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Signals Designed for Recovery After Clipping—II. Fourier Transform Theory of Recovery

By B. F. LOGAN, JR.*

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This paper develops a Fourier transform theory for the recovery of signals of a certain class from their zeros. The class, denoted by $S(b, c)$, consists of real-valued signals of the form $s(t) = g(t) + \cos ct$, where g is bandlimited to $[-b, b]$, $0 < b < c < \infty$, and such that $(-1)^k s(k\pi/c) > 0$, which is satisfied, for example, if $|g(t)| < 1$. A very simple method of recovery is given for the case $c > 3b$, and a somewhat more complicated method is given for the case $c > 2b$. The theory also suggests a novel method of effecting amplitude modulation, which has advantages in high-power applications.

I. INTRODUCTION

Reference 1 shows that there is a practical way to reconstruct signals $s(t)$ of a certain class $S(b, c)$ from their zeros. The class $S(b, c)$ consists of real-valued signals of the form

$$s(t) = g(t) + \cos ct, \tag{1}$$

where

$$g \text{ is bandlimited to } [-b, b],\ 0 < b < c < \infty, \tag{1a}$$

and such that

$$(-1^k s(k\pi/c) > 0, \qquad k = 0, \pm 1, \pm 2, \cdots . \tag{1b}$$

The alternation condition (1b) ensures that s has only real, simple zeros, one between each extremal of $\cos ct$. A practical sufficient

* AT&T Bell Laboratories.

condition for (1b) is

$$|g(t)| < 1, \quad -\infty < t < \infty. \tag{1c}$$

However, (1c) is not necessary.

In Ref. 1 we introduced the so-called "fundamental function" associated with the zeros $\{t_k\}$ of $s \in S(b, c)$; viz.,

$$h(t) = J(t) - ct, \tag{2}$$

where $J(t)$ is a jump function increasing by π at each zero t_k, $J(0) = 0$. We showed that $h(t)$ is a high-pass function with no spectrum in $(-\lambda, \lambda)$, where

$$\lambda = c - b > 0 \tag{3}$$

is the "gap frequency", i.e., the difference between the top frequency of g and the carrier or "bias" frequency. We also established that

$$-\pi < h(t) < \pi, \ -\infty < t < \infty, \tag{4}$$

and derived the important relation

$$\log|2s(t)| = \hat{h}(t) = \frac{1}{\pi} - \int_{-\infty}^{\infty} \frac{h(x)}{t - x}\, dx, \tag{5}$$

where $\hat{h}$ is the Hilbert transform of h. From (5) we obtained, using the fact that h is high-pass, an algorithm for approximating $s(t)$ just from a knowledge of the zeros of s in the interval $(t - T, t + T)$, the error in the approximation being of the order of $e^{-\lambda T}$.

Here we want to consider alternate ways of reconstructing $s(t)$ that involve additional facts about the Fourier transform of $h(t)$. Although the functions we deal with will not, in general, have Fourier transforms in the ordinary sense, we can still attach consistent meaning to the statement "the Fourier transform of f vanishes over E," where E is the union of a finite number of disjoint intervals. To do this (see Refs. 2 and 3) we simply require f to be orthogonal to a suitable class of functions that have ordinary Fourier transforms supported on E. Then we can say that "the Fourier transforms of f_1 and f_2 agree over E if and only if the Fourier transform of $(f_1 - f_2)$ vanishes over E." We will make free use of the results derived in Ref. 2 stemming from these basic definitions and also the results in Ref. 3 pertaining especially to high-pass functions. There is too much material needed for rigorous treatment to collect here. However, the reader should be able to follow most of the arguments by supposing the functions to have ordinary Fourier transforms or else by supposing the functions to be periodic.

We need not concern ourselves with the nature of the Fourier transform of a bounded function in the sense of Schwartz distributions.

We only need the concept of the Fourier transforms of f_1 and f_2 agreeing over intervals, which has all the utility we require.

Our first result is that for the case of large gap frequency, $c > 3b$, we have

$$g(t) = -\frac{\pi}{c} \sum_{-\infty}^{\infty} (\cos ct_k) K(t - t_k), \tag{6}$$

where K is in L_1 and

$$\begin{aligned} \int_{-\infty}^{\infty} K(t)e^{-i\omega t} &= 1, \qquad -b \le \omega \le b \\ &= 0, \qquad |\omega| \le \lambda - b = c - 2b > b. \end{aligned} \tag{7}$$

That is, if $c > 3b$, we can simply sample $-\cos ct$ at the zeros t_k and then filter the impulses to recover $g(t)$. Note that $g(t_k) = -\cos ct_k$.

We next establish that the Fourier transforms of the fundamental function h and the function h_n agree over $(-(n+1)\lambda, (n+1)\lambda)$, where

$$h_n(t) = -\text{Im}\left\{\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \gamma^k(t)\right\}, \qquad n = 1, 2, \cdots \tag{8}$$

and

$$\gamma(t) = 2e^{ict}g(t) + e^{2ict}. \tag{8a}$$

[In (8), Im denotes the imaginary part.] The spectrum of the function $\gamma(t)$ is one-sided, being confined to the interval $[\lambda, \lambda + 2b]$ and the point $2c$. Hence, the Fourier transform of $\gamma^k(t)$ vanishes over $(-\infty, k\lambda)$, $k = 1, 2, \cdots$. So in the case $2\lambda > c$, i.e., $c > 2b$, the Fourier transforms of h and h_1 agree over $(-2\lambda, 2\lambda)$, where

$$h_1(t) = -2g(t)\sin ct - \sin 2ct. \tag{9}$$

Now low-pass filtering of $h(t)$ is equivalent to filtering $h_1(t)$, provided the cutoff frequency of the filter is no greater than 2λ. In case $2\lambda > c$ we can filter $h(t)$ with a sideband filter (upper vestigial) to recover a little more than the lower sideband of the signal $-2g(t)\sin ct$, rejecting the higher frequency nonlinear components. Then standard sideband detection techniques allow us to recover $g(t)$, the details being given later.

In the case $b < c < 2b$, nonlinear operations are required to recover $g(t)$ or $s(t)$ from $h(t)$, i.e., from the zeros of s. The methods of Analytic Modulation Systems (see Ref. 2) are used here. We may take a filtered version of the analytic signal $H(t)$,

$$H(t) = h(t) + i\hat{h}(t), \tag{10}$$

namely,

$$H_\nu^*(t) = \int_{-\infty}^{\infty} H(x)K_\nu\,(t-x)dx, \tag{11}$$

where the filter kernel K_ν belongs to L_1 and

$$\int_{-\infty}^{\infty} K_\nu(t)e^{-i\omega t}\,dt = 1, \qquad \lambda \le \omega \le \nu, \tag{11a}$$

$$\nu > c. \tag{11b}$$

Both functions here have one-sided spectrum, and since h is high-pass, the Fourier transform of H vanishes over $(-\infty, \lambda)$. By virtue of (11a), the Fourier transforms of H_ν^* and H agree over $(-\infty, \nu)$.

Next we define

$$G_\nu^*(t) = \exp\{-iH_\nu^*(t)\}. \tag{12}$$

We then find that the Fourier transforms of G_ν^* and G agree over $(-\infty, \nu)$, where

$$G(t) = \exp\{-iH(t)\} = 1 + 2e^{ict}g(t) + e^{2ict}. \tag{13}$$

Now, since $\nu > c$, we can filter $G_\nu^*(t)$, using a vestigial sideband filter (cutoff frequency ν), which is equivalent to filtering $G(t)$, so as to recover a little more than the lower sideband of $2g(t)\sin ct$ (taking the imaginary part of $G_\nu^*(t)$). Then we can proceed as before with the standard sideband detection to recover $g(t)$. This method allows the minimum bandwidth requirement ($\nu > c$) in filtering operations on h, and thus eases the problem of obtaining the (unbounded) Hilbert transform $\hat{h}(t)$; i.e., $h(t) + i\hat{h}(t)$ is replaced by $h_\nu^*(t) + i\hat{h}_\nu^*(t)$. Note, however, that in practice two filtering operations are required here, $h \to h_\nu^*$ and $h \to \hat{h}_\nu^*$, to effect the operation indicated in (11).

Actually, the exponential function in (12) may be replaced by a partial sum of its power series, as detailed later. This option may be convenient in the case of moderately large gap frequencies where only a few terms are required.

The sideband detection required in the previous method can be avoided when (11b) is replaced by $\nu > 2c$. Then $G_\nu^*(t)$ can be low-pass filtered (cutoff frequency ν) to obtain (all of) $G(t) \equiv 2e^{ict}s(t)$. Equivalently, the Fourier transforms of $e^{-ict}G_\nu^*(t)$ and $2s(t)$ agree over $(-\mu, \mu)$, where $\mu = \nu - c > c$. Denoting the real part of $e^{-ict}G_\nu^*(t)$ by $2s_\mu^*(t)$, we have

$$s_\mu^*(t) = \frac{1}{2}\,\{\cos[h_\nu^*(t) + ct]\}\cdot\exp[\hat{h}_\nu^*(t)\}. \tag{14}$$

This is to be compared with the *basic formula*

$$s(t) = \frac{1}{2}\{\text{sgn } s(t)\}\exp\{\hat{h}(t)\}, \tag{15}$$

which follows from (5). Defining $J_\nu^*(t)$, analogous to $J(t)$ in (2), by

$$h_\nu^*(t) = J_\nu^*(t) - ct, \tag{16}$$

we may write

$$s_\mu^*(t) = \frac{1}{2}\{\cos[J_\nu^*(t)]\}\cdot\exp\{\hat{h}_\nu^*(t)\} \tag{17}$$

and then (15), in a similar form,

$$s(t) = \frac{1}{2}\{\cos[J(t)]\}\cdot\exp\{\hat{h}(t)\}. \tag{18}$$

The function $s_\mu^*(t)$ may be low-pass filtered (cutoff frequency μ) to obtain $s(t)$, the resulting formula being a generalization of the basic formula (15).

Finally, a novel method of effecting amplitude modulation is suggested by the fact that for s in $S(b, c)$, the Fourier transforms of the two functions, $g(t)\sin ct$ and $(\pi/4)$ $\{\text{sgn } s(t)\}\cdot\{\text{sgn } \sin ct\}$, agree over $(-2\lambda, 2\lambda)$. The square wave then may be filtered to obtain $g(t)\sin ct$, provided $2\lambda > c + b$ or $c > 3b$.

II. FILTERING THE INDUCED SAMPLES OF $g(t)$ FOR THE CASE $c > 3b$

The zeros of a signal s of the form (1), $s(t) = g(t) + \cos ct$, induce a certain "oversampling" of $g(t)$. That is, if we denote the zeros of s by $\{t_k\}$, $k\pi/c < t_k < (k+1)\pi/c$, we have

$$g(t_k) = -\cos ct_k, \; g \text{ bandlimited to } [-b, b]. \tag{19}$$

So, in effect, the zeros $\{t_k\}$ give us a nonuniform sampling of $g(t)$ at the rate c/π, and we only need a rate slightly larger than b/π to pin down $g(t)$. However, in the general case, nonuniform sampling complicates the recovery problem.

We have given in Ref. 1 a reconstruction formula for $s(t)$, based on (5), which may be regarded as one *practical* way of solving the nonuniform sampling problem (19). We could apply that formula to reconstructing only the uniform samples of $g(t)$, say $g(k\pi/c)$, and then use the fact that

$$g(t) = \frac{\pi}{c}\sum_{-\infty}^{\infty} g(k\pi/c)\, K\left(t - k\,\pi/c\right), \tag{20}$$

where K is any bandlimited function belonging to L_1 whose Fourier transform satisfies

$$\int_{-\infty}^{\infty} K(t)e^{-i\omega t}dt = 1, \qquad -b \leq \omega \leq b$$

$$= 0, \qquad |\omega| \geq 2c - b. \tag{20a}$$

In engineering terminology, we low-pass filter the uniform samples of g. The validity of (20) for bounded g whose Fourier transforms vanish outside $[-b, b]$ may be established from the fact (see Ref. 2) that K is a reproducing kernel for g,

$$g(t) = \int_{-\infty}^{\infty} g(x)K(t - x)dx, \tag{21}$$

and the fact that (for each t)

$$\int_{-\infty}^{\infty} g(x)K(t - x)e^{-i\omega x}dx = 0, \qquad |\omega| \geq 2c, \tag{22}$$

i.e., as a function of x, $g(x)K(t - x)$ is a function of L_1 whose Fourier transform vanishes (see Ref. 2) outside $(-2c, 2c)$. Then (20) follows by applying the Poisson sum formula to $g(x)K(t - x)$,

$$\int_{-\infty}^{\infty} g(x)K(t - x)dx = \frac{\pi}{c} \sum_{-\infty}^{\infty} g\ (k\pi/c)\ K\ (t - k\pi/c). \tag{23}$$

Now we would like to show that the nonuniform samples of g at the zeros of s can be treated in the same way as uniform samples in case $c > 3b$.

We consider the logarithmic derivative of s,

$$\frac{s'(\tau)}{s(\tau)} = \frac{g'(\tau) - c \sin c\tau}{g(\tau) + \cos c\tau}, \ \tau = t + iu, \tag{24}$$

which may be written, multiplying numerator and denominator by $2e^{ic\tau}$, as

$$\frac{s'(\tau)}{s(\tau)} = \frac{2e^{ic\tau}g'(\tau) + ice^{2ic\tau} - ic}{2e^{ic\tau}g(\tau) + e^{2ic\tau} + 1}. \tag{24a}$$

Then, since $g'(\tau)$ and $g(\tau)$ grow no faster than $A \cos b\tau$ (see Ref. 4, Theorem 6.2.6, p. 83), we have

$$\frac{s'(t + iu)}{s(t + iu)} + ic = 0(e^{-\lambda u}), \qquad u \to \infty. \tag{25}$$

From the infinite product representation of $s(\tau)$ we have

$$\frac{s'(\tau)}{s(\tau)} = \sum_{-\infty}^{\infty} \frac{1}{\tau - t_k}, \tag{26}$$

where the sum converges conditionally. But if we write

$$\frac{s'(t+iu)}{s(t+iu)} = \sum_{-\infty}^{\infty} \frac{1}{t - t_k + iu}$$

$$= \sum_{-\infty}^{\infty} \frac{(t-t_k)}{(t-t_k)^2+u^2} - i \sum_{-\infty}^{\infty} \frac{u}{(t-t_k)^2+u^2}, \qquad (27)$$

the second sum converges absolutely (see Ref. 4, p. 86). For any fixed $u > 0$ the second sum is a bounded function of t, since $k\,\pi/c < t_k < (k+1)\pi/c$. Then (25) implies that for any fixed $u > 0$, the function of t,

$$h'(t;\, u) = \sum_{-\infty}^{\infty} \frac{u}{(t-t_k)^2+u^2} - c, \qquad (28)$$

is a high-pass function whose Fourier transform vanishes over $(-\lambda, \lambda)$. The integral of each of the pulses in the sum is π, and the pulses concentrate around the points t_k as $u \to 0$. Then, if f belongs to L_1 and its Fourier transform vanishes outside $(-\lambda, \lambda)$, we have

$$\int_{-\infty}^{\infty} f(t)h'(t;\, u)dt = 0, \qquad u > 0. \qquad (29)$$

Then, letting $u \to 0$, we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{c} \sum_{-\infty}^{\infty} f(t_k). \qquad (30)$$

Now if $\lambda > 2b$, i.e., if $c > 3b$, we can apply (30) to

$$f(x) = g(x)K(t-x),$$

where K is in L_1 and

$$\int_{-\infty}^{\infty} K(t)e^{-i\omega t}dt = 1, \qquad -b < \omega < b$$

$$= 0, \qquad |\omega| > \lambda - b = c - 2b > b, \qquad (31)$$

to obtain

$$g(t) = \frac{\pi}{c} \sum_{-\infty}^{\infty} g(t_k)K(t-t_k)$$

$$= -\frac{\pi}{c} \sum_{-\infty}^{\infty} (\cos ct_k)K(t-t_k) \qquad (c > 3b). \qquad (32)$$

That is, the nonuniform samples of g at the zeros of s, occurring at the rate c/π, may be filtered as if they were uniform samples, provided $c > 3b$. The rate of nonuniform sampling must be three times greater than the Nyquist rate for g in order to obtain this simple method of recovery. We merely sample cos ct at the zeros of $s(t)$ and then use appropriate low-pass filtering to recover $g(t)$. In this connection, if the phase of the carrier, or high-frequency bias, cos ct is not known, it can in practice be obtained from phase-lock circuitry operating on sgn $s(t)$. It has already been noted in Ref. 1 that [cf. eqs. (82) and (83) here]

$$\{\text{sgn } s(t)\} \cdot \{\text{sgn } \sin ct\}$$

is high-pass, which gives a proper condition for phase lock.

In connection with the necessity of the condition $c > 3b$, we note that no matter how small g is, provided $g \not\equiv 0$, the nonuniform sampling, though almost uniform, is not perfect in the sense that (32) does not hold (exactly) unless $c > 3b$. We could estimate the error in (32) for $c > b$, which, with proper choice of K, will be small for small g (depending, of course, on how large the gap frequency is). Depending on the application, (32) may give a suitable approximation to $g(t)$ for c only somewhat larger than b.

We note that in (28), letting $u \to 0$, we may write, formally,

$$h'(t;\, 0+) = h'(t) = \pi \sum_{-\infty}^{\infty} \delta(t - t_k) - c, \tag{33}$$

where $\delta(t)$ is the Dirac delta function and $h'(t)$ is interpreted as the derivative of the fundamental function $h(t)$ defined in (2). So far we have described the Fourier transform of $h(t)$ only as vanishing over $(-\lambda, \lambda)$. We give a further description next.

III. THE FOURIER TRANSFORM OF THE FUNDAMENTAL FUNCTION

The complex-valued function,

$$H(t) = h(t) + i\hat{h}(t), \tag{34}$$

where h is the fundamental function and $\hat{h}$ is the Hilbert transform of h, has a one-sided spectrum, i.e., its Fourier transform vanishes over $(-\infty, \lambda)$. Equivalently (see Ref. 2), it extends as a function analytic in the upper half-plane of the complex variable $\tau = t + iu$, satisfying

$$H(t + iu) = 0(e^{-\lambda u}), \qquad u \to \infty \quad (-\infty < t < \infty). \tag{35}$$

As Ref. 1 establishes, $H(t)$ is related to $s(t)$ by

$$H(\tau) = i \log G(\tau), \tag{36}$$

where

$$G(\tau) = 2e^{ic\tau}\, s(\tau), \tag{37}$$

and we take the principal branch, $\log(1 + z) \to z$ as $z \to 0$.

The function $G(\tau)$ is an entire function. Let us set

$$G(\tau) = 1 + \gamma(\tau) = 1 + 2e^{ic\tau} g(\tau) + e^{2ic\tau}, \qquad (38)$$

where

$$\gamma(t + iu) = 0\{e^{-\lambda u}\}, \qquad u \to \infty. \qquad (38a)$$

For sufficiently large u, say $u > u_0$, we will have $|\gamma(t + iu)| < 1$ for all t so that the Taylor series for $\log G$ converges, i.e.,

$$H(\tau) = i\left[\gamma(\tau) - \frac{1}{2}\gamma^2(\tau) + \frac{1}{3}\gamma^3(\tau) + \cdots\right], \qquad (\text{Im } \tau > u_0). \quad (39)$$

The series does not converge everywhere on the real line because we do not have $|\gamma(t)| < 1$ for $-\infty < t < \infty$. However, if we define

$$H_n(\tau) = i \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \gamma^k(\tau), \qquad n = 1, 2, \cdots, \qquad (40)$$

we have

$$H(\tau) - H_n(\tau) = 0\ \{\gamma^{n+1}(\tau)\}, \qquad \gamma \to 0$$
$$= 0\ \{e^{-(n+1)\lambda u}\}, \qquad u \to \infty. \qquad (41)$$

It follows (see Ref. 2) that the Fourier transforms of $H(t)$ and $H_n(t)$ agree over $(-\infty, (n + 1)\lambda)$. Also, the Fourier transforms of $H(t)$ and $H_n(t)$ vanish over $(-\infty, \lambda)$. Then for any finite n we may write

$$H(t) = H_n(t) + R_{n+1}(t)$$
$$= h_n(t) + i\hat{h}_n(t) + R_{n+1}(t), \qquad (42)$$

where the Fourier transform of $R_{n+1}(t)$ vanishes over $(-\infty, (n + 1)\lambda)$, and $\hat{h}_n$ is the Hilbert transform of h_n.

So, despite the fact that the infinite series in (39) does not converge for $u = 0$, we can still equate the Fourier transforms of $H(t)$ and the partial sums $H_n(t)$ of the series over the frequency interval $(-\infty, (n + 1)\lambda)$. Here again we see the importance of a positive gap frequency λ. Since

$$\gamma(t) = 2e^{ict}g(t) + e^{2ict}, \qquad (43)$$

we have an effective way of describing the Fourier transforms of $h(t)$ and $\hat{h}(t)$ over intervals $(-(n + 1)\lambda, (n + 1)\lambda)$; i.e., they agree with the Fourier transforms of $h_n(t)$ and $\hat{h}_n(t)$, respectively, over such intervals. These functions in turn are simply related to g through γ. For example,

$$H_1(t) = i\gamma(t)$$
$$= 2ie^{ict}g(t) + ie^{2ict}, \qquad (44)$$

$$h_1(t) = \frac{1}{2} H_1(t) + \frac{1}{2} \bar{H}_1(t)$$

$$= -2g(t)\sin ct - \sin 2ct, \quad (45)$$

$$\hat{h}_1(t) = \frac{1}{2i} H_1(t) - \frac{1}{2i} \bar{H}_1(t)$$

$$= 2g(t)\cos ct + \cos 2ct. \quad (46)$$

Thus the Fourier transforms of $h(t)$ and $-2g(t) \sin ct$ agree over the frequency interval $(-2\lambda, 2\lambda)$. Similarly, the Fourier transform of $\hat{h}(t) = \log 2|s(t)|$ agrees over $(-2\lambda, 2\lambda)$ with the Fourier transform of $2g(t)\cos ct$. Since the Fourier transform of $g(t)\sin ct$ vanishes outside $[-b - c, b + c]$, we see that in case $2\lambda > b + c$, i.e., $c > 3b$, we can recover $-2g(t)\sin ct$ by appropriate filtering of $h(t)$. Actually, in case $c > 3b$, it is simpler to recover $g(t)$ by the sampling technique of the previous section.

Next we consider the case $2\lambda > c$, i.e., $c > 2b$, in which case there is a relatively simple way of recovering $g(t)$ from $h(t)$.

IV. A LINEAR RECOVERY METHOD FOR THE CASE $c > 2b$

Here we need only be concerned with the case $2b < c \leq 3b$, since for $c > 3b$ we have a simpler recovery method. However, in case $c > 2b$ we still have a method requiring only linear operations.

We have seen that the Fourier transform of the fundamental function $h(t)$, defined in (2), agrees over $(-2\lambda, 2\lambda)$ with that of $h_1(t)$ in (45) and hence, agrees over $(-2\lambda, 2\lambda)$ with that of $-2g(t)\sin ct$. In case $2\lambda > c$, i.e., $c > 2b$, the band $(-2\lambda, 2\lambda)$ includes the lower sideband of $-2g(t)\sin ct$ and at least a part of the upper sideband. Then, with proper filtering of $h(t)$ and subsequent demodulation, we can recover $g(t)$ by linear operations. In the first place the filtering must eliminate everything outside the band $(-2\lambda, 2\lambda)$; i.e., the higher-order nonlinear terms $h_n(t)$, $n \geq 2$, defined in (42). There is nothing in the band $(-\lambda, \lambda)$ since the Fourier transform of h_1 vanishes over $(-\lambda, \lambda)$. So let K_1 denote the kernel (impulse response) of a bandpass filter, where we first require

$$\int_{-\infty}^{\infty} K_1(t)e^{-i\omega t}dt = 0, \qquad |\omega| \geq 2\lambda, \quad (47)$$

$$\int_{-\infty}^{\infty} |K_1(t)|dt < \infty, \quad (47a)$$

and define

$$f_1(t) = \int_{-\infty}^{\infty} h(x)K_1(t - x)dx, \tag{48}$$

which is equivalent to

$$f_1(t) = -\int_{-\infty}^{\infty} 2g(x)\sin cx\ K_1(t - x)dx$$

$$= -\int_{-\infty}^{\infty} K_1(x)2g(t - x)\sin \{c(t - x)\}dx. \tag{48a}$$

Now if we multiply $f_1(t)$ by $-\sin ct$ and band limit the result, we should obtain a filtered version of $g(t)$. The spectrum of f_1 is confined to the intervals $[\lambda, 2\lambda]$ and $[-2\lambda, -\lambda]$. Then, defining

$$f_2(t) = -f_1(t)\sin ct, \tag{49}$$

we see that the spectrum of f_2 is confined to the union of the intervals $[\lambda + c, 2\lambda + c]$, $[\lambda - c, 2\lambda - c]$, $[-2\lambda - c, -\lambda - c]$, and $[-2\lambda + c, -\lambda + c]$. (Recall that $\lambda = c - b$.) The innermost intervals are the second and fourth; viz., $[-b, c - 2b]$ and $[-c + 2b, b]$. The outer intervals are $[2c - b, 3c - 2b]$ and $[-3c + 2b, -2c + b]$. We are interested in the case $2b < c \leq 3b$ and want to retain the spectrum in the inner intervals and eliminate the spectrum in the outer intervals by low-pass filtering. We have ample room for this. So let K_2 be a kernel (impulse response) of a low-pass filter, satisfying

$$\int_{-\infty}^{\infty} K_2(t)e^{-i\omega t}dt = 0, \qquad |\omega| \geq 2c - b > 3b \tag{50}$$

$$\int_{-\infty}^{\infty} |K_2(t)|dt < \infty, \tag{50a}$$

and then define

$$g_1(t) = \int_{-\infty}^{\infty} f_2(x)K_2(t - x)dx. \tag{51}$$

We have

$$g_1(t) = -\int_{-\infty}^{\infty} \sin cx\ f_1(x)K_2(t - x)dx$$

$$= -\int_{-\infty}^{\infty} \sin cx\ K_2(t - x)dx \int_{-\infty}^{\infty} K_1(y)2g(x - y)\sin\{c(x - y)\}dy.$$

The requirements (47a) and (50a) allow us to interchange the order of integration. Then

$$\int_{-\infty}^{\infty} K_2(t - x)g(x - y)2 \sin cx \sin\{c(x - y)\}dx$$

$$= \int_{-\infty}^{\infty} K_2(t - x)g(x - y) \{\cos cy - \cos(cx - cy)\}dx$$

$$= (\cos cy) \int_{-\infty}^{\infty} K_2(t - x)g(x - y)dx, \tag{51a}$$

the other term dropping out since, for each y, the Fourier transform of $g(x - y)\cos(2cx - cy)$ vanishes over $(-2c + b, 2c - b)$. Thus we have

$$g_1(t) = \int_{-\infty}^{\infty} g(x)K_3(t - x)dx, \tag{52}$$

where

$$K_3(t) = \int_{-\infty}^{\infty} \{K_1(y)\cos cy\}\cdot K_2(t - y)dy. \tag{52a}$$

So $g_1(t)$ is in fact a filtered version of $g(t)$. Denoting by $\overset{>}{K}_i(\omega)$ the Fourier transform of $K_i(t)$, the overall filter response is given by

$$\overset{>}{K}_3(\omega) = \frac{1}{2}\left\{\overset{>}{K}_1(\omega + c) + \overset{>}{K}_1(\omega - c)\right\}\cdot\overset{>}{K}_2(\omega). \tag{53}$$

So, if in addition to the requirements (47) and (50) we require

$$\frac{1}{2}\left\{\overset{>}{K}_1(w + c) + \overset{>}{K}_1(\omega - c)\right\}\cdot\overset{>}{K}_2(\omega) = 1, \qquad |\omega| < b, \tag{54}$$

then we will have, with g_1 defined in (51),

$$g_1(t) = g(t). \tag{55}$$

Conceptually, the simplest way to satisfy (54) is to require K_1 to be a vestigial sideband filter with linear cutoff characteristic, i.e., assuming $2b < c \leq 3b$,

$$\overset{>}{K}_1(\omega) = \frac{1}{2}\cdot\frac{2\lambda - \omega}{2\lambda - c}, \qquad |\omega - c| \leq c - 2b$$

$$= 1, \qquad \lambda \leq \omega \leq 2b, \tag{56}$$

and, of course,

$$\overset{>}{K}_1(\omega) = \overset{>}{K}_1(-\omega)$$

$$\overset{>}{K}_1(\omega) = 0, \qquad |\omega| > 2\lambda.$$

Then K_2 is a low-pass filter with gain of 2 in the passband; i.e.,

$$\begin{aligned} \overset{>}{K}_2(\omega) &= 2, \qquad |\omega| \leq b \\ &= 0, \qquad |\omega| \geq 2c - b > 3b. \end{aligned} \tag{57}$$

We should note that the multiplication of $f_1(t)$ by $-\sin ct$ in (49) can be replaced by multiplication with $-\pi/4 \operatorname{sgn}\{\sin ct\}$, since the additional high-frequency terms drop out in (51a). That is, a switch-type multiplier may be used in the demodulation process, thereafter making appropriate gain corrections.

V. A GENERAL RECOVERY METHOD

In case λ is not greater than b, i.e., c is not greater than $2b$, nonlinear operations are required to recover $s(t)$ or $g(t)$. The general method here, which makes further use of the ideas developed in the Theory of Analytic Modulation Systems (see Ref. 2), affords an alternative to the method in Ref. 1. In effect, the method here allows the Hilbert transform in (11) to be replaced by a filtered version. This modification must be taken into account in subsequent operations.

The method is best explained in terms of the functions with one-sided spectrum, i.e., the functions G and H in (36),

$$H(t) = i \log G(t) = i \log[1 + \gamma(t)] = h(t) + i\hat{h}(t), \tag{58}$$

where

$$G(t) = 2e^{ict}s(t)$$

and

$$\gamma(t) = 1 + 2e^{ict}g(t) + e^{2ict}.$$

We have already noted that the Fourier transforms of $H(t)$ and $H_n(t)$, defined by

$$H_n(t) = i \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \gamma^k(t), \tag{59}$$

vanish over $(-\infty, \lambda)$ and agree over $(-\infty, (n+1)\lambda)$.

Let us denote by $H_\nu^*(t)$ any function such that the Fourier transforms of $H(t)$ and $H_\nu^*(t)$ agree over $(-\infty, \nu)$. In particular, H_ν^* may be any function of the form

$$H_\nu^*(t) = \int_{-\infty}^{\infty} H(x)K_\nu(t - x)dx, \tag{60}$$

where K_ν, in the interesting case $\nu > \lambda$, satisfies

$$\int_{-\infty}^{\infty} K_\nu(t)e^{-i\omega t}dt = 1, \qquad \lambda \le \omega \le \nu, \tag{61}$$

with

$$\int_{-\infty}^{\infty} |K_\nu(t)|dt < \infty. \tag{61a}$$

Or we may take $H_\nu^*(t)$ to be any function of the form

$$H_\nu^*(t) = i \sum_{k=1}^{\infty} a_k\gamma^k(t), \tag{62}$$

where the a_k decrease rapidly enough and

$$a_k = \frac{(-1)^{k+1}}{k} \text{ for } 1 \le k \le n, \qquad \text{where} \quad (n+1)\lambda \ge \nu. \tag{62a}$$

Now let us denote by $G_\nu^*(t)$ any function such that the Fourier transforms of $G(t)$ and $G_\nu^*(t)$ agree over $(-\infty, \nu)$. Since

$$G(t) = 1 + 2e^{ict}g(t) + e^{2ict},$$

and $g(t)$ is real, we can recover $g(t)$ from any such function $G_\nu^*(t)$ provided $\nu > c$, which is to say all we need is a little more than the lower sideband of $g(t)\sin ct$ or $g(t)\cos ct$.

We have the inverse relation between $G(t)$ and $H(t)$; viz.,

$$G(t) = \exp\{-iH(t)\}, \tag{63}$$

and as shown in Ref. 2, the Fourier transforms of $G(t)$ and $\exp\{-iH_\nu^*(t)\}$ agree over $(-\infty, \nu)$. So we may take

$$G_\nu^*(t) = \exp\{-iH_\nu^*(t)\}, \qquad \nu > c, \tag{64}$$

where $H_\nu^*(t)$ is any function of the form (50).

Since the Fourier transforms of $H(t)$ and $H_\nu^*(t)$ vanish over $(-\infty, \lambda)$, the Fourier transform of the nth power of $H_\nu^*(t)$ vanishes over $(-\infty, n\lambda)$. So we may take, if we prefer,

$$G_\nu^*(t) = \sum_{k=0}^{n} \frac{1}{k!}[-iH_\nu^*(t)]^k, \qquad (n+1)\lambda \ge \nu > c. \tag{65}$$

This last form may be preferable to (64) for moderately large gap frequencies, for example, $3\lambda > c$. For small gap frequencies there would appear to be no practical reason for perferring a large-degree polynomial in (65) to the exponential in (64). In general, if $2\lambda \le c$, then nonlinear operations are required to obtain a suitable $G_\nu^*(t)$. We can modify this statement in case $g(t)$ itself has a spectral gap about the origin. Then we could recover $g(t)$ from $G_\nu^*(t)$ when ν is slightly less than c. For example, in this case if $2\lambda = c$, we could take $n = 1$ in (65).

Whether we choose (64) or (65), we have the Fourier transforms of $-\text{Im}\{G_\nu^*(t)\}$ and $-2g(t)\sin ct$ agreeing over $(-\nu, \nu)$, $\nu > c$. Then the recovery problem is essentially that of Section IV, except there the Fourier transforms of $h(t)$ and $-2g(t)\sin ct$ agree over $(-2\lambda, 2\lambda)$, $2\lambda > c$. So, with obvious modifications of the cutoff frequencies of the filters, the recovery process in Section III applies to the general case when $h(t)$ is replaced by $-\text{Im}\{G_\nu^*(t)\}$, $\nu > c$.

We have

$$H_\nu^*(t) = h_\nu^*(t) + i\hat{h}_\nu^*(t). \tag{66}$$

So the general recovery method allows $\hat{h}(t)$ to be replaced by a function $\hat{h}_\nu^*(t)$, whose Fourier transform agrees with that of $\hat{h}(t)$ only over the interval $(-\nu, \nu)$, $\nu > c$. In particular, $\hat{h}_\nu^*(t)$ could be bandlimited to a larger interval. Alternatively, $\hat{h}_\nu^*(t)$ could be obtained from $h(t)$ by a transversal filter having the appropriate periodic frequency response.

In practice, all the required operations can only be approximated. As contrasted to the method in Ref. 1, complexity in the recovery method here is exchanged for simplicity in approximating the Hilbert transform of $h(t)$.

We now want to specialize to the case $\nu \geq 2c$ and obtain a formula for $s(t)$ analogous to that in Ref. 1.

VI. A GENERALIZATION OF THE BASIC FORMULA

From (5) we have

$$s(t) = \frac{1}{2}\{\text{sgn}\, s(t)\}\cdot\exp\{\hat{h}(t)\}. \tag{67}$$

We call this the basic formula for recovery of $s(t)$ from sgn $s(t)$ or, equivalently, from the fundamental function $h(t)$ defined in (2).

In Ref. 1, the practicality of (67) was demonstrated by using the fact that the Fourier transform of $h(t)$ vanishes over $(-\lambda, \lambda)$. Here we give a generalization of (67), which may be preferred in analog implementations.

We take $H_\nu^*(t)$ to be any function of the form (60) where $\nu > 2c$, and set

$$G_\nu^*(t) = \exp\{-iH_\nu^*(t)\}, \tag{68}$$

as in (64). Now we have the Fourier transforms of $G(t)$ and $G_\nu^*(t)$ agreeing over $(-\infty, \nu)$ where $\nu > 2c$. Recall that

$$G(t) = 2e^{ict}s(t),$$

and, therefore, its Fourier transform vanishes outside $[0, 2c]$. So we can multiply $G_\nu^*(t)$ by e^{-ict} and then, with appropriate filtering, recover $2s(t)$.

Let us define then

$$2s_\mu^*(t) = \mathrm{Re}\{e^{-ict}G_\nu^*(t)\}. \tag{69}$$

Since $s(t)$ is real, the Fourier transforms of $s(t)$ and $s_\mu^*(t)$ agree over the interval $(-\mu, \mu)$ where $\mu = \nu - c > c$.

In (68) we set

$$H_\nu^*(t) = h_\nu^*(t) + i\hat{h}_\nu^*(t),$$

and then write (69) as

$$s_\mu^*(t) = \frac{1}{2}\{\cos[h_\nu^*(t) + ct]\}\cdot\exp\{\hat{h}_\nu^*(t)\}, \tag{70}$$

and note the similarity to (67). To carry this a bit further, we recall the definition (2) of h,

$$h(t) = J(t) - ct, \tag{71}$$

where $J(t)$ is a jump function increasing by π at each zero of $s(t)$, $J(0) = 0$. We may define $J_\nu^*(t)$ in an analogous way by setting

$$h_\nu^*(t) = J_\nu^*(t) - ct. \tag{72}$$

Since $h_\nu^*(t)$ is a filtered version of $h(t)$, their Fourier transforms agreeing over $(-\nu, \nu)$, we may think of $J_\nu^*(t)$ as a filtered version of $J(t)$, although the corresponding convolution with $J(t)$ alone need not make sense.

Now we may write

$$s_\mu^*(t) = \frac{1}{2}\{\cos[J_\nu^*(t)]\}\cdot\exp\{\hat{h}_\nu^*(t)\}, \tag{73}$$

and observing that $\{\mathrm{sgn}\ s(t)\} = \cos[J(t)]$,

$$s(t) = \frac{1}{2}\{\cos[J(t)]\}\cdot\exp\{\hat{h}(t)\}. \tag{74}$$

This is the analogy we seek. Now $s(t)$ may be obtained by bandlimiting $s_\mu^*(t)$; i.e.,

$$s(t) = \int_{-\infty}^{\infty} s_\mu^*(x)K_{c,\mu}(t-x)dx, \tag{75}$$

where $K_{c,\mu}(t)$ belongs to L_1 and satisfies

$$\int_{-\infty}^{\infty} K_{c,\mu}(t)e^{-i\omega t}dt = 1, \quad |\omega| \le c$$

$$= 0 \qquad |\omega| \ge \mu > c. \tag{76}$$

We may regard (74) as a limiting case ($\mu \to \infty$) of (75). That is, (75) is a generalization of (74).

VII. NOTE ON HIGH-POWER AMPLITUDE MODULATION

Compared to conventional modulators, switching-type modulators have the advantage that little power is dissipated in the switches (e.g., transistors). Of course they also have the advantage of easily effecting linear modulation. The disadvantage is the introduction of high-frequency components, which must be removed by filtering. However, if the filter has low losses, the high-frequency energy is, for the most part, stored in the reactive elements of the filter, resulting in an efficient modulation system.

If $g(t)$ is bandlimited to $[-b, b]$ and we wish to modulate sin ct with $g(t)$, then a straightforward method to do this with a switching-type modulator is to form

$$\begin{aligned} f(t) &= g(t)\,\{\text{sgn} \sin ct\} \\ &= \frac{4}{\pi}\left[g(t)\sin ct + \frac{1}{3}\,g(t)\sin 3ct + \cdots\right], \end{aligned} \tag{77}$$

and then, assuming $c > b$, an output filter may be designed to deliver $4/\pi\; g(t)\sin ct$ to the load.

In high-power applications, the disadvantage of this method is that the input $g(t)$ to the switches must furnish the power to the load. That is, high-power baseband amplifiers are required. An interesting alternative is suggested by the results in Section III.

There we showed that the Fourier transforms of $h(t)$ and $-2g(t)\sin ct$ agree over $(-2\lambda, 2\lambda)$ where $\lambda = c - b$.

Recall that

$$h(t) = J(t) - ct, \tag{78}$$

where $J(t)$ is a jump function with $J(0) = 0$, which increases by π at each zero of the function $s(t)$,

$$s(t) = g(t) + \cos ct. \tag{79}$$

Now denote by $\sigma(t)$ the periodic sawtooth function,

$$\begin{aligned} \sigma(t) &= \frac{\pi}{2} - ct, \qquad 0 < t \leq \pi/c \\ \alpha\left(t + \frac{\pi}{c}\right) &= \sigma(t). \end{aligned} \tag{80}$$

This function, associated with $\log(1 - e^{i2ct})$, has the Fourier series

$$\sigma(t) = \sum_{n=1}^{\infty} \frac{1}{n} \sin 2nct. \tag{81}$$

Since $s(t)$ has one and only one zero between each extremal of $\cos ct$, we have

$$J(k\pi/c) = k\pi, \qquad k = 0, \pm 1, \pm 2, \cdots .$$

Now the function $\phi(t)$, defined by

$$\phi(t) = h(t) - \sigma(t) = J(t) - ct - \sigma(t), \tag{82}$$

is a pure step function increasing by π at the zeros $\{t_k\}$ of $s(t)$ and decreasing by π at the zeros of $\sin ct$. We see that $\phi(0+) = -\pi/2$ and since the zeros of $s(t)$ and $\sin ct$ interlace, it is easy to see that we can express $\phi(t)$ in the form

$$\phi(t) = -\frac{\pi}{2}\{\text{sgn } s(t)\}\cdot\{\text{sgn} \sin ct\}. \tag{83}$$

Clearly, from (81) and (82), the Fourier transforms of $\phi(t)$ and $h(t)$ agree over $(-2c, 2c)$, and hence, the Fourier transforms of $\phi(t)$ and $-2g(t)\sin ct$ agree over $(-2\lambda, 2\lambda)$, since $2c > 2\lambda = 2(c - b)$.

In order to filter out $-2g(t) \sin ct$ from $\phi(t)$ we require $c + b < 2\lambda$ or $c > 3b$. Thus, if $c > 3b$ we can use (83) as a keying or switching voltage and generate

$$\begin{aligned} f(t) &= M\{\text{sgn } s(t)\}\cdot\{\text{sgn} \sin ct\} \\ &= \frac{4M}{\pi} g(t)\sin ct + \text{hi-freq}, \end{aligned} \tag{84}$$

where M represents a constant voltage that furnishes power to the load and "hi-freq" represents components of $f(t)$ outside the band $(-2\lambda, 2\lambda)$ that may be removed by filtering.

Thus, if we add $\cos ct$ to $g(t)$, and simply require $|g(t)| \le 1$, with $c > 3b$, where b is the "top-frequency" of g (i.e., add a high-frequency bias larger than $|g(t)|$) and then clip the result to obtain $\{\text{sgn } s(t)\}$ and then multiply by $\{\text{sgn} \sin ct\}$, we obtain a square wave, which we may use to switch transistors alternately between $\pm M$, the output having the desired "in-band" component $(4M/\pi)\, g(t)\sin ct$.

Alternatively, we may multiply $s(t)$ by $\{\text{sgn} \sin ct\}$ and then clip the result to obtain the keying or switching voltage.

This method clearly has an advantage over the straightforward method of (77) in high-power applications. The disadvantage of both methods is that filters capable of handling high power are required. Conceivably, in some applications (audio) the filtering may not be required.

VIII. DISCUSSION AND CONCLUSION

Here we have, in essence, detailed the remarkable properties of the zeros $\{t_k\}$ of any function $s(t)$ of the form (1). We may regard the description of the Fourier transform of the fundamental function $h(t) = J(t) - ct$ as summarizing the properties of the zeros $\{t_k\}$.

The rude property is that the Fourier transform of h vanishes over $(-\lambda, \lambda)$, where λ is the gap frequency in (1). This means that

$$\frac{1}{\pi} h'(t) = \sum_{-\infty}^{\infty} \delta(t - t_k) - c/\pi,$$

is high-pass, and so provides a quadrature formula for functions f in L_1 and bandlimited to $(-\lambda, \lambda)$, i.e.,

$$\int_{-\infty}^{\infty} g(t)dt = \frac{\pi}{c} \sum_{-\infty}^{\infty} f(t_k).$$

This formula is the basis of the simple recovery procedure for $c > 3b$, which is not an uninteresting case, e.g., in recording applications, where c is the bias frequency.

In the first stage of refinement, the zeros $\{t_k\}$ have the property that, insofar as regards spectrum in the interval $(-2\lambda, 2\lambda)$, the transformation $s(t) \rightarrow h(t)$ or $s(t) \rightarrow \sum_{-\infty}^{\infty} \delta(t - t_k)$ is linear. Then, when $2\lambda > c$, i.e., $c > 2b$, we have a linear recovery method that we would use in case c were not larger than $3b$. What is involved here is essentially vestigial sideband detection.

In further stages of refinement, the zeros $\{t_k\}$ have the property that we can equate the Fourier transform of $h(t)$ (or the pulse train) over frequency intervals $(-(n + 1)\lambda, (n + 1)\lambda)$ to the Fourier transform of a finite sum involving no more than the nth power of $g(t)$. For recovery of $g(t)$ in case $c < 2b$ the theory of analytic modulation and detection after bandlimiting as developed in Ref. 2 comes into play. The theory given there is specialized here to detection of analytic signals $z(t)$ of the special form $z(t) = 2e^{ict}s(t) \equiv G(t)$, where $s(t)$ is a real-valued bandlimited signal of still more special form. The analytic "modulation law" $f(z)$ in this case is $i \log z$. We are given

$$h(t) = \text{Re}[i \log\{z(t)\}],$$

and then can obtain $z(t)$ from

$$z(t) = \exp\{-iH(t)\}, \qquad H(t) = h(t) + i\hat{h}(t).$$

This is essentially the basic formula in Ref. 1. However, to allow us as much latitude as possible in approximating $\hat{h}(t)$, we assume, in effect, that we are given $h_\nu^*(t)$, a function whose Fourier transform agrees with that of $h(t)$ over the frequency interval $(-\nu, \nu)$. We only require

$\nu > c$, half the bandwidth of $z(t)$, because of the special form of $z(t)$. The general recovery method assumes no more than $\nu > c$, which requires the vestigial sideband detection techniques in recovery of $g(t)$ from

$$z_\nu^*(t) \equiv G_\nu^*(t) = \exp\{-iH_\nu^*(t)\}, \qquad H_\nu^*(t) = h_\nu^*(t) + i\hat{h}_\nu^*(t).$$

The general recovery method is interesting from a mathematical viewpoint, since we assume no more than is necessary. However, the simplification obtained by requiring $\nu > 2c$, as in Section VI, appears to be well worth the price paid in filter requirements, i.e., the filters are required to have prescribed characteristics over the band $(-\nu, \nu)$. The resulting recovery formula may be preferred to the basic formula given in Ref. 1, especially for analog implementation.

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AUTHOR

Benjamin F. Logan, Jr., B.S. (Electrical Engineering), 1946, Texas Technological College; M.S., 1951, Massachusetts Institute of Technology; Eng.D.Sc. (Electrical Engineering), 1965, Columbia University; AT&T Bell Laboratories, 1956—. While at MIT, Mr. Logan was a research assistant in the Research Laboratory of Electronics, investigating characteristics of high-power electrical discharge lamps. Also at MIT he engaged in analog computer development at the Dynamic Analysis and Control Laboratory. From 1955 to 1956 he worked for Hycon-Eastern, Inc., where he was concerned with the design of airborne power supplies. He joined AT&T Bell Laboratories as a member of the Visual and Acoustics Research Department, where he was concerned with the processing of speech signals. Currently, he is a member of the Mathematical Research Department. Member, Sigma Xi, Tau Beta Pi.