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Signals Designed for Recovery After Clipping—III: Generalizations

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Let $S(b, c)$ be the class of all real-valued bounded functions $s(t)$ of the form

$$s(t) = g(t) + \cos ct, \tag{i}$$

where g is bandlimited to $[-b, b]$ and $0 \le b < c < \infty$ and such that

$$(-1)^k s(k\pi/c) > 0, \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{ii}$$

a condition that is always satisfied if $|g(t)| < 1$. In earlier papers we showed that such functions could be reconstructed from a knowledge of their zeros in the interval $(t - T, t + T)$ to within an accuracy $0(e^{-\lambda T})$, where $\lambda = c - b$. This paper generalizes these results to functions of the form (i) satisfying the condition that $s(t)$ have only real zeros, a condition which is weaker than (ii). The bounds on the accuracy of the reconstruction obtained are weaker. This paper also shows that every interval of length greater than $2\pi/\lambda$, where $\lambda = c - b > 0$, must contain at least one zero of $s(t)$, and that $s(t)$ satisfies

$$|s(t)| \le 2^{p-1}, \; -\infty < t < \infty,$$

where $p = 2c/\lambda$.

I. INTRODUCTION

References 1 and 2 present various practical means for recovery of signals $s(t)$ in a certain class $S(b, c)$ from their zeros. The class $S(b, c)$ consists of all real-valued bounded signals $s(t)$ of the form

$$s(t) = g(t) + \cos ct, \tag{1a}$$

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where g is bandlimited to $[-b, b]$, $0 \leq b < c < \infty$, and such that

$$(-1)^k \, s(k\pi/c) > 0, \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{1b}$$

which is satisfied, for example, if $|g(t)| < 1$.

The alternation condition (1a) ensures that $s(t)$ has only real simple zeros $\{t_k\}$,

$$\frac{k\pi}{c} < t_k < (k+1)\frac{\pi}{c}, \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{1c}$$

i.e., the zeros of $s(t)$ are interlaced with the zeros of sin ct.

The recovery procedures involve operations on the so-called *fundamental function* associated with the zeros $\{t_k\}$ of s,

$$h(t) = J(t) - ct, \tag{2}$$

where $J(t)$ is a jump function increasing by π at each zero t_k, $J(0) = 0$. The linearly decreasing term $-ct$ just offsets the growth of $J(t)$ so that

$$-\pi < h(t) < \pi, \qquad -\infty < t < \infty. \tag{3}$$

The practicality of the recovery procedures owes to the fact, established in Ref. 1, that $h(t)$ is a high-pass function, having no spectrum in the (angular) frequency interval $(-\lambda, \lambda)$, where

$$\lambda = c - b \tag{4}$$

is the "gap frequency" associated with the class $S(b, c)$. The basic recovery formula given in Ref. 1 is

$$s(t) = \frac{1}{2}\{\text{sgn}\ s(t)\}\exp[\hat{h}(t)], \tag{5}$$

where $\hat{h}$ is the Hilbert transform of h. Since $h(t)$ is a bounded high-pass function, a good estimate $\hat{h}_T(t)$ for $\hat{h}(t)$ can be made from the knowledge of h in a finite moving interval $(t - T, t + T)$; i.e., from the zeros t_k in the interval $(t - T, t + T)$, leading to an estimate $s_T(t)$,

$$s_T(t) = \frac{1}{2}\{\text{sgn}\ s(t)\}\exp[\hat{h}_T(t)] \tag{6}$$

such that

$$|s(t) - s_T(t)| \leqslant \frac{2e^{-\lambda T}}{(1 - e^{-\lambda T})^2}\,|s(t)|. \tag{7}$$

In Ref. 2, generalizations of the basic recovery formula were developed, showing how $s(t)$ could be recovered from bandlimited versions of $h(t)$ or, equivalently, from bandlimited versions of

$$h'(t) = \pi \sum_{-\infty}^{\infty} \delta(t - t_k) - c. \tag{8}$$

Here we wish to extend the validity of the previous results by removing the alternation condition (1b), simply requiring that $s(t)$ of the form (1) have only real zeros $\{t_k\}$. For practical purposes we would also require the zeros to be simple, but here we allow each zero t_k to have *multiplicity* m_k.

Several interesting questions now arise:

1. What is the longest possible zero-free interval for $s(t)$?
2. How many zeros, counted according to multiplicity, can $s(t)$ have in a closed interval of length T?
3. How large can $|h(t)|$ be?
4. How large can $|s(t)|$ be?

The basic question here is the third question. A bound on $|h(t)|$ is needed to establish that $h(t)$ is a high-pass function with no spectrum in $(-\lambda, \lambda)$. In order for $|h(t)|$ to be bounded it is necessary, of course, for $s(t)$ to have a limited zero-free interval and a limited number of zeros in any interval of fixed length. The fourth question is one of corollary interest.

Once we establish that

$$-M \leqslant h(t) \leqslant M$$

we conclude that h is high-pass, so the results in Ref. 2 remain valid and the reconstruction algorithm in Ref. 1 remains valid, with (7) replaced by

$$|s(t) - s_T(t)| \leqslant \frac{1}{2} |s(t)| \left[\left(\frac{1 + e^{-\lambda T}}{1 - e^{-\lambda T}} \right)^{\frac{2M}{\pi}} - 1 \right].$$

In order to obtain a bound on $|h(t)|$ we consider

$$h'(t) = \pi \sum_{-\infty}^{\infty} m_k \delta(t - t_k) - c \tag{9}$$

and show that $h'(t)$ is a high-pass distribution with no spectrum in $(-\lambda, \lambda)$. To do this we must first show that the total mass of the distribution in any interval of fixed length T is uniformly bounded. A crude bound is obtained which, together with the exponential decay in the upper half plane $u > 0$ of

$$\frac{s'(t + iu)}{s(t + iu)} - ic,$$

establishes that $h'(t)$ has no spectrum in $(-\lambda, \lambda)$; that is,

$$\int_{-\infty}^{\infty} f_\lambda(t) h'(t) dt = 0$$

or

$$\frac{\pi}{c} \sum_{-\infty}^{\infty} m_k f_\lambda(t_k) = \int_{-\infty}^{\infty} f_\lambda(t)\, dt, \tag{10}$$

where $f_\lambda(t)$ is any bandlimited function of L_1 whose Fourier transform vanishes outside $(-\lambda, \lambda)$.

The "quadrature formula" (10), together with appropriately chosen f_λ, gives an upper bound for the longest possible zero-free interval of $s(t)$.

The fundamental function $h(t)$, under the less restrictive condition, is of the form

$$h(t) = J(t) - ct, \tag{11}$$

where now $J(t)$ is a jump function increasing by $m_k\pi$ at each zero t_k of multiplicity m_k. The levels of $J(t)$ are still multiples of π but we do not (necessarily) have $J(0) = 0$ as before. If, for example, $s(0) \neq 0$, then $J(0) = n\pi$, where n is determined by the condition that $h(t)$ have zero average value. To obtain an upper bound on $|h(t)|$ we write $h(t)$ as an "unbiased" integral of $h'(t)$. An unbiased integral of a high-pass function is a particular integral that is also high-pass; e.g., the unbiased integral of cos λt is λ^{-1}sin λt. Owing to the spectral gap $(-\lambda, \lambda)$ the unbiased integral may be obtained by convolution with an integrating kernel (see Ref. 3) $I_\lambda(t)$ belonging to L_1, satisfying

$$\int_{-\infty}^{\infty} I_\lambda(t) e^{-i\omega t} dt = \frac{1}{i\omega}, \qquad |\omega| \geq \lambda. \tag{12}$$

There is, then, an equivalence class of kernels $\{I_\lambda(t)\}$. If we further require

$$\int_{-\infty}^{\infty} I_\lambda(t) dt = 0, \tag{13}$$

for example, by requiring $I_\lambda(t)$ to be an odd function, then

$$h(t) = \pi \sum_{-\infty}^{\infty} m_k I_\lambda(t - t_k). \tag{14}$$

We choose a particular integrating kernel $I_\lambda(t)$ and a particular $f_\lambda(t)$ [for which (10) is valid] such that

$$-f_\lambda(t) \leq I_\lambda(t) \leq f_\lambda(t), \qquad -\infty < t < \infty, \tag{15}$$

and hence obtain

$$-c \int_{-\infty}^{\infty} f_\lambda(x) dx \leq h(t) \leq c \int_{-\infty}^{\infty} f_\lambda(x) dx, \qquad -\infty < t < \infty. \tag{16}$$

The upper and lower bounds for $h(t)$ readily give an upper bound for the number of zeros of $s(t)$, counted according to multiplicity, in a closed interval of length T.

It is of interest to determine an upper bound for $|s(t)|$, which amounts to determining an upper bound for $\hat{h}(t)$, the Hilbert transform of $h(t)$, in (5). Now $|\hat{h}(t)|$ is not bounded, since $\hat{h}(t)$ has logarithmic singularities at the discontinuities of $h(t)$, i.e., at the zeros of $s(t)$. However, $\hat{h}(t)$ is bounded above. The Hilbert transform is given by (see Ref. 4)

$$\hat{h}(t) = \int_{-\infty}^{\infty} h(x) K_\lambda(t - x) dx, \tag{17}$$

where K_λ is any function of the form

$$K_\lambda(t) = \frac{f_\lambda(t)}{\pi t} \tag{17a}$$

and

$$f_\lambda(t) \text{ is bandlimited to } [-\lambda, \lambda] \tag{17b}$$

$$f_\lambda(0) = 1 \tag{17c}$$

$$\int_{|t|>1} |f_\lambda(t)| \frac{dt}{|t|} < \infty. \tag{17d}$$

The integral in (17) is interpreted as a Cauchy principal value. With suitable further restrictions on f_λ we can write (17) as

$$\begin{aligned} \hat{h}(t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} h'(x) L_\lambda(t - x) dx \\ &= \sum_{-\infty}^{\infty} m_k L_\lambda(t - t_k) - \frac{c}{\pi} \int_{-\infty}^{\infty} L_\lambda(x) dx, \end{aligned} \tag{18}$$

where

$$L_\lambda(t) = \int_{-\infty}^{t} \pi K_\lambda(x) dx, \qquad t < 0, \tag{18a}$$

$$L_\lambda(t) = L_\lambda(-t). \tag{18b}$$

In accord with (17d) the integral in (18a) is absolutely convergent for $t < 0$ and we obtain (18b) by choosing f_λ even, and such that L_λ is integrable. For the problem here we further choose f_λ so that

$$L_\lambda(t) \leqslant 0, \qquad -\infty < t < \infty, \tag{18c}$$

$$\int_{-\infty}^{\infty} L_\lambda(x)dx = 2\int_{-\infty}^{0} L_\lambda(x)dx \text{ (integrate by parts)}$$
$$= \lim_{t\to 0}\left\{tL_\lambda(t) - \pi\int_{-\infty}^{t} xK_\lambda(x)dx\right\}$$
$$= -2\int_{-\infty}^{0} f_\lambda(x)dx = -\int_{-\infty}^{\infty} f_\lambda(x)dx. \qquad (18d)$$

Then from (18), with (18c) and (18d), we have

$$\hat{h}(t) \leqslant \frac{c}{\pi}\int_{-\infty}^{\infty} f_\lambda(x)dx, \qquad -\infty < t < \infty. \qquad (19)$$

Here, we choose an appropriate f_λ, subject to (17b) through (17d) and (18c), so as to minimize the integral in (19).

The results stated below are sharp only for the case $2c/\lambda = m$, an integer. It is not at all clear how one would proceed to improve the results when $2c/\lambda$ is not an integer. Still, the results show that signals of the form (1) having only real zeros are very "nice" in that they behave, at worst, much like polynomials of order $[2c/\lambda]$ in $\cos \lambda t/2$.

II. RESULTS

Theorem 1: Let $s(t)$ be any function of the form

$$s(t) = \cos ct + g(t),$$

where $g(t)$ is a bounded real-valued function bandlimited to $[-b, b]$, $0 \leqslant b < c$, such that $s(t)$ has only real zeros $\{t_k\}$ with corresponding multiplicities $\{m_k\}$. Then the distribution

$$h'(t) = \pi\sum_{-\infty}^{\infty} m_k\delta(t - t_k) - c$$

has no spectrum in $(-\lambda, \lambda)$, where

$$\lambda = c - b.$$

That is, for every f_λ in L_1 of the form

$$f_\lambda(t) = \int_{-\lambda}^{\lambda} F(\omega)e^{i\omega t}d\omega$$

we have

$$\int_{-\infty}^{\infty} f_\lambda(t)h'(t)dt = 0$$

or

$$\frac{\pi}{c}\sum_{-\infty}^{\infty} m_k f_\lambda(t_k) = \int_{-\infty}^{\infty} f_\lambda(t)\, dt.$$

Theorem 2: Suppose

$$\sum_{-\infty}^{\infty} \mu_k f_\lambda(t_k) = \int_{-\infty}^{\infty} f_\lambda(t)dt,$$

where

$$\mu_k > 0, \qquad t_{k+1} > t_k,$$

holds for every f_λ in L_1 of the form

$$f_\lambda t) = \int_{-\lambda}^{\lambda} F(\omega)e^{i\omega t}d\omega.$$

Then

$$t_{k+1} - t_k \leqslant 2\pi/\lambda, \qquad k = 0, \pm 1, \pm 2, \cdots,$$

where equality holding for any k implies

$$t_k = \frac{2\pi}{\lambda} k + \theta, \qquad k = 0, \pm 1, \pm 2, \cdots$$

with

$$\mu_k = \frac{2\pi}{\lambda}, \qquad k = 0, \pm 1, \pm 2, \cdots.$$

Corollary 2: Let $s(t)$ satisfy the hypotheses of Theorem 1. Then every interval of length $>2\pi/\lambda$ contains at least one zero of $s(t)$. Furthermore, if $s(t)$ has a zero-free open interval of length $2\pi/\lambda$, then

$$\frac{2c}{\lambda} = m, \qquad \text{an integer } (\geqslant 2),$$

and

$$s(t) = \frac{(-1)^n}{2}\left\{2 \cos\left(\frac{\lambda t}{2} + \frac{n\pi}{m}\right)\right\}^m,$$

where n is some integer.

Theorem 3: Under the hypotheses of Theorem 2, the distribution

$$\mu'(t) = \sum_{-\infty}^{\infty} \mu_k \delta(t - t_k) - 1$$

has an (unbiased) integral $\mu(t)$ satisfying

$$\frac{-\pi}{\lambda} \leqslant \mu(t) \leqslant \frac{\pi}{\lambda}, \qquad -\infty < t < \infty,$$

where equality holding (on either side) for one value of t implies

$$t_k = \frac{2\pi}{\lambda} k + \theta, \qquad k = 0, \pm 1, \pm 2, \cdots$$

$$\mu_k = \frac{2\pi}{\lambda}, \qquad k = 0, \pm 1, \pm 2, \cdots .$$

Corollary 3.1: Under the hypotheses of Theorem 1 there is a function of the form

$$h(t) = J(t) - ct,$$

where $J(t)$ is a jump function increasing by $m_k\pi$ at each zero t_k of multiplicity m_k, satisfying

$$\frac{-\pi c}{\lambda} \leqslant h(t) \leqslant \frac{\pi c}{\lambda}, \qquad -\infty < t < \infty,$$

and

$$\int_{-\infty}^{\infty} h(t) f_\lambda(t) dt = 0$$

for every f_λ in L_1 whose Fourier transform vanishes outside $(-\lambda, \lambda)$,

$$\lambda = b - c > 0.$$

Furthermore, equality for one value of t in

$$|h(t)| \leqslant \frac{\pi c}{\lambda}$$

gives the same conclusion as Corollary 2.

Corollary 3.2: Let $s(t)$ satisfy the hypotheses of Theorem 1 and denote by $N_T(x)$, the number of zeros of $s(t)$, counted according to multiplicity, in the closed interval $[x, x + T]$. Then

$$N_T(x) \leqslant \frac{cT}{\pi} + \frac{2c}{\lambda}, \qquad (-\infty < x < \infty)$$

with equality possible if, and only if,

$$\frac{2c}{\lambda} = m, \qquad \text{an integer,}$$

and

$$s(t) = \frac{(-1)^n}{2} \left\{ 2 \cos\left(\frac{\lambda}{2} t + \frac{n\pi}{m}\right) \right\}^m, \qquad n \text{ an integer,}$$

and x and $x + T$ are zeros of $s(t)$.

Theorem 4: Let $s(t)$ satisfy the hypotheses of Theorem 1. Then

$$|s(t)| \leqslant 2^{(2c/\lambda)-1}, \qquad -\infty < t < \infty,$$

where equality for any t gives the conclusion of Corollary 2.

III. PROOF OF THEOREM 1

The basic definition of the class $\mathbf{H}(\lambda)$ of high-pass functions is as follows.

Definition: $\mathbf{H}(\lambda)$ *consists of all functions* $h(t)$ *satisfying*

$$1. \int_x^{x+T} |h(t)|\, dt \leqslant M(T), \qquad -\infty < x < \infty$$

$$2. \int_{-\infty}^{\infty} h(t) f_\lambda(t) dt = 0, \qquad \textit{for all } f_\lambda \textit{ in } L_1$$

whose Fourier transforms vanish outside $(-\lambda, \lambda)$.

The basic representation theorem for $\mathbf{H}(\lambda)$ established in Ref. 5 is as follows.

Representation Theorem: A real-valued function $h(t)$ *belongs to* $\mathbf{H}(\lambda)$, *if, and only if,*

$$h(t) = \lim_{u \to 0+} h(t + iu) \textit{ almost all } t,$$

where

$$h(t + iu) = \mathrm{Re}\{H(t + iu)\}$$

and $H(\tau)$ *is analytic in the upper-half plane* $u > 0$,

$$|H(\tau)| = 0(e^{-\lambda u}), \qquad u \to \infty,$$

$$\int_x^{x+T} |h(t + iu)|\, dt \leqslant M(T), \qquad -\infty < x < \infty, \qquad u \geqslant 0.$$

Actually, $\mathbf{H}(\lambda)$ could have been defined to include distributions having limited mass in any interval of fixed length. As it stands, the closure of $\mathbf{H}(\lambda)$ includes such distributions.

We consider

$$H'(\tau) = i\,\frac{s'(\tau)}{s(\tau)} - c, \qquad \tau = t + iu. \tag{20}$$

We have

$$\frac{s'(\tau)}{s(\tau)} = \frac{-c \sin c\tau + g'(\tau)}{\cos c\tau + g(\tau)}.$$

Since $g(t)$ and $g'(t)$ are bandlimited to $[-b, b]$, the growth of $g(\tau)$ and $g'(\tau)$ is no faster than that of $\cos b\tau$. Thus,

$$\frac{s'(t+iu)}{s(t+iu)} = -c\tan c(t+iu) + 0(e^{-\lambda u}), \qquad u \to \infty,$$

$$= -ic + 0(e^{-\lambda u}), \qquad u \to \infty (\lambda = c - b). \tag{21}$$

We have

$$\frac{s'(\tau)}{s(\tau)} = \lim_{n\to\infty} \sum_{-n}^{n} \frac{m_k}{t - t_k}. \tag{22}$$

So

$$H'(t+iu) = \lim_{n\to\infty} i \sum_{-n}^{n} \frac{(t-t_k) - iu}{(t-t_k)^2 + u^2} m_k - c. \tag{23}$$

Setting

$$h'(t+iu) = \mathrm{Re}\{H'(t+iu)\} \tag{24}$$

we have

$$h'(t+iu) = \sum_{-\infty}^{\infty} \frac{m_k u}{(t-t_k)^2 + u^2} - c, \tag{25}$$

where the sum is absolutely convergent (p. 86, Ref. 6).

Now let $N_T(x)$ denote the number of zeros of $s(t)$, counted according to multiplicity, in the interval $[x - T, x + T]$. Then

$$\sum_{-\infty}^{\infty} \frac{m_k u}{(x-t_k)^2 + u^2} > N_T(x) \frac{u}{T^2 + u^2}.$$

Thus

$$N_T(x) \frac{u}{T^2 + u^2} < c + h'(x+iu).$$

Setting $u = T$ we have

$$N_T(x) < 2cT + 2Th'(x+iT). \tag{26}$$

Since $h'(x+iT) = 0(e^{-\lambda T})$, $T \to \infty$, we have

$$N_T(x) < 2cT + \epsilon, \quad \text{for sufficiently large } T. \tag{27}$$

Now

$$\int_{x-T}^{x+T} |h'(t+iu)|\,dt \leqslant 2cT + \int_{x-T}^{x+T} dt \left\{\sum_{-\infty}^{\infty} \frac{m_k u}{(t-t_k)^2 + u^2}\right\}$$

$$\leqslant 2cT + \int_{-\infty}^{\infty} \frac{u}{(t-x)^2 + u^2} N_T(t)\,dt$$

$$\leqslant 2cT + \pi(2cT + \epsilon). \tag{28}$$

It follows from (28) and (21) and the Representation Theorem that

$$\int_{-\infty}^{\infty} h'(t+iu)f_\lambda(t)dt = 0, \qquad u > 0, \tag{29}$$

where f_λ is any function of L_1 given by a finite Fourier integral over $(-\lambda, \lambda)$. The integral in (29) is absolutely convergent.

$$\int_{-\infty}^{\infty} |h'(t+iu)||f_\lambda(t)|dt$$

$$\leq M(T) \sum \max_{kT \leq t \leq (k+1)T} |f_\lambda(t)| \quad \text{(see p. 101, Ref. 6)}$$

with the bound independent of u.

Thus we may write

$$\int_{-\infty}^{\infty} h'(t+iu)f_\lambda(t)dt = \sum_k m_k \int_{-\infty}^{\infty} \frac{uf_\lambda(t)}{(t-t_k)^2+u^2}\,dt - c\int_{-\infty}^{\infty} f_\lambda(t)dt$$

and let $u \rightarrow 0$ to obtain the quadrature formula

$$\frac{\pi}{c}\sum_{-\infty}^{\infty} m_k f_\lambda(t_k) = \int_{-\infty}^{\infty} f_\lambda(t)dt. \tag{30}$$

This completes the proof of Theorem 1.

IV. PROOF OF THEOREM 2

We are given for the stipulated f_λ,

$$\sum_{-\infty}^{\infty} \mu_k f_\lambda(t_k) = \int_{-\infty}^{\infty} f_\lambda(t)dt$$

$$\mu_k > 0, \qquad t_{k+1} > t_k, \tag{31}$$

and wish to prove that $t_{k+1} - t_k \leq 2\pi/\lambda$, all k. To do this we take

$$f_\lambda(t) = \frac{\left(\cos\frac{\lambda}{2}t\right)^2}{\frac{\pi^2}{\lambda^2} - t^2}, \tag{32}$$

which satisfies the stipulation and

$$f_\lambda(t) > 0, \qquad \frac{-\pi}{\lambda} < t < \frac{\pi}{\lambda}, \tag{32a}$$

$$\leq 0, \qquad |t| \geq \frac{\pi}{\lambda},$$

$$\int_{-\infty}^{\infty} f_\lambda(t)dt = 0. \tag{32b}$$

Now suppose in (31) that

$$t_{n+1} - t_n > \frac{2\pi}{\lambda}. \tag{33}$$

Setting $T_n = (t_{n+1} + t_n)/2$, we have for f_λ given by (32),

$$\sum_{-\infty}^{\infty} \mu_k f_\lambda(t_k - T_n) = 0, \tag{34}$$

where in accord with (33), $|t_k - T_n| > \pi/\lambda$, all k. Since

$$f_\lambda(t) \leqslant 0 \quad \text{for} \quad |t| > \pi/\lambda$$

we conclude, since $\mu_k > 0$, that $t_k - T_n$ is a zero of $f_\lambda(t)$, greater in magnitude than the first; i.e., (33) implies

$$|t_k - T_n| = (2\upsilon_k + 1)\frac{\pi}{\lambda}, \qquad \upsilon_k \text{ a positive integer (all } k). \tag{35}$$

But now if we apply the formula (31) to $g_\lambda(t - T_n)$, where

$$g_\lambda(t) = \frac{f_\lambda(t)}{\frac{\pi^2}{\lambda^2} - t^2} = \left\{ \frac{\cos\frac{\lambda}{2}t}{\frac{\pi^2}{\lambda^2} - t^2} \right\}^2 \geqslant 0 \tag{36}$$

we obtain, if (35) is true,

$$\int_{-\infty}^{\infty} g_\lambda(t - T_n)dt = \sum_{-\infty}^{\infty} \mu_k g_\lambda(t_k - T_n) = 0, \tag{37}$$

which is obviously false. Hence (33) is false, i.e., we must have

$$t_{k+1} - t_k \leqslant \frac{2\pi}{\lambda}, \qquad \text{all } k. \tag{38}$$

Now suppose

$$t_{n+1} - t_n = \frac{2\pi}{\lambda}. \tag{39}$$

With the same f_λ as before, we conclude that (39) implies

$$t_k - T_n = (2\upsilon_k + 1)\pi/\lambda, \qquad \upsilon_k \text{ an integer (all } k). \tag{40}$$

In other words, (39) implies that $(t_k - t_j)$ is an even multiple of π/λ, all k and j. We have established (38),

$$0 < t_{k+1} - t_k \leqslant \frac{2\pi}{\lambda} \qquad \text{for all } k$$

Then (39) implies

$$t_k = \frac{2\pi}{\lambda} k + \theta, \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{41}$$

With t_k given by (41) we have for

$$s_\lambda(t) = \left\{ \frac{\sin \frac{\lambda}{2} t}{\frac{\lambda}{2} t} \right\}^2, \tag{42}$$

$$\int_{-\infty}^{\infty} s_\lambda(t - t_j) dt = \sum_{-\infty}^{\infty} \mu_k s_\lambda(t_k - t_j) = \mu_j s_\lambda(0)$$
$$= \mu_j = \frac{2\pi}{\lambda}, \qquad \text{all } j, \tag{43}$$

which completes the proof of Theorem 2.

V. PROOF OF COROLLARY 2

Under the hypotheses of Theorem 1 we have the quadrature formula,

$$\frac{\pi}{c} \sum_{-\infty}^{\infty} m_k f_\lambda(t_k) = \int_{-\infty}^{\infty} f_\lambda(t) dt,$$

where $\{t_k\}$ are the zeros of $s(t)$ having associated multiplicities $\{m_k\}$, $m_k \geqslant 1$, $t_{k+1} - t_k > 0$. It follows from Theorem 2 that $t_{k+1} - t_k \leqslant 2\pi/\lambda$, i.e., that every interval of length $>2\pi/\lambda$ contains at least one zero of $s(t)$. Furthermore, if $t_{k+1} - t_k = 2\pi/\lambda$ for some k, then (Theorem 2)

$$t_k = \frac{(2k+1)\pi}{\lambda} + T, \qquad \text{all } k,$$

$$\mu_k = \frac{\pi}{c} m_k = \frac{2\pi}{\lambda}, \qquad \text{all } k,$$

i.e., the zeros have a common multiplicity

$$m_k = \frac{2c}{\lambda} = m \quad \text{an integer } (\geqslant 2).$$

Thus we can have $t_{k+1} - t_k = 2\pi/\lambda$ only if $2c/\lambda$ is an integer m (necessarily $\geqslant 2$, since $\lambda = c - b \leqslant c$) and then only if all zeros have multiplicity m on a lattice of spacing $2\pi/\lambda$; i.e., if

$$s(t) = A \left\{ 2 \cos \frac{\lambda}{2} (t - T) \right\}^m.$$

The possibilities for A and T are determined by the top frequency content;

$$s(t) = 2A \cos \frac{m\lambda}{2}(t - T) + g(t) = \cos \frac{m\lambda t}{2} + g(t),$$

i.e.,

$$\frac{m\lambda T}{2} = n\pi$$

$$2A = \cos n\pi.$$

VI. PROOF OF THEOREM 3

The high-pass distribution

$$\mu'(t) = \sum_{-\infty}^{\infty} \mu_k \delta(t - t_k) - 1, \qquad \mu_k > 0, \tag{44}$$

has an unbiased integral given by

$$\mu(t) = \sum_{-\infty}^{\infty} \mu_k I_\lambda(t - t_k) - \int_{-\infty}^{\infty} I_\lambda(t)dt, \tag{45}$$

$$\int_{-\infty}^{\infty} |I_\lambda(t)|dt < \infty, \tag{45a}$$

$$\int_{-\infty}^{\infty} I_\lambda(t)e^{-i\omega t}dt = \frac{1}{i\omega}, \qquad |\omega| \geq \lambda. \tag{45b}$$

It was shown in Ref. 3 that an equivalent description of an integrating kernel $I_\lambda(t)$ is

$$I_\lambda(t) = \frac{1}{2} \operatorname{sgn} t - g_\lambda(t), \tag{46}$$

where g_λ is any function bandlimited to $[-\lambda, \lambda]$ such that (45a) holds. A useful construction for g_λ is as follows.

Suppose $f_\lambda(t)$ is bandlimited to $[-\lambda, \lambda]$, even, and its analytic continuation $f(\tau)$ is positive on the imaginary axis. Then for each $\xi \geq 0$,

$$g_\lambda(t; \xi) = \frac{t\left\{1 - \dfrac{f_\lambda)t)}{f_\lambda(i\xi)}\right\}}{\pi(t^2 + \xi^2)} \tag{47}$$

is bandlimited to $[-\lambda, \lambda]$.

Now we define the bandlimited function g_λ as

$$g_\lambda(t) = \int_0^{\infty} g_\lambda(t; \xi)d\xi$$

$$= \frac{1}{\pi} \int_0^\infty \frac{t}{t^2 + \xi^2}\, d\xi - \frac{t f_\lambda(t)}{\pi} \int_0^\infty \frac{d\xi}{(t^2 + \xi^2) f_\lambda(i\xi)}$$

$$= \frac{1}{2} \operatorname{sgn} t - \frac{t f_\lambda(t)}{\pi} \int_0^\infty \frac{d\xi}{(t^2 + \xi^2) f_\lambda(i\xi)}. \qquad (48)$$

This gives the useful representation for integrating kernels

$$I_\lambda(t) = t f_\lambda(t) \int_0^\infty \frac{d\xi}{\pi(t^2 + \xi^2) f_\lambda(i\xi)}. \qquad (49)$$

We have

$$\int_0^\infty \frac{d\xi}{\pi(t^2 + \xi^2) f_\lambda(i\xi)} < \frac{1}{t^2} \int_0^\infty \frac{d\xi}{f_\lambda(i\xi)}. \qquad (50)$$

So I_λ will belong to L_1 if (and only if)

$$\int_{|t|>1} \frac{|f_\lambda(t)|}{|t|}\, dt < \infty. \qquad (51)$$

Now if

$$f_\lambda(i\xi) > f_\lambda(0) \qquad \text{for} \qquad \xi > 0, \qquad (52)$$

as is always the case for even f_λ having only real zeros,

$$f_\lambda(i\xi) = f_\lambda(0) \prod_k^\pi \left(1 + \frac{\xi^2}{t_{k^2}}\right),$$

then

$$|t| \int_0^\infty \frac{d\xi}{\pi(t^2 + \xi^2) f_\lambda(i\xi)} < \frac{1}{f_\lambda(0)} \frac{1}{\pi} \int_0^\infty \frac{|t|\, d\xi}{t^2 + \xi^2}$$

$$= \frac{1}{2 f_\lambda(0)},\ (t \neq 0). \qquad (53)$$

Thus if (51) and (52) hold, we have

$$|I_\lambda(t)| \leqslant \frac{1}{2} \frac{|f_\lambda(t)|}{|f_\lambda(0)|},\ -\infty < t < \infty, \qquad (54)$$

with equality for some nonzero t_k if and only if $f_\lambda(t_k) = 0$.

Now we choose

$$f_\lambda(t) = \frac{1}{2} \left\{ \frac{\sin \frac{\lambda}{2} t}{\frac{\lambda}{2} t} \right\}^2 \qquad (55)$$

so that the corresponding integrating kernel (49) is odd and satisfies

$$-f_\lambda(t) \leqslant I_\lambda(t) \leqslant f_\lambda(t). \tag{56}$$

We then have for this I_λ

$$\mu(t) = \sum_{-\infty}^{\infty} \mu_k I_\lambda(t - t_k), \tag{57}$$

and since $\mu_k > 0$, we have

$$\mu(t) \geqslant -\sum_{-\infty}^{\infty} \mu_k f_\lambda(t - t_k) \tag{57a}$$

$$\leqslant \sum_{-\infty}^{\infty} \mu_k f_\lambda(t - t_k). \tag{57b}$$

But

$$\sum_{-\infty}^{\infty} \mu_k f_\lambda(t - t_k) = \int_{-\infty}^{\infty} f_\lambda(t - x)dx = \frac{\pi}{\lambda}.$$

Therefore,

$$\frac{-\pi}{\lambda} \leqslant \mu(t) \leqslant \frac{\pi}{\lambda}, \qquad -\infty < t < \infty. \tag{58}$$

We have equality holding on either side in (56) only for $t = 2k\pi/\lambda$, $k \neq 0$, with

$$I_\lambda(0+) = f_\lambda(0) = 1/2$$

$$I_\lambda(0-) = -f_\lambda(0) = -1/2.$$

So equality can hold in (57a) as a limit from the left, and in (57b) as a limit from the right, if and only if

$$t_k = \frac{2k\pi}{\lambda} + \theta, \qquad \text{all } k, \tag{59}$$

implying as before

$$\mu_k = \frac{2\pi}{k}, \qquad \text{all } k. \tag{60}$$

VII. PROOF OF COROLLARY 3.1

We set $h'(t)/c = \mu'(t)$ and apply Theorem 3 to obtain

$$\frac{-\pi c}{\lambda} \leqslant h(t) \leqslant \frac{\pi c}{\lambda}. \tag{61}$$

If f_λ is any function of L_1 whose Fourier transform vanishes outside $(-\lambda, \lambda)$, then

$$\int_{-\infty}^{\infty} f_\lambda(t)h(t)dt = \int_{-\infty}^{\infty} g_\lambda(t)h'(t)dt,$$

where

$$g_\lambda(t) = \int_{-\infty}^{\infty} f_\lambda(x)I_\lambda(t - x)dx \tag{62a}$$

$$\int_{-\infty}^{\infty} |g_\lambda(t)|dt \leq \int_{-\infty}^{\infty} |f_\lambda(t)|dt \cdot \int_{-\infty}^{\infty} |I_\lambda(t)|dt. \tag{62b}$$

Thus g_λ belongs to L_1 and its Fourier transform

$$\overset{>}{g}_\lambda(\omega) = \overset{>}{f}_\lambda(\omega)\overset{>}{I}_\lambda(\omega)$$

vanishes outside $(-\lambda, \lambda)$. Hence

$$\int_{-\infty}^{\infty} f_\lambda(t)h(t)dt = \int_{-\infty}^{\infty} g_\lambda(t)h'(t)dt = 0. \tag{63}$$

We could also establish this result by using the Representation Theorem for high-pass functions in Section III, and identifying $h(t)$ as

$$h(t) = \lim_{u \to 0+} \operatorname{Re}\{H(t + iu)\}, \tag{64}$$

where

$$H(\tau) = i \log G(\tau) \tag{64a}$$

and

$$G(\tau) = 2e^{ic\tau}s(\tau) = 1 + 2e^{ic\tau}g(\tau) + e^{2ic\tau} \tag{65}$$

$$G(t + iu) = 1 + 0(e^{-\lambda u}), \qquad u \to \infty. \tag{65a}$$

Here $H(\tau)$ is a particular integral of $H'(\tau)$ defined in (20), viz.,

$$H(\tau) = -\int_{\tau}^{i\infty} H'(z)dz. \tag{66}$$

However, bounds on $h(t)$ are not readily available from the representation (64) or (66), but more so from the unbiased integral of $h'(t)$.

The conditions for equality are argued as before in the proof of Corollary 2.

VIII. PROOF OF COROLLARY 3.2

We have

$$\mu'(t) = \frac{1}{c}\, h'(t) = \frac{\pi}{c} \sum_{-\infty}^{\infty} m_k\delta(t - t_k) - 1. \tag{67}$$

Then

$$\int_{x-0}^{x+T+0} \mu'(t)dt = \mu(x + T + 0) - \mu(x - 0). \tag{68}$$

Now the number of zeros of $s(t)$ in $[x, x + T]$ is

$$N_T(x) = \sum_{k:x \leqslant t_k \leqslant x+T} m_k. \tag{69}$$

Thus

$$\int_{x-0}^{x+T+0} \mu'(t)dt = \frac{\pi}{c} N_T(x) - T. \tag{70}$$

According to Theorem 3

$$\frac{-\pi}{\lambda} \leqslant \mu(t) \leqslant \frac{\pi}{\lambda}.$$

So

$$\frac{\pi}{c} N_T(x) - T \leqslant \frac{2\pi}{\lambda},$$

or

$$N_T(x) \leqslant \frac{2c}{\lambda} + \frac{c}{\pi} T. \tag{71}$$

The implications of equality are argued again as in Corollary 2.

IX. PROOF OF THEOREM 4

We have [cf. (64) and (65)]

$$h(t) + i\hat{h}(t) = \lim_{u \to 0+} i \log\{e^{ic(t+iu)} 2s(t + iu)\}, \tag{72}$$

where $\hat{h}$ is the Hilbert transform of h.

Then

$$|s(t)| = \frac{1}{2} e^{\hat{h}(t)}. \tag{73}$$

To obtain an upper bound on $\hat{h}$ we use (18); i.e.,

$$\hat{h}(t) = \sum_{-\infty}^{\infty} m_k L_\lambda(t - t_k) - \frac{c}{\pi} \int_{-\infty}^{\infty} L_\lambda(x)dx, \tag{74}$$

where

$$L_\lambda(t) = \int_{-\infty}^{t} f_\lambda(x)dx, \qquad t < 0, \tag{74a}$$

$$L_\lambda(t) = L_\lambda(-t) \tag{74b}$$

$$f_\lambda \text{ is bandlimited to } [-\lambda, \lambda] \tag{74c}$$

$$\text{with } f_\lambda(0) = 1,$$

$$f_\lambda(t) = f_\lambda(-t),$$

$$\text{and such that } \int_{-\infty}^{\infty} |L_\lambda(t)|\, dt < \infty.$$

We want an upper bound on $\hat{h}(t)$, say $\hat{h}(0)$. Since $\{m_k\}$ are positive (integers), we would like

$$L_\lambda(t) \leq 0, \qquad -\infty < t < \infty, \tag{75}$$

so that

$$\hat{h}(t) \leq -\frac{c}{\pi} \int_{-\infty}^{\infty} L_\lambda(x)dx, \qquad -\infty < t < \infty. \tag{76}$$

Then equality, e.g.,

$$\hat{h}(0) = -\frac{c}{\pi} \int_{-\infty}^{\infty} L_\lambda(x)dx,$$

would imply $L_\lambda(-t_k) = 0$, all k.

We expect the lattice distribution is extremal again, requiring $L_\lambda(t)$ to vanish with $(\cos \lambda t/2)^2$ in order to obtain the least upper bound for $\hat{h}$. We need an alternate construction for L_λ. To obtain this we write

$$L_\lambda(t) = -\int_t^\infty \frac{f_\lambda(x)}{x}\, dx, \qquad t > 0,$$

$$= \log t + \int_t^1 \frac{1 - f_\lambda(x)}{x}\, dx - \int_1^\infty \frac{f_\lambda(x)}{x}\, dx.$$

Then, since L_λ is even we have

$$L_\lambda(t) = \log|t| + F_\lambda(t), \tag{77}$$

where F_λ is an even entire function of exponential type λ such that $L_\lambda(t)$ is absolutely integrable,

$$F_\lambda(t) = \int_t^1 \frac{1 - f_\lambda(x)}{x}\, dx - \int_1^\infty \frac{f_\lambda(x)}{x}\, dx, \tag{77a}$$

$$F_\lambda'(t) = \frac{f_\lambda(t) - 1}{t}. \tag{77b}$$

Now we obtain an alternate construction for F_λ.

We suppose that g_λ is an even entire function of exponential type λ, positive on the imaginary axis and define for each $\xi \geq 0$

$$G_\lambda(t;\,\xi) = \frac{\xi\left(1 - \dfrac{g_\lambda(t)}{g_\lambda(i\xi)}\right)}{t^2 + \xi^2} - \frac{\xi}{1 + \xi^2}. \tag{78}$$

Then we set

$$\begin{aligned} F_\lambda(t) &= \int_0^\infty G_\lambda(t;\,\xi)d\xi \\ &= \int_0^\infty \left(\frac{\xi}{t^2 + \xi^2} - \frac{\xi}{1 + \xi^2}\right) d\xi - g_\lambda(t) \int_0^\infty \frac{\xi}{(t^2 + \xi^2)g_\lambda(i\xi)}\, d\xi \\ &= -\log|t| - g_\lambda(t) \int_0^\infty \frac{\xi d\xi}{(t^2 + \xi^2)g_\lambda(i\xi)} \end{aligned} \tag{79}$$

and then

$$L_\lambda(t) = \log|t| + F_\lambda(t) = -g_\lambda(t) \int_0^\infty \frac{\xi d\xi}{(t^2 + \xi^2)g_\lambda(i\xi)}. \tag{80}$$

We have

$$L_\lambda(t) \sim -\frac{g_\lambda(t)}{t^2} \int_0^\infty \frac{\xi d\xi}{g_\lambda(i\xi)}, \qquad t \to \infty. \tag{81}$$

So L_λ will be absolutely integrable if (and only if)

$$\int_{|t|>1} |g_\lambda(t)|\, \frac{dt}{t^2} < \infty. \tag{82}$$

The particular L_λ we want is

$$L_\lambda(t) = -\cos^2 \frac{\lambda t}{2} \int_0^\infty \frac{\xi d\xi}{(t^2 + \xi^2)\cosh^2 \dfrac{\lambda\xi}{2}}. \tag{83}$$

We can evaluate the integral of L_λ indirectly by considering

$$\frac{2c}{\lambda} = m,$$

$$s(t) = \frac{1}{2}\left(2 \cos \frac{\lambda t}{2}\right)^m.$$

Then

$$\log 2|s(t)| = \sum_{-\infty}^{\infty} mL_\lambda(t - t_k) - \frac{c}{\pi} \int_{-\infty}^\infty L_\lambda(x)dx,$$

where

$$t_k = (2k + 1)\,\frac{\pi}{\lambda},$$

$$\log 2|s(0)| = -\frac{c}{\pi}\int_{-\infty}^{\infty} L_\lambda(x)dx,$$

$$m \log\ 2 = -\frac{m\lambda}{2\pi}\int_{-\infty}^{\infty} L_\lambda(x)dx,$$

i.e., for L_λ given by (83) we have

$$\int_{-\infty}^{\infty} L_\lambda(t)dt = -\frac{2\pi}{\lambda}\log 2. \tag{84}$$

Thus we have from (84) and (76)

$$\hat{h}(t) \leqslant \frac{2c}{\lambda}\log 2, \qquad -\infty < t < \infty, \tag{85}$$

with equality attainable only for the lattice distribution, giving

$$|s(t)| \leqslant 2^{(2c/\lambda)-1}, \qquad -\infty < t < \infty, \tag{86}$$

where equality for any t implies the conclusion of Corollary 2, by the same argument.

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