π. We can conclude that $h(k\pi/c) = 0$ from the fact that h is high pass and from the fact that the argument (phase) of $s(t)$ is a decreasing function of t, decreasing by π at each zero of s (indent into the *upper* half plane around the zeros). There is one, and only one, zero between each extremal of $\cos ct$, and $h'(t) = -c$ except when t is a zero of $s(t)$. Thus,

$$h(t) = h(0) - ct + J(t), \tag{26}$$

where $J(t)$ is a nondecreasing staircase function (jump function), $J(0) = 0$, which increases by π at t_k, the zeros of $s(t)$; i.e.,

$$\frac{k\pi}{c} < t_k < \frac{(k+1)\pi}{c}, \tag{26a}$$

$$J'(t) = 0, \qquad t \neq t_k, \tag{26b}$$

and

$$J\left(\frac{k\pi}{c}\right) = k\pi. \tag{26c}$$

Thus, we have from (26) and (26c)

$$h\left(\frac{k\pi}{c}\right) = h(0) = 2n\pi, \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{27}$$

Since $h(t)$ can increase or decrease at most by π in each interval $k\pi/c < t < (k+1)\pi/c$, we have

$$-\pi < h(t) - h(k\pi/c) < \pi$$

or

$$(2n-1)\pi < h(t) < (2n+1)\pi.$$

But, $h(t)$ is high pass and must change sign, so we must have $n = 0$, and hence,

$$-\pi < h(t) < \pi \qquad (-\infty < t < \infty). \tag{28}$$

In other words, $h(0) = 0$ is the only value of $\arg\{e^{ict}s(t)\}$ at $t = 0$ consistent with $H(t + iu) = 0(e^{-\lambda u})$, where we took $\log\{2e^{ic\tau}s(\tau)\}$ to be the principal branch, $\log\{1 + 2e^{ic\tau}g(\tau)\} \to 2e^{ic\tau}g(\tau)$, $u \to \infty$. Thus, we have from (17), (23), and (28)

$$\log 2|s(t)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(x)}{t-x}\,dx, \qquad |h(x)| < \pi, \tag{29}$$

where h is the function defined in (11). Now, since h is a bounded high-pass function, we can obtain an exponential localization of the Hilbert transform in (29).

IV. LOCALIZATION OF THE HILBERT TRANSFORM

If the spectrum of a bounded function $h(t)$ omits the interval $(-\lambda, \lambda)$, there are many equivalent representations for its Hilbert transform. We have[4]

$$h(t) = \int_{-\infty}^{\infty} h(x)K(t-x)dx, \tag{30}$$

where*

$$K(t) = \frac{f(t)}{\pi t} \tag{30a}$$

and

$f(t)$ is any bandlimited function whose spectrum is confined to $[-\lambda, \lambda]$, satisfying (30b)

$$f(0) = 1 \tag{30c}$$

and

$$\int_{|t|>1} \frac{|f(t)|}{|t|} dt < \infty. \tag{30d}$$

Although $\hat{h}(t)$ is not generally bounded, the singular kernel $K(t)$ can be truncated to an interval $(-T, T)$ giving $K_T(t)$, such that

$$\hat{h}_T(t) = \int_{-\infty}^{\infty} h(x)K_T(t-x)dx = \int_{-T}^{T} K_T(x)h(t-x)dx \tag{31}$$

is a good approximation to $\hat{h}(t)$; i.e.,

$$h(t) - h_T(t) = \int_{|x|>T} \frac{f(x)}{\pi x} h(t-x)dx. \tag{32}$$

Then

$$|\hat{h}(t) - \hat{h}_T(t)| \leq M \int_{|x|>T} \frac{|f(x)|}{\pi |x|} dx, \tag{33}$$

where

$$M = \sup_t |h(t)|.$$

The bandlimited function f that minimizes the integral in (33), with $f(0) = 1$ and spectrum confined to $[-\lambda, \lambda]$, is (see Ref. 4)

$$f(t) = \frac{T}{\sinh \lambda T} \frac{\sin \lambda \sqrt{t^2 - T^2}}{\sqrt{t^2 - T^2}} \tag{34}$$

or, equivalently,

$$f(t) = \frac{T}{\sinh \lambda T} \frac{\sinh \lambda \sqrt{T^2 - t^2}}{\sqrt{T^2 - t^2}}. \tag{34a}$$

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* Formally, the Fourier transform of K is the convolution of $F(\omega)$, the Fourier transform of f, with $-i \operatorname{sgn} \omega$, which then agrees with $-i \operatorname{sgn} \omega$ for $|\omega| > \lambda$.

So, we will set

$$K_T(t) = \frac{1}{\pi t} \cdot \frac{T}{\sinh \lambda T} \frac{\sinh \lambda\sqrt{T^2 - t^2}}{\sqrt{T^2 - t^2}}, \qquad |t| < T$$
$$= 0, \qquad |t| \geq T. \tag{35}$$

Then, (34) in (33) gives (see Ref. 4)

$$|\hat{h}(t) - \hat{h}_T(t)| \leq \frac{2M}{\pi} \log \frac{1 + e^{-\lambda T}}{1 - e^{-\lambda T}}. \tag{36}$$

So K_T, defined in (35), gives us the desired exponential localization of the Hilbert transform.

V. LOCALIZATION OF THE RECONSTRUCTION OF $s(t)$

Now we are prepared to obtain a good approximation to $s(t)$ belonging to $S(b, c)$, $c - b = \lambda$, from a knowledge of the zeros of s in $(t - T, t + T)$. From (29) we have

$$s(t) = (1/2)\{\text{sgn } s(t)\}\exp \hat{h}(t), \tag{37}$$

where $\hat{h}$ is the Hilbert transform of the high-pass function h defined in (11). Now we set

$$s_T(t) = (1/2)\{\text{sgn } s(t)\}\exp \hat{h}_T(t), \tag{38}$$

where

$$\hat{h}_T(t) = \int_{-\infty}^{\infty} h(x)K_T(t - x)dx \tag{39}$$

and K_T is defined in (35). Since $|h(x)| < \pi$, we have from (36)

$$|\hat{h}(t) - \hat{h}_T(t)| < 2 \log \frac{1 + e^{-\lambda T}}{1 - e^{-\lambda T}}. \tag{40}$$

Owing to the special nature of $h(x)$ (cf. [11]), we can manipulate the integral (39) into a more convenient form. We may write

$$\hat{h}_T(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} dh(x)L_T(t - x)dx$$
$$= \sum_{-\infty}^{\infty} L_T(t - t_k) - \frac{c}{\pi} \int_{-\infty}^{\infty} L_T(t)dt, \tag{41}$$

where $\{t_k\}$ are the zeros of $s(t)$ and

$$L_T(t) = \pi \int_{-\infty}^{t} K_T(x)dx, \qquad -\infty < t < 0, \tag{42}$$

$$L_T(t) = L_T(-t), \tag{42a}$$

$$L_T(t) = 0, \qquad |t| > T, \tag{42b}$$

and

$$L_T(t) = -\int_t^T \frac{f(x)}{x}\,dx, \qquad 0 < t < T. \tag{42c}$$

The function $L_T(t)$ is a sort of truncated log $|t/T|$, depending both on λ and T, although in the notation we have suppressed the dependence on λ. Using the fact that $f(0) = 1$, we have

$$\begin{aligned} L_T(t) &= \log\left|\frac{t}{T}\right| + \int_t^T \frac{1 - f(x)}{x}\,dx, \qquad |t| < T \\ &= 0, \qquad |t| \geq T, \end{aligned} \tag{42d}$$

and furthermore, since $f(x) > 0$ for $|x| \leq T$ (cf. [42c]),

$$L_T(t) \leq 0, \qquad -\infty < t < \infty. \tag{42e}$$

Although it may be simpler to compute $\hat{h}_T(t)$ from (41) and then compute $s_T(t)$ according to (38), it is interesting to express $s_T(t)$ in product form. So we define

$$V_T(t) = \exp L_T(t). \tag{43}$$

Here, $V_T(t)$ is a sort of V-shaped function that we may write using (42d),

$$\begin{aligned} V_T(t) &= \left|\frac{t}{T}\right| \exp\left\{\int_t^T \frac{1 - f(x)}{x}\,dx\right\}, \qquad |t| < T \\ &= 1, \qquad |t| \geq T, \end{aligned} \tag{43a}$$

and we note that

$$V_T'(0+) = \frac{1}{T}\exp\int_0^T \frac{1 - f(x)}{x}\,dx \tag{44}$$

and

$$V_T'(T-) = \frac{f(T)}{T} = \frac{\lambda}{\sinh \lambda T}. \tag{44a}$$

Now we may write

$$s_T(t) = A\{\operatorname{sgn} s(t)\} \prod_{k=-\infty}^{\infty} V_T(t - t_k), \tag{45}$$

where, from (41) and (38),

$$A = A(b, c, T) = \frac{1}{2} \exp \left\{ -\frac{c}{\pi} \int_{-\infty}^{\infty} L_T(t)dt \right\}. \tag{45a}$$

We have

$$\int_{-\infty}^{\infty} L_T(t)dt = -\int_{-\infty}^{\infty} tL_T'(t)dt = -\int_{-T}^{T} f(t)dt. \tag{46}$$

From a table of Fourier transforms,[5] we find

$$\int_{-\infty}^{\infty} \frac{\sin \lambda\sqrt{t^2 - T^2}}{\sqrt{t^2 - T^2}} \cos \omega t dt = \pi I_0(T\sqrt{\lambda^2 - \omega^2}), \qquad |\omega| < \lambda$$

$$= 0, \qquad |\omega| > \lambda \tag{47}$$

and

$$\int_{-T}^{T} \frac{\cosh \lambda\sqrt{t^2 - T^2}}{\sqrt{t^2 - T^2}} \cos \omega t dt = \pi J_0(T\sqrt{\omega^2 - \lambda^2}), \tag{48}$$

where $I_0(x) = J_0(ix)$ is the modified Bessel function. Setting $\omega = 0$, we have

$$\int_{-\infty}^{\infty} \frac{\sin \lambda\sqrt{t^2 - T^2}}{\sqrt{t^2 - T^2}} dt = \int_{-T}^{T} \frac{\cosh \lambda\sqrt{t^2 - T^2}}{\sqrt{t^2 - T^2}} dt = \pi I_0(\lambda T),$$

from which we find

$$\int_{|t|>T} \frac{\sin \lambda\sqrt{t^2 - T^2}}{\sqrt{t^2 - T^2}} dt = 2 \int_0^T \frac{e^{-\lambda\sqrt{t^2-T^2}}}{\sqrt{t^2 - T^2}} dt$$

$$= 2 \int_0^{\pi/2} e^{-\lambda T \sin\theta} d\theta \tag{49}$$

and

$$\int_{-T}^{T} \frac{\sinh \lambda\sqrt{T^2 - t^2}}{\sqrt{T^2 - t^2}} dt = \pi I_0(\lambda T) - 2 \int_0^{\pi/2} e^{-\lambda T \sin\theta} d\theta. \tag{50}$$

Thus,

$$\int_{-\infty}^{\infty} L_T(t)dt = -\frac{T}{\sinh \lambda T} \left\{ \pi I_0(\lambda T) - 2 \int_0^{\pi/2} e^{-\lambda T \sin\theta} d\theta \right\}, \tag{51}$$

and

$$A = A(b, c, T) = \frac{1}{2} \exp \left\{ \frac{c}{\lambda} \mu(\lambda T) \right\}, \tag{52}$$

where $\lambda = c - b$, and

$$\mu(x) = \frac{xI_0(x)}{\sinh x} - \frac{x}{\sinh x} \cdot \frac{2}{\pi} \int_0^{\pi/2} e^{-x \sin\theta} d\theta \sim \sqrt{\frac{2x}{\pi}},$$

$$x \to \infty. \quad (52a)$$

We observe, since $V_T(t) = 1$ for $|t| \geq T$, that the infinite product in (45) is, in effect, for each t, a finite product. That is, we may write

$$s_T(t) = A\{\mathrm{sgn}\ s(t)\} \cdot \prod_{k:|t-t_k|<T} V_T(t - t_k) \quad (53)$$

to emphasize the localization of the reconstruction.

Now, to estimate the error, we write, using (37) and (38),

$$\begin{aligned} s_T(t) - s(t) &= (1/2)\{\mathrm{sgn}\ s(t)\}\{\exp[\hat{h}_T(t)] - \exp[\hat{h}(t)]\} \\ &= (1/2)s(t) \cdot \{\exp[\hat{h}_T(t) - \hat{h}(t)] - 1\}. \end{aligned} \quad (54)$$

Now,

$$\begin{aligned} |e^x - 1| &= \left| x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right| \\ &\leq \left| |x| + \frac{|x|^2}{2!} + \frac{|x|^3}{3!} + \cdots \right| = e^{|x|} - 1. \end{aligned}$$

Setting $x = \hat{h}_T(t) - \hat{h}(t)$ and using the estimate (40)

$$|\hat{h}_T(t) - \hat{h}(t)| < 2 \log \frac{1 + e^{-\lambda T}}{1 - e^{-\lambda T}},$$

we have

$$\begin{aligned} |s(t) - s_T(t)| &\leq \frac{1}{2} |s(t)| \cdot \left\{ \frac{1 + e^{-\lambda T}}{1 - e^{-\lambda T}} - 1 \right\} \left\{ \frac{1 + e^{-\lambda T}}{1 - e^{-\lambda T}} + 1 \right\} \\ &\leq \frac{2e^{-\lambda T}}{(1 - e^{-\lambda T})^2} |s(t)| \end{aligned} \quad (55)$$

with equality only for $t = t_k$, where $s(t_k) = 0$.

The error estimate in (55) is conservative since equality in (33) and (36) is attained only for (see Ref. 4)

$$\begin{aligned} h(t) &= \pm M(\mathrm{sgn}\ t) \cdot \{\mathrm{sgn}\ \sin \lambda \sqrt{t^2 - T^2}\}, \qquad |t| > T \\ &= 0, \qquad t \leq T. \end{aligned}$$

Here we have applied the inequality to a high-pass function $h(t)$, the "meandering" sawtooth function of (26) with $h(0) = 0$, which is quite different from the above extremal function. However, the exponent λT is probably sharp.

A series for computing $L_T(t)$ is given in the appendix.

VI. CONCLUSION

We have seen that signals of a certain class can be reconstructed at any point t with a relative error of the order of $e^{-\lambda T}$ from the knowledge of their zeros in an interval $(t - T, t + T)$. The reconstruction algorithm can be applied to practical signals of the form

$$s(t) = g_p(t) + \cos ct,$$

where $g_p(t)$ can be closely approximated by a bandlimited signal, $g(t)$, whose spectrum is confined to the interval $[-b, b]$. We would like the zeros of $s(t)$ not to change much when g_p is replaced by g. First, we require

$$|g_p(t)| \leq M < 1$$

for some suitable choice of M, say $M = 0.75$. We also need a condition on $|g_p(t)|$ *and* $|g'_p(t)|$ to ensure that $s(t)$ has only one zero between each extremal of cos ct, e.g.,

$$\{g_p(t)\}^2 + \{g'_p(t)/c\}^2 < 1.$$

This condition alone gives $|g_p(t)| < 1$, and hence, $(-1)^k s(k\pi/c) > 0$, from which we conclude that $s(t)$ must have an odd number of zeros in $(k\pi/c, (k + 1)\pi/c)$. It is sufficient to consider $(0, \pi/c)$. Suppose the number of zeros (counting multiplicity) in $(0, \pi/c)$ is more than 1. Then, at some point x_k in this interval (say, the second zero), we must have $s(x_k) = 0$ and $s'(x_k) \geq 0$; i.e.,

$$g_p(x_k) = -\cos cx_k,$$

$$g'_p(x_k) \geq c \sin cx_k$$

and then

$$\{g_p(x_k)\}^2 + \{g'_p(x_k)/c\}^2 \geq 1,$$

which is a contradiction. Then, with the two above conditions, we may regard the zeros of $s(t)$ as perturbations of the zeros of

$$s(t; b) = g(t) + \cos ct,$$

assuming that

$$|g(t) - g_p(t)| \leq \epsilon(b) < 1 - M.$$

Though the calculation is complicated, estimates can be made for the error in reconstructing $g_p(t)$ from the zeros of $s(t)$. Of course, the relative error in $|s(t) - s_T(t)|$ will not be of the order of $e^{-\lambda T}$ but of the order of $|g(t) - g_p(t)|$ (provided λT is sufficiently large). The algorithm has much in common with the reconstruction of almost band-limited signals from their samples. That is, if we were given

$g_p(k\pi/c)$ and knew that

$$|g(t) - g_p(t)| \le \epsilon(b),$$

we would use the fact that

$$g(t) = \frac{\pi}{c} \sum_{-\infty}^{\infty} g\left(\frac{k\pi}{c}\right) K\left(t - \frac{k\pi}{c}\right)$$

for any function $K(t)$ of the form

$$K(t) = \frac{\sin ct}{ct} f_\lambda(t),$$

where $f_\lambda(t)$ is a bandlimited function whose Fourier transform vanishes outside $[-\lambda, \lambda]$, $\lambda = c - b > 0$, and $f_\lambda(0) = 1$. Although the series would converge conditionally for $f(t) \equiv 1$, it would not necessarily converge when $g(k\pi/c)$ is replaced by $g_p(k\pi/c)$. So, it would be far better to take, for example,

$$f_\lambda(t) = \frac{T}{\sinh \lambda T} \frac{\sin \lambda\sqrt{t^2 - T^2}}{\sqrt{t^2 - T^2}}$$

and obtain an exponential localization in the reconstruction of $g(t)$ from its samples and then apply the formula to $g_p(t)$. The basic difference in the problem solved here is that the zeros of $s(t)$ give a *nonuniform* sampling of $g(t)$, but we make use of the fact that the fundamental function $h(t)$, given by (11), is high pass to obtain the exponential localization of the reconstruction.

We should note that the validity of the reconstruction could be extended to signals of the form (3) with implicit restrictions on $g(t)$ such that $s(t)$ has only real simple zeros. This does not gain us anything from a practical viewpoint since we do not have a practical way to determine all admissible functions $g(t)$, but it does say something interesting about the zeros of such functions, viz., that the zeros allow the exponential localization of the infinite products. The difference now is that we do not have the zeros occurring so regularly and we have to replace the upper bound for $|h(t)|$ in (16) by something larger than π. This raises an interesting mathematical question: if $s(t)$ is of the form

$$s(t) = g(t) + \cos ct$$

and g is bandlimited to $[-b, b]$, $0 < b < c < \infty$, and such that $s(t)$ has only real zeros, what is the longest possible zero-free interval for $s(t)$? It is probably something like π/λ, where $\lambda = c - b$. Also, what is the optimal truncation of the Hilbert transform kernel in order to estimate $\hat{h}(t)$ for the special class of $h(t)$ encountered in this problem?

Once a reasonable reconstruction algorithm has been found, one can ask for the best function $V_T(t)$, where $V_T(t) = 1$ for $|t| \geq T$ and corresponding constant A such that

$$s_T(t) = A\{\text{sgn } s(t)\} \prod_{k=-\infty}^{\infty} V_T(t - t_k)$$

best approximates $s(t)$ of the form

$$s(t) = g(t) + \cos ct$$

for some reasonably smooth class of functions $g(t)$, $|g(t)| \leq M < 1$. For example, one might take typical speech signals $g(t)$ and, with suitable c and T, find V_T empirically.

Some interesting applications may arise when, instead of $s(t)$, one transmits $p(t)s(t)$, where $p(t)$ is a low-frequency positive (nonnegative) signal. Then, $s(t)$ can be recovered by clipping $p(t)s(t)$ and applying the reconstruction algorithm. For example, in single-sideband transmission of $x(t)$ it is advantageous to transmit the carrier cos ct for purposes of demodulation; i.e., $s(t)$ is of the form (lower sideband)

$$s(t) = (1 + x(t))\cos ct + \hat{x}(t)\sin ct,$$

where $x(t)$ is bandpass, $[a, b]$ and $[-b, -a]$, $0 < a < b < c$, and $|x(t)| < 1$. Here, if $x(t)$ is a speech signal, one could take $p(t)$ to be a low-frequency (syllabic-frequency) envelope of $x(t)$ and transmit $p(t)s(t)$ to reduce the power in the carrier, and then recover $s(t)$ at the receiver. The disadvantage is more sensitivity to noise, but this added sensitivity occurs only when $p(t)$ is small, so $p(t)$ could be detected at the receiver and used as a squelch control to mute the noise portions.

It appears feasible to construct analog circuitry for effecting the reconstruction; the singularity in $L_T(t)$ is not bad, and for practical purposes, $L_T(t - T)$ can be approximated by the impulse response of a realizable filter. The pulse train $\sum \delta(t - t_k)$ can be ac coupled into the filter to obtain $\hat{h}_T(t - T)$ at the output, since the ac coupling gives (for sufficiently large time constant) very nearly $(\sum \delta(t - t_k) - c/\pi)$ as input to the filter. The fact that (the approximation to) $\hat{h}_T$ remains finite means that exp $\hat{h}_T(t)$ is not a good approximation to $|2s(t)|$ near the zeros of $s(t)$; however, this is not important. Also, the problem of delaying sgn $s(t)$ for multiplication by exp $\hat{h}_T(t - T)$ can be avoided by using a threshold circuit to detect the "near-zeros" (approximately $t_k - T$) of exp $\hat{h}_T(t - T)$, and then, pulses derived from the threshold circuit can be used to generate sgn $s(t - T)$.

It may be desirable to use a feedback circuit consisting of an operational integrating amplifier and a multiplier connected so that the output $y(t)$ of the amplifier satisfies

$$y'(t) \doteq \hat{h}'_T(t)y(t).$$

Then,

$$y \doteq M \exp \hat{h}_T(t).$$

Here,

$$\hat{h}'_T(t) = \pi \sum K_T(t - t_k),$$

and one uses a filter whose impulse response approximates $K_T(t - T)$, except for t very near T. Additional means will probably have to be provided for controlling drift in the integrator (or equivalent offset in the multiplier).

Given the state of the art, a specialized digital computer may be more practical than an analog computer for effecting the reconstruction. The problem encountered here is the quantization of the location of the zeros. Some analysis is required, then, to determine admissible bounds on the quantization errors.

For practical purposes, one need only compute $s_T(t)$ on a lattice, say $t = k\pi/c$. This is a good choice since $s(k\pi/c) \neq 0$, and so, $\hat{h}_T(k\pi/c)$ will be more easily computed in practice. One then would compute

$$g_T(k\pi/c) = s_T(k\pi/c) - \cos(k\pi)$$

and then filter the samples to obtain an approximation to $g(t)$. In analog computation this method eases the problem of approximating $\hat{h}_T(t)$ or $L_T(t)$ since one does not require the value of L_T for small arguments.

In connection with this last method, one may be given an unknown translate of $\mathrm{sgn}\{s(t)\}$, so some method is required to determine the proper phase for sampling. It can be shown that

$$\{\mathrm{sgn}\ s(t)\}\cdot\{\mathrm{sgn}\ \sin(ct + \theta)\} - \frac{2\theta}{\pi}, \qquad \text{where} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

is high pass, provided θ is small enough to allow the zeros of $s(t)$ and $\sin(ct + \theta)$ to interlace. This implies that phase-lock circuitry can be used to adjust θ to zero, and thus determine the proper phase for sampling, viz., at the zeros of $\sin ct$.

Alternatively, one could construct the fundamental function $h(t)$ by forming an unbiased integral of

$$h'(t) = \pi \sum \delta(t - t_k) - c,$$

where $\delta(t)$ is the Dirac delta function. An unbiased integral of a high-pass function is a particular integral that is also high pass. For practical purposes, a good approximation to $h(t)$ can be obtained by forming a narrow pulse $p(t - t_k)$ at each zero t_k of s and then ac coupling $\sum p(t - t_k)$ into an RC integrator. The zeros of h that occur between the pulses are then the proper sampling points, the zeros of sin ct (see Refs. 11 and 27).

VII. ACKNOWLEDGMENTS

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APPENDIX

Series for Computing $L_T(t)$

Replacing t in (42d) by xT, we have

$$\begin{aligned} L_T(xT) &= \log|x| + F(x;\lambda T), \qquad |x| < 1 \\ &= 0, \qquad |x| > 1, \end{aligned} \tag{56}$$

where

$$F(x;\beta) = \int_x^1 \frac{\sqrt{1-t^2} - (\sinh \beta\sqrt{1-t^2})/\sinh\beta}{t\sqrt{1-t^2}}\,dt \quad |x| < 1. \tag{57}$$

The integral in (57) can be evaluated with the aid of the identity

$$\frac{\sinh \beta\sqrt{1-t^2}}{\beta\sqrt{1-t^2}} = \sqrt{\pi/2\beta}\sum_{n=0}^{\infty}\frac{(-\beta t^2/2)^n}{n!} I_{n+1/2}(\beta), \tag{58}$$

where

$$\sqrt{\pi/2\beta}I_{1/2}(\beta) = \frac{\sinh\beta}{\beta},$$

$$\sqrt{\pi/2\beta}I_{3/2}(\beta) = -\frac{\sinh\beta}{\beta^2} + \frac{\cosh\beta}{\beta},$$

$$\sqrt{\pi/2\beta}I_{5/2}(\beta) = \left(\frac{3}{\beta^3} + \frac{1}{\beta}\right)\sinh\beta - \frac{3}{\beta^2}\cosh\beta,$$

and, in general,

$$\sqrt{\pi/2\beta}I_{n+1/2}(\beta) = (2\beta)^{-1}\{R[n + 1/2, -\beta]e^{\beta} - (-1)^n R[n + 1/2, \beta]e^{-\beta}\},$$

where

$$R\left\{n + \frac{1}{2}, \beta\right\} = 1 + \frac{(n+1)!}{1!\Gamma(n)}(2\beta)^{-1} + \frac{(n+2)!}{2!\Gamma(n-1)}(2\beta)^{-2} + \cdots + \frac{(2n)!(2\beta)^{-n}}{n!}$$

$$= \sum_0^n \frac{(n+k)!}{k!(n-k)!}(2\beta)^{-k}$$

and

$$R\{1/2, \beta\} = 1.$$

Alternatively, if we set

$$\phi_n(\beta) = \sqrt{\pi/2\beta}I_{n+1/2}(\beta) \equiv \beta^n \left(\frac{1}{\beta}\frac{d}{d\beta}\right)^n \frac{\sinh\beta}{\beta}, \tag{59}$$

we have the recurrence relation,

$$\phi_{n+1}(\beta) = \phi_{n-1}(\beta) - \frac{(2n+1)}{\beta}\phi_n(\beta). \tag{60}$$

In this notation we have

$$\frac{\sinh \beta\sqrt{1-t^2}}{(\sinh\beta)\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(-\beta t^2/2)^n}{n!}\frac{\phi_n(\beta)}{\phi_0(\beta)}, \tag{61}$$

and

$$F(0, \beta) = -\sum_{n=1}^{\infty} \frac{(-\beta/2)^n}{(2n)n!}\frac{\phi_n(\beta)}{\phi_0(\beta)}, \tag{62}$$

and

$$F(x, \beta) = F(0, \beta) + \sum_{n=1}^{\infty} \frac{(-\beta x^2/2)^n}{(2n)n!}\frac{\phi_n(\beta)}{\phi_0(\beta)}. \tag{63}$$

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