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# Click Modulation

By B. F. LOGAN, JR.*

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In this paper we show how one may determine a sequence of equal intensity impulses or clicks

$$\pi \sum_{-\infty}^{\infty} \delta(t - t_k)$$

such that a desired bandpass signal, $f'(t)$, may be obtained by filtering the clicks; i.e.,

$$f'(t) = \pi \sum_{-\infty}^{\infty} K(t - t_k),$$

where $K(t)$ is the impulse response of a suitable bandpass filter. The $\{t_k\}$ are found as the zeros of a bandlimited signal $s(t)$, where if $f(t)$, the bandpass signal whose derivative is $f'(t)$, is sufficiently small, we also have

$$f(t) = \int_{-\infty}^{\infty} q(x)K(t - x)dx,$$

where $q(x)$ is a square wave simply related to $s(t)$.

## I. INTRODUCTION

Click modulation describes a sort of pulse-position modulation leading to a sequence of equal-intensity impulses or clicks,

$$\pi \sum_{-\infty}^{\infty} \delta(t - t_k) \tag{1}$$

* AT&T Bell Laboratories.

such that a desired signal, say $f'(t)$, may be obtained by filtering the clicks; i.e.,

$$f'(t) = \pi \sum_{-\infty}^{\infty} K(t - t_k). \tag{2}$$

Here we suppose that $f'(t)$ is a bandpass signal with spectrum confined to $[\lambda, \mu]$ and $[-\mu, -\lambda]$, $0 < \lambda < \mu < \infty$, and $K(t)$ is any function satisfying

$$\int_{-\infty}^{\infty} |K(t)| dt < \infty \tag{3a}$$

$$\int_{-\infty}^{\infty} K(t)dt = 0 \tag{3b}$$

$$\begin{aligned} \tilde{K}(\omega) = \int_{-\infty}^{\infty} K(t)e^{-i\omega t}dt &= 1, \begin{cases} \lambda \leqslant \omega \leqslant \mu \\ -\mu \leqslant \omega \leqslant -\lambda \end{cases} \\ &= 0, \ |\omega| > \alpha. \end{aligned} \tag{3c}$$

The filter characterized by $K(t)$ is required to reproduce $f'(t)$, reject dc, and reject frequencies greater than $\alpha$, where $\alpha$ is some specified frequency ($\alpha > \mu$). (See Fig. 1.) In other words, we are supposing some constant $c$, such that the "Fourier transforms" of $f'(t)$ and $h'(t)$ agree over $(-\alpha, \alpha)$, where

$$h'(t) = \pi \sum_{-\infty}^{\infty} \delta(t - t_k) - c. \tag{4}$$

The distribution $h'(t)$ then has no spectrum in $(-\lambda, \lambda)$ nor in the guard band $(\mu, \alpha)$ and its reflection. In some applications a large guard band may be required to ease the filtering problem, while in others, e.g., audio, a small guard band may be tolerable.

## II. THE SOLUTION OF THE PROBLEM OF CLICK MODULATION

The problem of click modulation is given $f'(t)$, $\lambda$, $\mu$, and $\alpha$, to find a set $\{t_k\}$ such that (2) holds. The basis for solving this problem is found in Refs. 1 through 4. The $\{t_k\}$ are assumed to be the zeros (real and simple) of a signal of the form

$$s(t) = \cos ct + g(t), \tag{5}$$

where $g$ is bandlimited to $[-b, b]$ and

$$c - b = \lambda > 0 \tag{5a}$$

$$c > \alpha > \mu. \tag{5b}$$

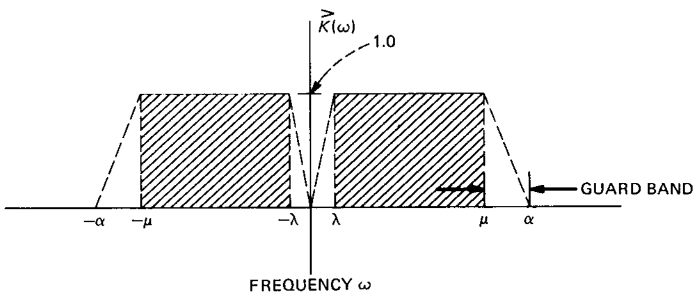


Fig. 1—Filter characteristic $\hat{K}(\omega)$ for bandlimiting impulse train (clicks) to recover desired signal $f'(t)$ of spectral support $[-\mu, -\lambda] \cup [\lambda, \mu]$.

Then the function defined by

$$H(\tau) = i \log[2e^{ic\tau}s(\tau)] \tag{6}$$

is analytic in the upper half-plane of the complex variable $\tau = t + iu$, where the principal branch of log is taken;

$$\log(1 + z) = z + O(z^2), \qquad z \to 0$$

giving

$$|H(t + iu)| = O(e^{-\lambda u}), \qquad u \to \infty. \tag{7}$$

The function $h(t)$ defined by

$$h(t) = \lim_{u \to 0+} \mathrm{Re}\{H(t + iu)\} \tag{8}$$

has the form

$$h(t) = J(t) - ct, \tag{9}$$

where $J(t)$ is a jump function increasing by $\pi$ at each zero $t_k$ of $s$, and $h$ is high-pass with no spectrum in $(-\lambda, \lambda)$.

We then suppose that $f(t)$, with spectrum confined to $[\lambda, \mu]$ and $[-\lambda, -\mu]$, is given by

$$f(t) = \int_{-\infty}^{\infty} h(x)K(t - x)dx, \tag{10}$$

where $K$ satisfies (3a), (3b), and (3c). Then differentiating, we obtain

$$f'(t) = \int_{-\infty}^{\infty} h'(x)K(t - x)dx$$

$$= \pi \sum_{-\infty}^{\infty} K(t - t_k). \tag{11}$$

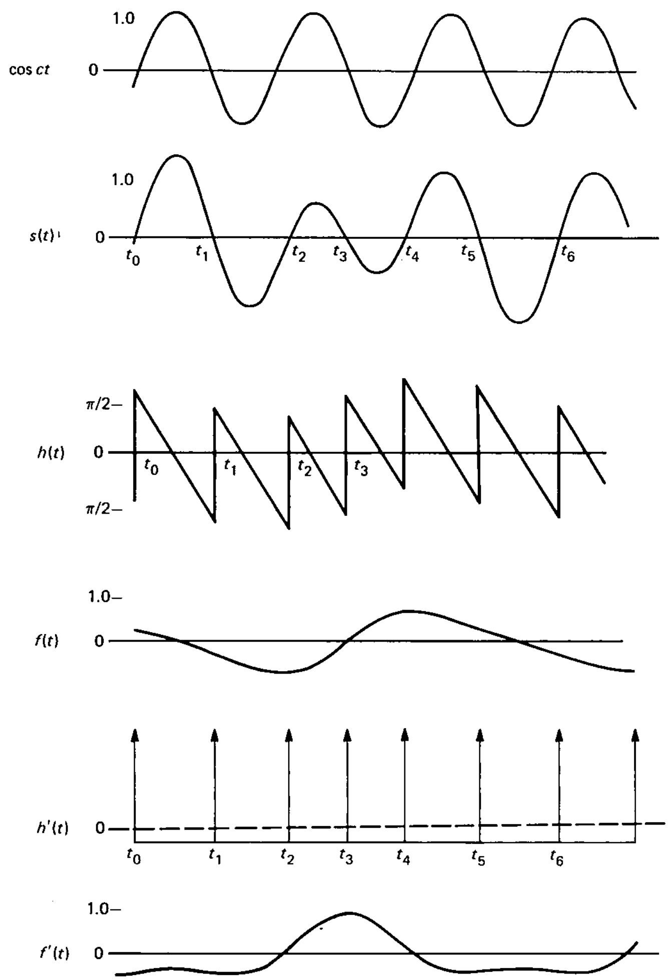


Fig. 2—Waveform relations in click modulation.

The relationships of $s(t)$, $h(t)$, $f(t)$, $h'(t)$, and $f'(t)$ are illustrated in Fig. 2.

Now we would like to find $s(t)$ and hence $h(t)$ so that (10) holds. To do this it is convenient to work with "analytic signals" having no negative frequency content, or in the terminology of Ref. 4, functions whose "Fourier transforms vanish over $(-\infty, 0)$". (Refer to Fig. 3 in the course of the following development.) Thus we introduce

$$F(t) = f(t) + i\hat{f}(t), \tag{12}$$

where $\hat{f}$ is the Hilbert transform of $f$. Then the Fourier transform of

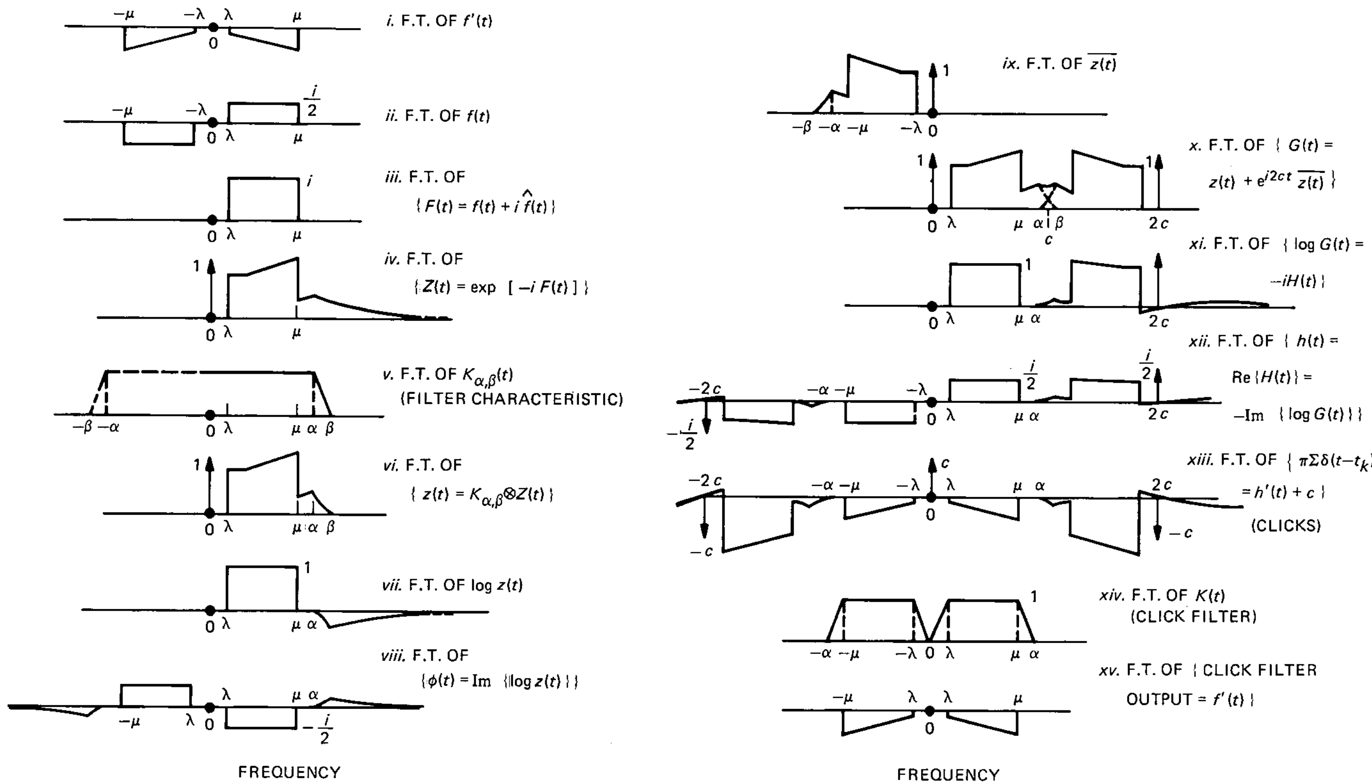


Fig. 3—Depiction of Fourier Transform (F.T.) relations in derivation of click modulation.

$F$ vanishes outside the single interval $[\lambda, \mu]$. [It is important to note that a bounded bandpass signal $f$ always has a bounded Hilbert transform $\hat{f}$.] Now we require that the Fourier transforms of $H(t)$ and $F(t)$ agree over $(-\infty, \alpha)$, i.e., in accord with (10),

$$F(t) = \int_{-\infty}^{\infty} H(x)K(t - x)dx. \tag{13}$$

We define

$$Z(t) = \exp[-iF(t)] \tag{14}$$

and then bandlimit $Z$ to obtain

$$z(t) = \int_{-\infty}^{\infty} Z(x)K_{\alpha,\beta}(t - x)dx, \tag{15}$$

where the filter kernel $K_{\alpha,\beta}$ is absolutely integrable and

$$\begin{aligned} K^{>}_{\alpha,\beta}(\omega) &= \int_{-\infty}^{\infty} K_{\alpha,\beta}(t)e^{-i\omega t}dt \\ &= 1 \quad \text{for} \quad 0 \leqslant \omega \leqslant \alpha \\ &= 0 \quad \text{for} \quad \omega \geqslant \beta, \qquad (\beta > \alpha). \end{aligned} \tag{15a}$$

Thus $z(t)$ is bandlimited to $[0, \beta]$, and the Fourier transforms of $z(t)$ and $Z(t)$ agree over $(-\infty, \alpha)$.

Now we set

$$\begin{aligned} G(t) = G(t; \theta, c) &= z(t) + e^{i2(ct+\theta)}\overline{z(t)} \\ &= e^{i(ct+\theta)}[z(t)e^{-i(ct+\theta)} + \overline{z(t)}e^{i(ct+\theta)}], \end{aligned} \tag{16}$$

where

$$2c - \beta \geqslant \alpha, \tag{16a}$$

$$\theta \text{ is any fixed angle,} \tag{16b}$$

and $\overline{z(t)}$ is the complex conjugate of $z(t)$. Now the Fourier transform of $\overline{z(t)}$ vanishes outside $[-\beta, 0]$. Thus the Fourier transform of

$$e^{i2(ct+\theta)}\overline{z(t)}$$

vanishes outside $[2c - \beta, 2c]$, and since $2c - \beta \geqslant \alpha$, the Fourier transforms of $Z(t)$, $z(t)$, and $G(t)$ agree over $(-\infty, \alpha)$. These relations are depicted in Fig. 3, parts (iv), (vi), and (x). Note that

$$Z(t) = \exp[-iF(t)] = 1 - iF(t) - \frac{F^2(t)}{2!} + \cdots, \tag{17}$$

and since the Fourier transform of $\{F(t)\}^n$ vanishes outside $[n\lambda, n\mu]$, $Z(t)$ has a spectral gap $(0, \lambda)$.

We may write (16) as

$$G(t) = 2e^{i(ct+\theta)}s(t), \tag{18}$$

where

$$s(t) = \frac{1}{2}\,[z(t)e^{-i(ct+\theta)} + \bar{z}(t)e^{i(ct+\theta)}] \tag{19}$$

is real-valued and bandlimited to $[-c, c]$, and is of the form

$$s(t) = \cos(ct + \theta) + g(t), \tag{19a}$$

where $g$ is bandlimited to $[-b, b]$, $b = c - \lambda$.

Now we suppose $s(\tau)$ has only real zeros. Then $\log G(\tau)$ is analytic in the upper half-plane. We compare it with

$$\log Z(\tau) = -iF(\tau),$$

which is certainly analytic in the upper half-plane. Since the Fourier transforms of $Z(t)$ and $G(t)$ agree over $(-\infty, \alpha)$ and both $\log Z(\tau)$ and $\log G(\tau)$ are analytic in the upper half-plane, it follows from the theory in Ref. 4 that the Fourier transforms of $\log G(t)$ and $\log Z(t)$ also agree over $(-\infty, \alpha)$; i.e., the Fourier transforms of

$$H(t) = i \log G(t)$$

and

$$F(t) = i \log Z(t)$$

agree over $(-\infty, \alpha)$ provided $s(\tau)$, the analytical continuation of $s(t)$ given by (19), has only real zeros $\{t_k\}$. We also require the zeros to be simple so that $h(t) = \mathrm{Re}\{H(t)\}$ has the form (9). It was shown in Ref. 5 that a sufficient condition for $s$ of the form (19), $(c > \beta/2)$, to have only real simple zeros is that $z(\tau)$ be zero-free in the (closed) upper half-plane, $\mathrm{Im}\ \tau \geq 0$.

Thus if $z(\tau)$, the analytic continuation of $z(t)$ obtained by bandlimiting $Z(t)$ where $Z(t) = \exp[-iF(t)]$, is zero-free in the upper half-plane $\mathrm{Im}\ \tau \geqslant 0$, then $\log z(\tau)$ is analytic in the upper half-plane and then the Fourier transforms of

$$H(t) = i \log G(t),$$

$$F(t) = i \log Z(t),$$

and

$$i \log z(t)$$

agree over $(-\infty, \alpha)$, which means that the Fourier transforms of the real (imaginary) parts of these functions agree over $(-\alpha, \alpha)$.

Writing

$$z(t) = A(t)e^{i\phi(t)}, \tag{20}$$

where

$$A(t) = |z(t)|,$$

$$\phi(t) = \text{Im}\{\log z(t)\} = \text{phase } z(t),$$

we have the Fourier transforms of $h(t)$, $f(t)$, and $-\phi(t)$ agreeing over $(-\alpha, \alpha)$ whenever $z(\tau)$ is zero-free in the upper half-plane, Im $\tau \geq 0$.

Using (20), we may write

$$s(t) = A(t)\cos[ct + \theta - \phi(t)]. \tag{21}$$

Thus the zeros $t_k$ of $s(t)$ are the zeros of the phase-modulated cosine,

$$\Phi(t) = \cos[ct + \theta - \phi(t)], \tag{22}$$

where $\phi(t)$ is the phase function of an analytic signal $z(t)$, with $z(t)$ bandlimited to $[0, \beta]$ and $z(t + iu)$ zero-free for $u \geq 0$, such that the Fourier transforms of $-\phi(t)$ and $f(t)$ agree over $(-\alpha, \alpha)$, $(\mu < \alpha < \beta)$, and

$$c \geq \frac{\alpha + \beta}{2}.$$

Now $\phi(t)$ is not bandlimited, but

$$-\phi(t) = f(t) + \epsilon(t), \tag{23}$$

where $\epsilon(t)$ is high-pass with no spectrum in $(-\alpha, \alpha)$ and $|\epsilon|$ may be small compared to $|f|$ if $|f|$ is small, or if $\alpha$, $\beta$, and $c$ are large compared to $\mu$.

We have noted that the Fourier transforms of $-\phi(t)$, $f(t)$, and $h(t)$ agree over $(-\alpha, \alpha)$. It is interesting to observe that

$$h(t_k) = -\phi(t_k) \tag{24}$$

provided we take

$$h(t_k) = \frac{1}{2}[h(t_k+) + h(t_k-)], \tag{24a}$$

which, incidentally, follows by defining

$$h(t) = \lim_{u \to 0+} \text{Re}\{H(t + iu)\}.$$

To see (24) we consider

$$\log G(t) - \log z(\tau) = \log \left\{1 + \frac{\overline{z(\bar{\tau})}}{z(\tau)} e^{i2(c\tau+\theta)}\right\}$$
$$= \log[1 + U(\tau)], \qquad (25)$$

where

$$U(t) = e^{i2(ct+\theta-\phi(t))}$$

and

$$|U(t + iu)| \leqslant e^{-(2c-\beta)u}, \qquad u \geqslant 0. \qquad (25a)$$

We have shown elsewhere[5] that

$$\phi'(t) \leqslant \beta/2$$

so that the total phase of $U(t)$ is an increasing function. With $|U(t)| = 1$, we have for the principal branch of the logarithm

$$-\pi/2 \leqslant \arg\{1 + U(t)\} \leqslant \pi/2. \qquad (26)$$

We have $U(t_k) = -1$ and since the phase is increasing

$$\arg\{1 + U(t_k-)\} = \pi/2$$
$$\arg\{1 + U(t_k+)\} = -\pi/2.$$

Thus

$$\operatorname{Re}\{i \log G(t_k-) - i \log z(t_k-)\} = h(t_k-) + \phi(t_k) = -\pi/2$$
$$\operatorname{Re}\{i \log G(t_k+) - i \log z(t_k+)\} = h(t_k+) + \phi(t_k) = \pi/2$$

from which (24) follows. In fact, we have from (25) and (26)

$$-\pi/2 \leqslant h(t) + \phi(t) \leqslant \pi/2. \qquad (27)$$

Then (24) follows from (27) and the fact that $\phi(t)$ is continuous with $h(t)$ increasing by $\pi$ at $t_k$.

The condition that $z(\tau)$ be zero-free in the upper half-plane is difficult to quantify precisely in terms of all the parameters. Generally speaking, for fixed $\alpha$ and $\beta$, $z(\tau)$ will be zero-free in the upper half-plane if $\sup|F(t)|$ is sufficiently small. The problem is much the same as that of determining the bandwidth requirements for exponential single-sideband modulation.[4] The problem will be treated in detail in a future paper. Suffice it here to say that a sufficient condition is

$$\operatorname{Re}\{z(t)\} > 0, \qquad -\infty < t < \infty, \qquad (28)$$

or

$$-\pi/2 < \phi(t) < \pi/2, \qquad -\infty < t < \infty. \qquad (29)$$

This condition is also of interest in obtaining a square wave representation of $f(t)$.

## III. SQUARE WAVE REPRESENTATION OF $f(t)$

When $|\phi(t)| < \pi/2$, the zeros $t_k$ of

$$s(t) = A(t)\cos[ct + \theta - \phi(t)] \tag{30}$$

interlace with the zeros of

$$\sin(ct + \theta).$$

For simplicity of discussion, we assume $\theta = 0$ (consider a translate). Then, as shown in Ref. 1, we have

$$h(k\pi/c) = 0 \qquad k = 0, \pm 1, \pm 2, \cdots \tag{31}$$

$$-\pi < h(t) < \pi. \tag{32}$$

Then if we subtract from $h(t)$ the periodic sawtooth function defined by

$$\begin{aligned} \sigma(t) &= \pi/2 - ct, \qquad 0 < t < \pi/c, \\ \sigma(t) &= \sigma(t + \pi/c) \\ \sigma(0) &= 0, \end{aligned} \tag{33}$$

we obtain a square wave,

$$q(t) = h(t) - \sigma(t) = -\frac{\pi}{2}\{\mathrm{sgn}\ s(t)\}\cdot\{\mathrm{sgn}\sin ct\} \tag{34}$$

(see Fig. 4).

The Fourier series for $\sigma(t)$ is

$$\sigma(t) = \sum_1^\infty \frac{1}{n}\sin 2nct;$$

so the Fourier transforms of $q(t)$ and $h(t)$ agree over $(-2c, 2c)$, and since $c > \alpha$, the Fourier transforms of $q(t)$ and $f(t)$ agree over $(-\alpha, \alpha)$.

Thus we may filter the square wave to obtain $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} q(x)K(t - x)dx, \tag{35}$$

where $K$ is a reproducing kernel for $f$, which rejects all frequencies outside $(-\alpha, \alpha)$.

The square wave representation of $f(t)$ is of practical interest. First, one may regard the formulas (34) and (35) as a practical way of demodulating $s(t)$ after it is transmitted through a nonlinear medium.

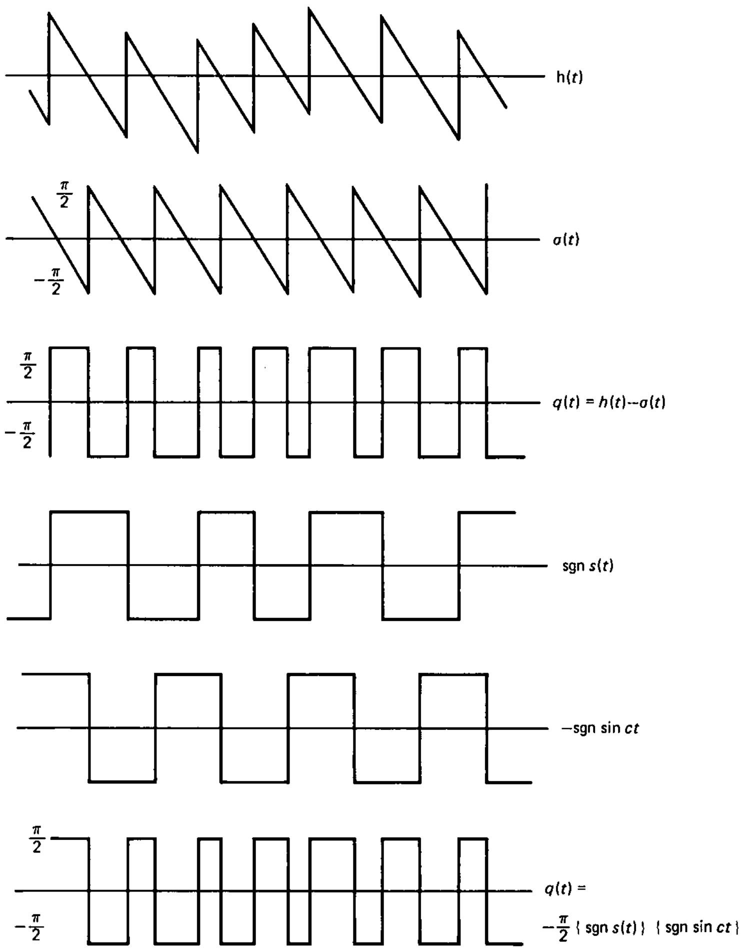


Fig. 4—Sum and product representation of square wave $q(t)$.

That is, sgn $s(t)$ is formed and then multiplied by sgn$\{\sin(ct + \Delta)\}$, where $\Delta$ is adjusted (to some multiple of $\pi$) by a phase-lock loop so that the average value of the product is zero. Then, filtering the resultant square wave gives a signal proportional to $f(t)$. In this application one may think of $c$ large, with $\alpha$ and $\beta$ not much larger than $\mu$, the top frequency of $f$, so that $s(t)$ has the character of a single-sideband (lower-sideband) signal with spectrum confined to $[c, c - \beta]$ and $[-c, -(c - \beta)]$.

In another application, $f(t)$ may be thought of as an audio signal that is to be reproduced by driving a loudspeaker with a switching-type (class D) power amplifier having very good efficiency. The Sony

Corporation has marketed a switching-type amplifier using an approximate method of obtaining the square wave.[6] Their approximation is equivalent to taking $t_k$ to be the zeros of [cf. (22), (23)]

$$\cos[ct + \theta + f(t)],$$

where

$$\pi/2 < f(t) < -\pi/2,$$

which results in a good approximation for $c \gg \mu$ (in their case, $c = 2\pi \cdot 500$ kHz) but requiring an unnecessarily high switching frequency. [In the Sony system the square wave is generated by clipping the sum of the sawtooth $\sigma(t)$ and the signal $f(t)$; i.e.,

$$\frac{\pi}{2}\,\mathrm{sgn}\{\sigma(t) + f(t)\},$$

which is a conventional way of obtaining analog pulse-width modulation. This is to be compared with (34), which may be equivalently written

$$q(t) = -\frac{\pi}{2}\,\mathrm{sgn}\{\sigma(t) + \phi(t)\}.]$$

## IV. IMPLEMENTATION

Figure 5 is a block diagram of a click modulation system, including the optional square wave output. The input is the bandpass signal $f'(t + \Delta)$, which is fed to a Hilbert transform network, incurring a delay $\Delta$, to obtain $\hat{f}'(t)$. The input is correspondingly delayed to obtain $f'(t)$. Then $f'(t)$ and $\hat{f}'(t)$ are fed to an Analytic Exponential Modulator (AEM), which furnishes the outputs

$$X(t) = e^{\hat{f}(t)}\cos[f(t)] \tag{36}$$

$$Y(t) = -e^{\hat{f}(t)}\sin[f(t)], \tag{37}$$

where

$$X(t) + iY(t) = Z(t) = \exp[-iF(t)] \tag{38}$$

and

$$F(t) = f(t) + i\hat{f}(t). \tag{39}$$

These outputs are then bandlimited with identical low-pass filters having unity transmission over the band $(-\alpha, \alpha)$ and zero transmission outside the band $(-\beta, \beta)$ to obtain the signals $x(t)$ and $y(t)$, where

$$x(t) + iy(t) = z(t). \tag{40}$$

Then these signals are multipled by $\cos(ct + \theta)$ and $\sin(ct + \theta)$ and

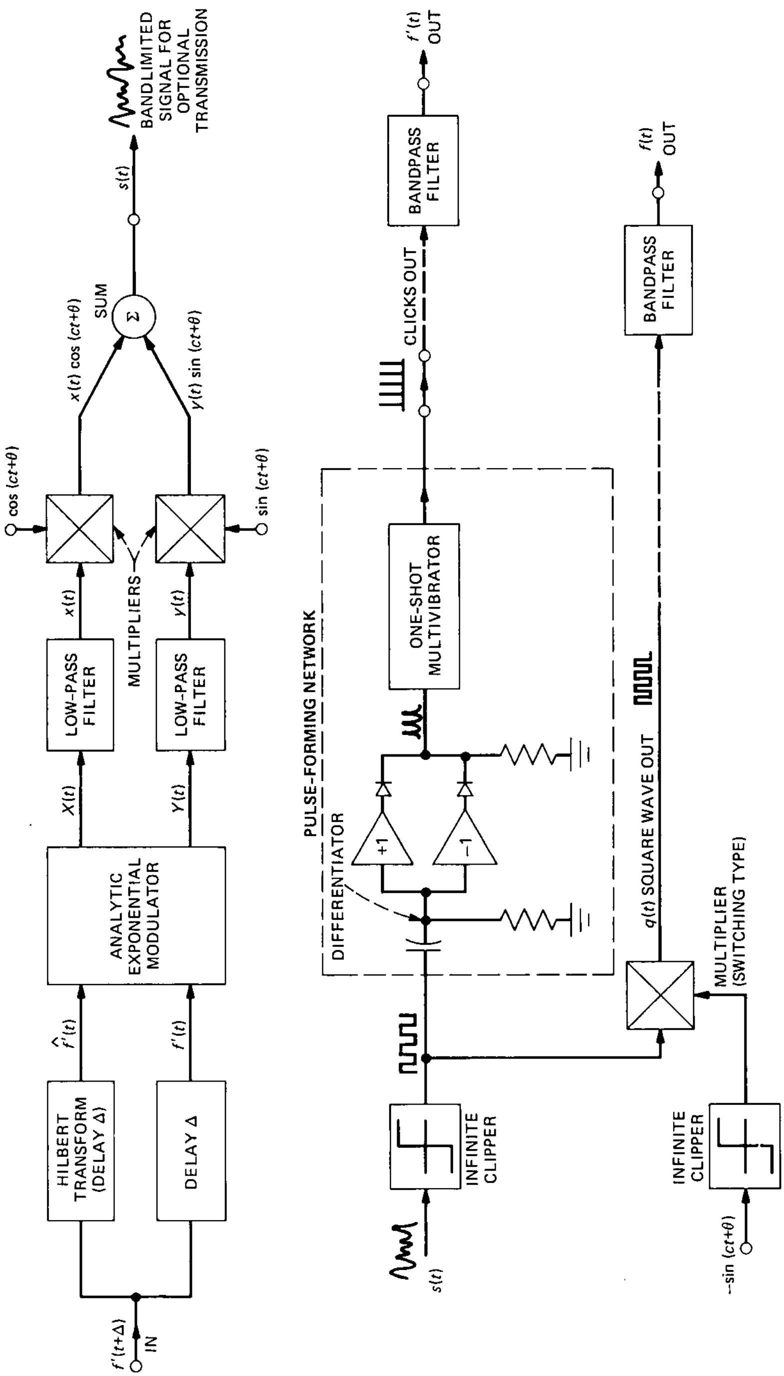


Fig. 5—A click modulation system.

then added to obtain the signal

$$s(t) = x(t)\cos(ct + \theta) + y(t)\sin(ct + \theta), \tag{41}$$

where the carrier frequency $c$ must satisfy

$$c \geqslant \frac{\alpha + \beta}{2}.$$

The level of the input signal should be adjusted so that consecutive zeros of $s(t)$ never coalesce (in loud passages), or alternatively, to use the square wave output, the input level should be adjusted (made sufficiently small) so that $x(t)$ is always positive.

To obtain the click output, uniform pulses are formed at the zeros of $s(t)$. This may be accomplished as indicated by infinitely clipping $s(t)$, then differentiating, rectifying the resulting pulses, and using these to trigger a one-shot multivibrator. These pulses may be filtered to obtain $f'(t)$, the derivative of $f(t)$. To obtain the square wave output, the clipped signal $s(t)$, i.e., sgn $s(t)$, is multiplied by the clipped sine wave, $-\text{sgn}\{\sin(ct + \theta)\}$. When it is filtered, the resulting square wave, with suitable scaling (multiplied by $\pi/2$), will give $f(t)$, provided the input level is small enough so that $x(t)$ is always positive.

## V. THE ANALYTIC EXPONENTIAL MODULATOR

The functions $X(t)$ and $Y(t)$, given by (36) and (37), may be generated using function generators and multipliers. However, they may be generated more simply in a feedback loop, which can be made stable, using the fact that $\{X(t) - 1\}$ and $Y(t)$ are high-pass signals containing no frequencies lower than $\lambda$, the lower frequency of $f(t)$.

Differentiating (38) we obtain

$$Z'(t) = iF'(t)Z(t)$$

or

$$X'(t) + iY'(t) = \{\hat{f}'(t) - if'(t)\}\{X(t) + iY(t)\}.$$

Then, equating real and imaginary parts we have

$$X'(t) = \hat{f}'(t)X(t) + f'(t)Y(t) \tag{42}$$

$$Y'(t) = \hat{f}'(t)Y(t) - f'(t)X(t). \tag{43}$$

Now integrators and multipliers can be connected so as to solve this pair of differential equations for $X(t)$ and $Y(t)$, given *arbitrary* functions $f(t)$ and $\hat{f}(t)$. However, drifts and offsets would soon cause trouble, resulting in exponential growth of the functions. This can be avoided when $f$ and $\hat{f}$ are high-pass Hilbert transform pairs.

In the theory we have treated all signals as dimensionless quantities.

Now, for clarity in implementing the analog circuitry, we attach the dimension of volts to all the signals and write

$$Z(t) = B \exp[-iF(t)/E] = X(t) + iY(t), \tag{44}$$

where $B$ and $E$ have the dimensions of volts. Then (42) and (43) become

$$\frac{dX}{dt} = \frac{d\hat{f}(t)}{dt}\frac{X(t)}{E} + \frac{df(t)}{dt}\frac{Y(t)}{E} \tag{45}$$

$$\frac{dY}{dt} = \frac{d\hat{f}(t)}{dt}\frac{Y(t)}{E} - \frac{df(t)}{dt}\frac{X(t)}{E}. \tag{46}$$

All the signals here are high-pass (lower frequency $= \lambda$), with the exception of $X(t)$, which we write as

$$X(t) = B + X_0(t), \tag{47}$$

where $X_0(t)$ is high-pass (lower frequency $= \lambda$). Then we rewrite eqs. (45) and (46) as

$$\frac{dX}{dt} = \frac{dX_0}{dt} = \frac{B}{E}\frac{d\hat{f}(t)}{dt} + \frac{d\hat{f}(t)}{dt}\frac{X_0(t)}{E} + \frac{df(t)}{dt}\frac{Y(t)}{E} \tag{48}$$

$$\frac{dY}{dt} = \frac{d\hat{f}(t)}{dt}\frac{Y(t)}{E} - \frac{B}{E}\frac{df(t)}{dt} - \frac{df(t)}{dt}\frac{X_0(t)}{E}. \tag{49}$$

We multiply all derivatives by some $T$, which has the dimensions of $t$ (time), since analog circuitry for differentiating $f(t)$ will give $Tf'(t)$ in volts.

The analog implementation of the two differential eqs. (48) and (49) is shown in Fig. 6.

High-gain (negative) ac amplifiers are connected as ac integrators with feedback capacitors $C$ and input resistors $R$, $R$, and $(M/B)R$ to give

$$RC\frac{dY}{dt} = \frac{T\hat{f}'(t)}{M}\,Y(t) - \frac{B}{M}\,Tf'(t) - \frac{T}{M}\,f'(t)X_0(t) \tag{50}$$

$$RC\frac{dX}{dt} = \frac{T\hat{f}'(t)}{M}\,X_0(t) + \frac{B}{M}\,T\hat{f}'(t) + \frac{T}{M}\,f'(t)Y(t). \tag{51}$$

Here $M$ is the multiplier scale factor (volts); it is the output of the multiplier when both inputs are 1 volt. Comparing these equations with (48) and (49) we see that the "normalization factor" $E$ introduced in (44) is

$$E = \frac{RCM}{T}. \tag{52}$$

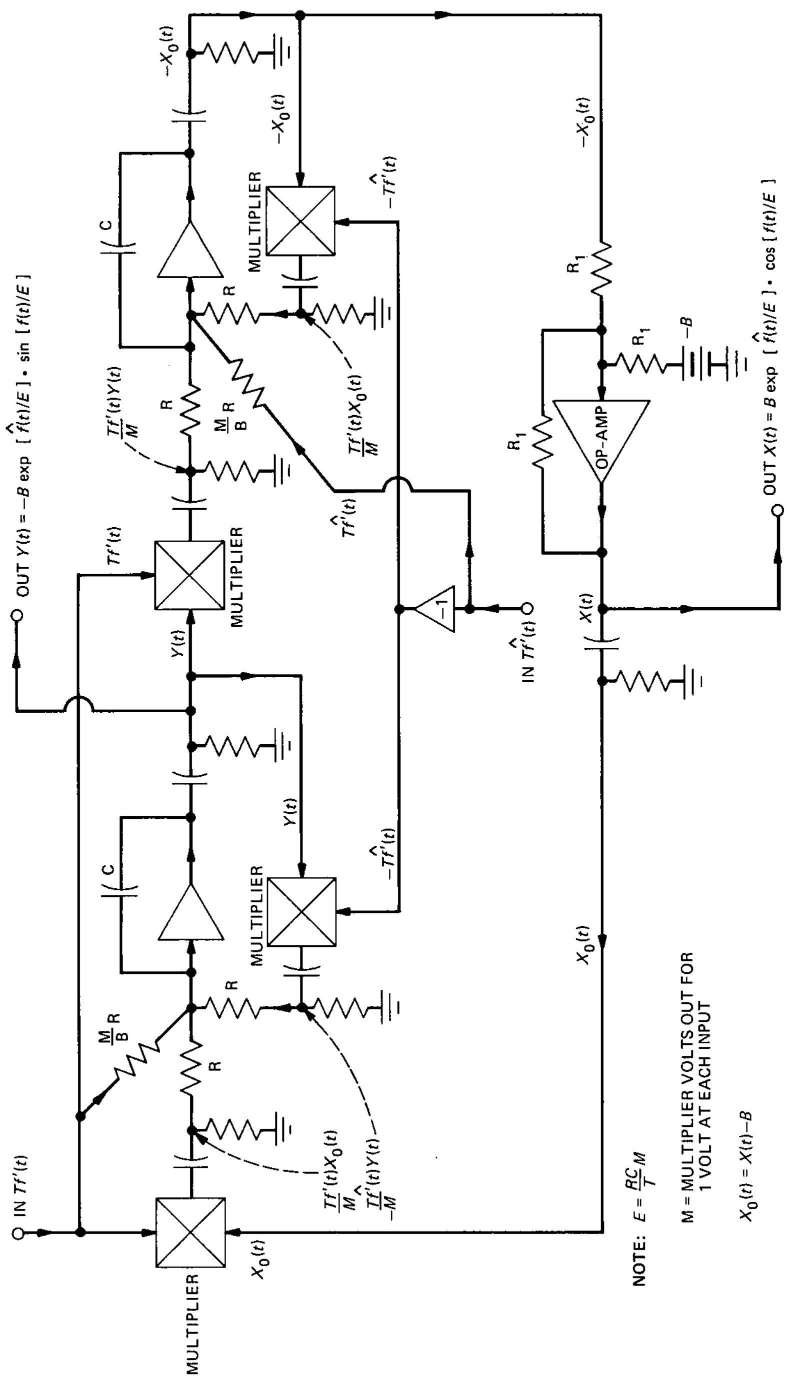


Fig. 6—Analytic exponential modulator.

Coupling networks employing dc blocking condensers are shown at the inputs and outputs of amplifiers and multipliers. The time constant of these networks should be large compared to the reciprocal of the lower frequency $\lambda$. Some of these networks may be incorporated in the ac amplifiers. The important point is that the inputs to the multipliers should not have dc components. Then when inputs $Tf'(t)$ and $T\hat{f}'(t)$ are zero (steady state), the circuit is in a quiescent condition with $X(t) = B$ and $Y(t) = 0$. If $f'(t)$ and $\hat{f}'(t)$ were not high-pass, and Hilbert transform pairs, then ac coupling could not be used, because the corresponding signals $X(t) - B$ and $Y(t)$ would not be high-pass.

A dc operational amplifier is shown in the feedback loop, connected to give a gain of $-1$ to the inputs $-X_0(t)$ and $-B$. This is not necessary, but provides a convenient way to add $B$ to $X_0(t)$ and give a low impedance output for $X(t)$. The dc amplifier can be replaced by an ac amplifier (gain $-1$) and the summing of $B$ with $X_0(t)$ incorporated elsewhere in the external circuitry.

Switching-type multipliers may be used in the AEM if $f'(t)$ and $\hat{f}'(t)$ are replaced by binary-valued switching signals, which, if integrated, give very close approximations to $f(t)$ and $\hat{f}(t)$. These switching signals may be obtained from a "delta mod", as shown in Fig. 7. The resulting error is mainly high frequency in $f(t)$ and $\hat{f}(t)$, which translates into high-frequency error in $X(t)$ and $Y(t)$. This error will subsequently be removed by bandlimiting $X(t)$ and $Y(t)$ to obtain $x(t)$ and $y(t)$.

## VI. NUMERICAL EXAMPLE

To illustrate the theory we take a simple example:

$$\lambda = 1, \quad \mu = 2, \quad \alpha = 2.2, \quad \beta = 2.8, \quad c = 5/2,$$

$$f'(t) = -\frac{1}{2}\cos t + \frac{1}{4}\cos 2t,$$

$$f(t) = -\frac{1}{2}\sin t + \frac{1}{8}\sin 2t.$$

We have

$$\hat{f}(t) = \frac{1}{2}\cos t - \frac{1}{8}\cos 2t$$

so that

$$iF(t) = \hat{f}(t) - if(t) = \frac{1}{2}e^{it} - \frac{1}{8}e^{i2t}.$$

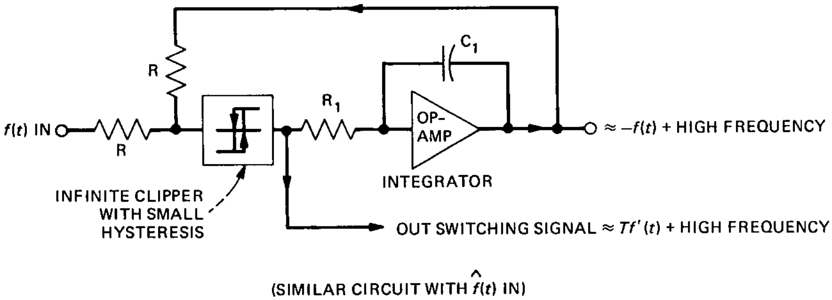


Fig. 7—Delta-mod circuit for deriving switching signals for multipliers in analytic exponential modulator.

Then

$$Z(t) = \exp[-iF(t)] = 1 - iF(t) + \frac{(-iF(t))^2}{2!} \cdots$$

$$= 1 + \frac{1}{2} e^{it} - \frac{1}{8} e^{i2t} + \frac{1}{2!} \left\{\frac{1}{2} e^{it} - \frac{1}{8} e^{2it}\right\}^2$$

$$+ \frac{1}{3!} \left\{\frac{1}{2} e^{it} - \frac{1}{8} e^{2it}\right\}^3 + \cdots$$

$$= 1 + \frac{1}{2} e^{it} - \frac{1}{8} e^{i2t} + \frac{1}{2!} \left(\frac{1}{4} e^{i2t} - \frac{1}{8} e^{i3t} + \frac{1}{64} e^{i4t}\right)$$

$$+ \frac{1}{6} \left(\frac{1}{8} e^{i3t} - \frac{3}{32} e^{i4t} \cdots\right) + \frac{1}{24 \cdot 16} e^{i4t} + \cdots$$

$$= 1 + \frac{1}{2} e^{it} + 0 - \frac{1}{24} e^{i3t} - \frac{1}{192} e^{i4t} + \cdots.$$

Now we bandlimit $Z(t)$ by convolution with $K_{\alpha,\beta}(t)$ to preserve the spectrum in $[0, \alpha]$ and eliminate frequencies above $\beta$. (Here we assume $\alpha = 2.2$ and $\beta = 2.8$.) We thus obtain

$$z(t) = 1 + \frac{1}{2} e^{it}.$$

Clearly, $z(t + iu)$ is zero-free in the upper half-plane, $u \geqslant 0$. Then

$$\log z(t) = \frac{1}{2} e^{it} - \frac{1}{2} \left(\frac{1}{2} e^{it}\right)^2 + \frac{1}{3} \left(\frac{1}{2} e^{it}\right)^3 + \cdots,$$

the first two terms agreeing with

$$\log Z(t) = -iF(t) = \frac{1}{2} e^{it} - \frac{1}{8} e^{i2t},$$

i.e., the Fourier transforms of log $z(t)$ and log $Z(t)$ agree over $(-\infty, \alpha)$.

Now we take $\theta = 0$, $c = 5/2$, and form

$$\begin{aligned} G(t;\, \theta,\, c) &= z(t) + e^{i2ct}\overline{z(t)} \\ &= 1 + \frac{1}{2} e^{it} + e^{i5t}\left(1 + \frac{1}{2} e^{-it}\right) \\ &= 1 + \frac{1}{2} e^{it} + \frac{1}{2} e^{i4t} + e^{i5t}. \end{aligned}$$

Then $G(\tau)$ has only real simple zeros, since $z(\tau)$ is zero-free in the (closed) upper half-plane. We have

$$G(t) = 2e^{i5t/2}s(t),$$

where

$$\begin{aligned} s(t) &= \frac{1}{2} \cos \frac{3t}{2} + \cos \frac{5t}{2} \\ &= 4\left(-\frac{3}{4} + \cos \frac{t}{2}\right)\left(\frac{1}{2} + \cos \frac{t}{2}\right) \cos \frac{t}{2}. \end{aligned}$$

Writing $G(t)$ in factored form we have

$$G(t) = (1 + e^{it}) \prod_{k=1}^{2} (1 - e^{i\theta_k}e^{it})(1 - e^{-i\theta_k}e^{it}),$$

where

$$\begin{aligned} \cos \theta_1 &= 3/4 \\ \cos \theta_2 &= -1/2. \end{aligned}$$

Since $G(\tau)$ and $z(\tau)$ are zero-free in the upper half-plane and the Fourier transforms of $G(t)$ and $z(t)$ agree over $(-\infty, \alpha)$, it follows that the Fourier transforms of log $G(t)$ and log $z(t)$ agree over $(-\infty, \alpha)$. We have

$$\log G(t) = \log(1 + e^{it}) + \sum_{k=1}^{2} \log(1 - e^{i\theta_k}e^{it}) + \sum_{k=1}^{2} \log(1 - e^{-i\theta_k}e^{it})$$

or

$$\log G(t) = e^{it} - \frac{1}{2} e^{i2t} + \frac{1}{3} e^{i3t} - \frac{1}{4} e^{i4t} + \cdots$$

$$- 2(\cos \theta_1 + \cos \theta_2)e^{it} - 2(\cos 2\theta_1 + \cos 2\theta_2) \frac{e^{i2t}}{2}$$

$$- 2(\cos 3\theta_1 + \cos 3\theta_2) \frac{e^{i3t}}{3} + \cdots$$

$$= \frac{1}{2} e^{it} - \frac{1}{8} e^{i2t} + \frac{1}{24} e^{i3t} + \frac{31}{64} e^{i4t} + \frac{121}{160} e^{i5t}$$

$$- \frac{145}{384} e^{i6t} + \frac{169}{7 \cdot 128} e^{i7t} - \frac{449}{8 \cdot 256} e^{i8t} - \frac{1511}{9 \cdot 512} e^{i9t}$$

$$- \frac{1201}{10 \cdot 1024} e^{i10t} + \frac{4489}{11 \cdot 2048} e^{i11t} - \frac{6305}{12 \cdot 4096} e^{i12t} + \cdots.$$

We have defined

$$H(t) = i \log G(t) = h(t) + i\hat{h}(t).$$

Then

$$h(t) = -\mathrm{Im}\{\log G(t)\} = -\frac{1}{2} \sin t + \frac{1}{8} \sin 2t$$

$$- \frac{1}{24} \sin 3t - \frac{31}{64} \sin 4t + \cdots.$$

This is the "meandering" sawtooth function, which increases by $\pi$ at the zeros of $s(t)$. (The waveforms in Fig. 1 correspond to the example here.)

The phase $\phi(t)$ of the analytic bandlimited signal $z(t)$ is

$$\phi(t) = \mathrm{Im}\{\log z(t)\} = \frac{1}{2} \sin t - \frac{1}{8} \sin 2t + \frac{1}{24} \sin 3t + \cdots.$$

We see that the Fourier transforms of $f(t)$, $h(t)$, and $-\phi(t)$ agree over $(-\alpha, \alpha)$. Then the Fourier transforms of $f'(t)$ and

$$h'(t) = \pi \sum_{-\infty}^{\infty} \delta(t - t_k) - c$$

$$\sim -\frac{1}{2} \cos t + \frac{1}{4} \cos 2t - \frac{1}{8} \cos 3t - \frac{31}{16} \cos 4t + \cdots$$

agree over $(-\alpha, \alpha)$, where $\{t_k\}$ are the zeros of $s(t)$. [The Fourier series of $h'(t)$ does not converge, of course.] Thus, if $K(t)$ is the kernel of a bandpass filter that reproduces frequencies between $\lambda$ and $\mu$, rejecting dc and frequencies greater than $\alpha$, we have

$$f'(t) = h'(t) \otimes K(t) = \pi \sum_{-\infty}^{\infty} K(t - t_k)$$

$$= -\frac{1}{2} \cos t + \frac{1}{2} \cos 2t.$$

Writing

$$z(t) = A(t)e^{i\phi(t)}$$

and

$$s(t) = \frac{1}{2} e^{-ict}G(t) = A(t)\cos\{ct - \phi(t)\}$$

we see that $\{t_k\}$ are the zeros of the phase-modulated signal

$$\cos[ct - \phi(t)].$$

Note that $\phi(t)$ is not bandlimited;

$$\phi(t) = \mathrm{Im}\{\log[\mathbf{B}_{\alpha,\beta}e^{-i\{f(t)+i\hat{f}(t)\}}]\},$$

where $\mathbf{B}_{\alpha,\beta}$ is the bandlimiting operator defined by convolution with $K_{\alpha,\beta}(t)$, so that

$$-\phi(t) = f(t) + \epsilon(t),$$

where $\epsilon(t)$ is a high-pass function whose Fourier transform vanishes over $(-\alpha, \alpha)$. Had we taken $\alpha$ and $\beta$ much larger, then $\epsilon(t)$ would have been extremely small, but this would have required a much larger $c$ [$c \geqslant (\alpha + \beta/2)$] in forming $s(t)$, i.e., a much larger pulse rate in obtaining

$$f'(t) = \pi \sum_{-\infty}^{\infty} K(t - t_k).$$

In the example here we have

$$x(t) = \mathrm{Re}\{z(t)\} = 1 + \frac{1}{2} \cos t \geqslant 0$$

and therefore

$$-\pi/2 < \phi(t) < \pi/2,$$

so that the zeros of $s(t)$ and $\sin ct$ interlace. Then [cf. (34)] with $c = 5/2$,

$$q(t) = -\frac{\pi}{2}\{\operatorname{sgn} s(t)\}\cdot\{\operatorname{sgn}\sin ct\} = h(t) - \sigma(t)$$

$$= -\frac{1}{2}\sin t + \frac{1}{8}\sin 2t - \frac{1}{24}\sin 3t - \frac{31}{64}\sin 4t - \frac{121}{160}\sin 5t - \sin 5t + \cdots$$

is a square wave that may be bandlimited to obtain

$$f(t) = -\frac{1}{2}\sin t + \frac{1}{8}\sin 2t.$$

If we replace $f(t)$ by

$$f(t) = -a(\sin t - \frac{1}{4}\sin 2t)$$

we find that

$$z(t) = 1 + ae^{it} + \frac{a}{2}\left(a - \frac{1}{2}\right)e^{i2t}.$$

Then $z(\tau)$ will be zero-free in the upper half-plane for

$$-\frac{(\sqrt{33} - 1)}{4} < a < \frac{\sqrt{33} + 1}{4}$$

$$(-1.186141 < a < 1.68614).$$

However, the real part of $z(t)$ will be positive, i.e., the square wave representation will be valid, only for the reduced range

$$-1.06 < a < 1.38.$$

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## AUTHOR

**Benjamin F. Logan, Jr.**, B.S. (Electrical Engineering), 1946, Texas Technological College; M.S., 1951, Massachusetts Institute of Technology; Eng.D.Sci. (Electrical Engineering), 1965, Columbia University; AT&T Bell Laboratories, 1956—. While at MIT, Mr. Logan was a research assistant in the Research Laboratory of Electronics, investigating characteristics of high-power electrical discharge lamps. Also at MIT he engaged in analog computer development at the Dynamic Analysis and Control Laboratory. From 1955 to 1956 he worked for Hycon-Eastern, Inc., where he was concerned with the design of airborne power supplies. He joined AT&T Bell Laboratories as a member of the Visual and Acoustics Research Department, where he was concerned with the processing of speech signals. Currently, he is a member of the Mathematical Research Department. Member, Sigma Xi, Tau Beta Pi.