

Probabilistic Analysis of Interframe Tie Requirements for Cross-Connect Systems

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Cross-connect system frames provide the capability for remote access and cross-connection of circuits in digital format. Their usefulness has resulted in a rapidly growing number of existing and planned offices with multiple frames. A key problem in the planning of these offices is the determination of the frame capacity that should be reserved for tying purposes. Ties, or connections between frames, are necessary to cross-connect two-point circuits whose segments are terminated on different frames. This paper provides the theoretical basis for the computation of interframe tie requirements. We describe two scenarios for the assignment of circuits to facilities, and facilities to frames. For each case we derive the mean and variance of the number of tied circuits per frame. These quantities can be used to determine the number of ports that must be reserved for ties so that circuits requiring ties between frames will only be blocked with a prescribed small probability.

I. INTRODUCTION

Automated cross-connect systems, engineered to be effectively non-blocking, are often used in special service applications. Legs of circuits are assigned to transmission facilities that are linked to a cross-connect frame via ports. These frames allow remote access to digital circuits for testing purposes and provide cross-connection of legs of a circuit remotely. Multiframe offices, necessary because of the limited capacity

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of an individual frame, require planning so that sufficient capacity is reserved for interframe connections. This paper describes a method to compute the capacity required for this purpose.

One such system is the Digital Access and Cross-Connect System (DACS). For an introduction to DACS, see Refs. 1 through 7. The analysis presented here is applicable to similar cross-connect systems.

Each frame has a fixed capacity of ports. Currently, many offices require more than one frame because of their size, and the need for such multiframe offices is growing rapidly. Multiframe offices may be configured in two different ways: fully interconnected and tandem (see Figs. 1 and 2).

Legs of two-point circuits that terminate on different frames must be cross-connected by ties (special connections between the frames). These connections are either made via facilities linked directly between the frames in the fully interconnected case, or through a tandem frame. Since ties consume additional termination capacity on frames (as well as additional hardware), they should be minimized as much as possible. On the other hand, it is also important for planners to reserve enough tie ports on a frame so that circuit legs are not blocked because of lack of ties.

Legs of a multipoint circuit are connected by means of a multipoint bridge, which is external to the frames, eliminating the need for ties. For the remainder of the paper, we assume that all ties are due to two-point circuits. The analysis is easily extended to treat the case of

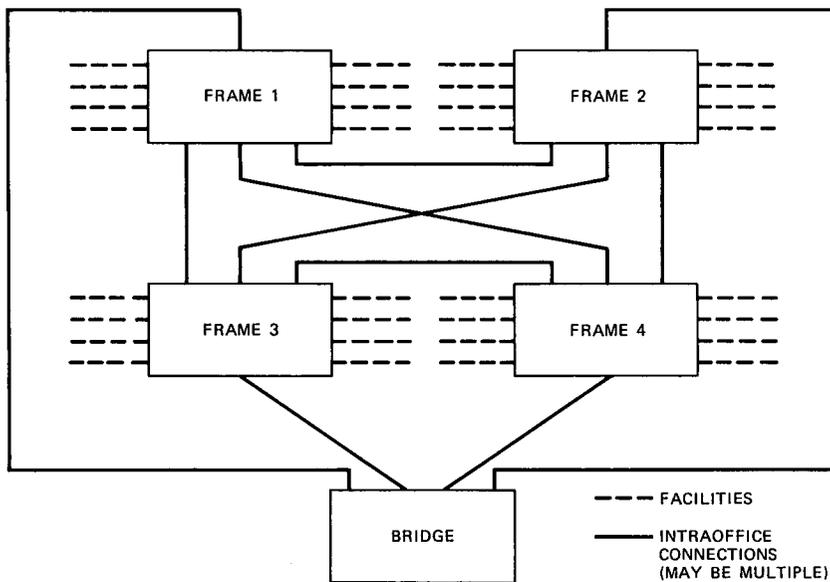


Fig. 1—Fully interconnected office configuration.

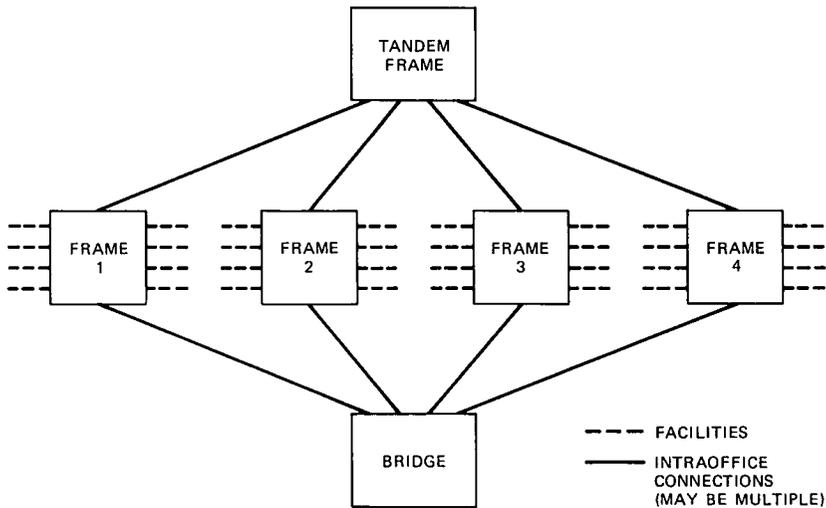


Fig. 2—Tandem office configuration.

multipoint bridges that are inside the frames. When such bridges are available, ties may be necessary to bring the legs of a multipoint circuit to the proper frame for bridging.

This paper describes the theoretical basis for the computation of interframe tie requirements for an office for each of two scenarios. These scenarios assume different procedures for the assignments of circuits to facilities. We have developed a model for an office that incorporates these two scenarios as subcases, and we refer to this model as the "semicontrol model".

1.1 The semicontrol model

In the semicontrol model, destinations are of two types: control destinations, for which the office administrator controls the assignment of circuit legs to facilities; and no-control destinations, for which the assignment of circuit legs to facilities is performed by someone else. It is assumed that circuit legs going to a no-control destination appear at random on facilities with capacity to that destination. This model also assumes that facilities to control destinations are balanced, that is, the numbers of facilities to a given destination that are terminated on different frames are as equal as is possible (they differ by at most one). Each two-point circuit is assumed to connect one control and one no-control destination. Multipoint circuits may connect any combination of destination types. The semicontrol model is consistent with networks of separately administered offices, since the assignment of a circuit leg between two points cannot be controlled by both end offices.

In this paper we analyze two subcases of the semicontrol model. In the first, tie requirements are computed for a mature frame subject to "churn" (circuits are connected and disconnected with time) without the possibility of planned rearrangement. Thus, once a circuit is cross-connected by a tie, it remains in this configuration throughout its lifetime, even if the possibility of a nontied path becomes available later (due to the disconnection of another circuit). In this subcase we develop a stochastic model for tied circuits, and solve for the mean and variance of the number of ties of the equilibrium distribution.

In the second subcase of the semicontrol model, we assume simultaneous assignment of circuit legs to the control facilities after observation of the assignment of circuit legs to the no-control facilities. (We do not use this knowledge in the assignment of facilities to frames.) We compute tie requirements immediately after this assignment—before the effects of churn are felt. This subcase of the model is appropriate immediately after the cutover of a new office (when knowledge of circuits on facilities is not used in assignment of facilities to frames), or immediately after a planned rearrangement of the circuit legs to control destinations within an existing office.

One should note that as time passes, if circuits are not rearranged, the total tie requirement will increase from the value computed assuming the second scenario until it reaches the value computed assuming the first scenario of the model. The speed with which this occurs depends on the value of the disconnect rate (sometimes referred to as churn rate), but the eventual value of the total tie requirement does not.

The present two subcases do not adequately model a third scenario involving a growing office. In this scenario, demand increases, and new frames and facilities are added from time to time. The policies used to select the times for the introduction of new frames and new facilities are crucial in determining tie requirements during growth. It is conceivable that phased introduction of equipment can lead to tie requirements higher than those given by either of the two subcases described earlier. This phenomenon has been verified by simulation work done by P. Soni.⁸ Eventual tie requirements for mature offices without planned rearrangements are still given by the first subcase.

For both of the subcases, we compute the mean and variance of the number of tied circuits (on a frame) associated with each control destination. The mean of total tie requirements per frame can be found by summing the previously described means over the control destinations. The variance of total tie requirements per frame can be found by summing the previously described variances over the control destinations, under the assumption that circuits to different control destinations on a frame behave independently. The number of ports

necessary to guarantee that the tie requirement is met with a given probability is then obtained for the various configurations of frames, assuming that total number of tied circuits is normally distributed. It would also be possible to compute tie requirements by the equivalent-random method,⁹ but the large number of control destinations typically encountered results in satisfactory accuracy of the normal approximation.

The methodology used here assumes that there is a large number of frames within the office. Care must be taken when interpreting the results for small offices. The calculations described herein will overestimate tie requirements for offices of two or three frames. When four or more frames are involved, the tendency to overestimate tie requirements is probably minimal.

The rest of this paper is organized as follows: Section II describes and analyzes the subcase of a mature frame affected by churn without planned rearrangements, and Section III describes and analyzes the subcase of a frame after an initial assignment of circuits. Section IV contains concluding remarks.

II. THE "MATURE" FRAME UNDER CHURN WITHOUT PLANNED REARRANGEMENTS

Here we consider an individual frame within an office and compute the distribution of the number of tied circuits on that frame when the churn assumption is appropriate. We assume the previously described semicontrol model. That is, there are control and no-control destinations, circuit legs to no-control destinations appear at random on facilities to those destinations, two-point circuits connect one no-control and one control destination, and facilities to control destinations are balanced as much as possible on frames. We also assume that there are no planned rearrangements, i.e., once a circuit is cross-connected, it remains in that configuration for the duration of its lifetime.

The lifetime of a circuit is assumed to be exponentially distributed. We consider a "mature" frame, so that the rate of replacement of circuits on a frame is essentially constant and equal to the long-run rate of circuit disconnects (the disconnect rate per circuit multiplied by the average number of circuits on the frame). The arrivals of connect orders having a leg to a given destination are assumed to be given by a Poisson process. We assume that the long-run average of the proportion of circuits to any given destination is constant, as is the proportion of two-point circuits. We also assume that these proportions are maintained on each of the no-control facilities, independent of their destinations. The last assumption will tend to give higher values for the computed number of ties because it ignores the com-

munities of interest that may exist between the no-control and control destinations.

The entire analysis of this section is concerned with circuits that have a leg to a given control destination and appear on a given frame. The mean and variance of the number of tied circuits on a frame due to these circuits will be determined. The mean (variance) of total tie requirements per frame will be found by summing the means (variances) of the tie requirements for all destinations.

We keep track of four types of circuits (see Fig. 3) for the given control destination on the given frame:

1. Two-point circuits cross-connected through the frame—Here the two legs of the circuit are assigned to facilities appearing on the same frame. Obviously, no ties are used for these circuits.

2. Two-point circuits tied out of the frame—Here the circuit has one leg on a no-control facility terminated on the given frame, but is tied to another frame for cross-connection to a facility going to the proper (control) destination. One tie circuit is needed on this frame for this type of circuit.

3. Two-point circuits tied into the frame—Here the circuit has one leg on a no-control facility terminated on another frame, but is tied to the given frame for cross-connection to a facility going to the proper (control) destination. One tie circuit is needed on this frame for this type of circuit.

4. Multipoint legs on the frame—Here a cross-connection appears on the given frame between a circuit going to the multipoint bridge and a circuit on a facility going to the proper (control) destination.

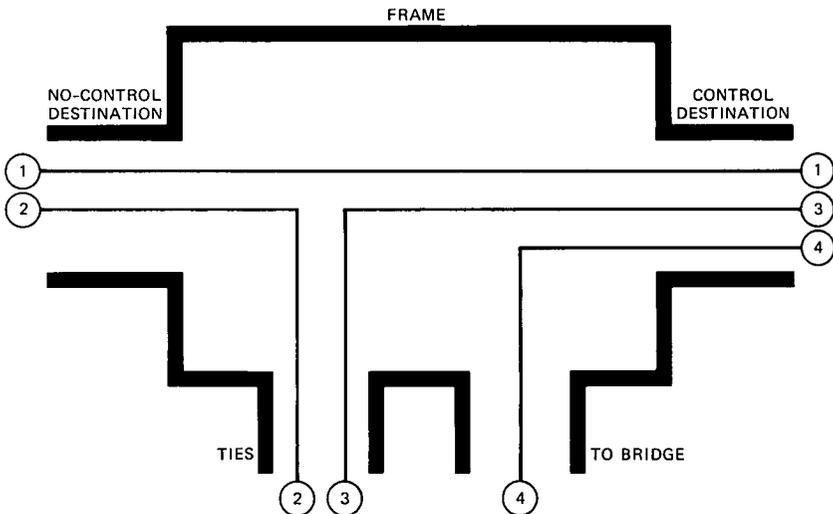


Fig. 3—Circuit types on a given frame.

No ties are needed for circuits of this type, although port capacity is used to connect these circuits to the bridge.

We denote by X_i the number of circuits of type i (described above) to a particular control destination. We show that the evolution of the state vector (X_1, X_2, X_3, X_4) is given by a continuous-time Markov chain that is closely related to the evolution of the state vector for a classical overflow process. The latter has a known solution for the means and variances of its two random variables, which enables us to calculate the mean and variance of the equilibrium tie requirements $(X_2 + X_3)$.

We assume that the total capacity of facilities to the control destination, which are terminated on the given frame, is k circuits. Clearly, $X_1 + X_3 + X_4$ is the number of these circuits in use, which cannot exceed k .

State transitions change one of the components of the state vector by one unit (either up or down). These transitions correspond to the connecting or disconnecting of a circuit of the type represented by that component.

We consider disconnect transitions first. Without loss of generality, we scale time in units of the mean lifetime of a circuit, so that the disconnect rate per circuit is unity. There is a transition corresponding to the disconnection of a circuit of type i at rate X_i for each of the four types of circuits, since the lifetimes are assumed to be exponentially distributed.

Now consider transitions corresponding to the connecting of circuits. There are two kinds of these transitions. The first comprises those two-point circuits that have one leg connected to a no-control destination on the given frame (corresponding to circuits of types 1 and 2). The second comprises those circuits having a leg elsewhere, which we assign to the given frame for cross-connection to the control destination (corresponding to circuits of types 3 and 4).

The first kind corresponds to arrivals of circuit legs on no-control facilities of the given frame that require cross-connection to a particular destination. We assume that these occur according to a Poisson process with rate λ_1 . If there is capacity to the destination ($X_1 + X_3 + X_4 < k$), these circuits are cross-connected through the frame to avoid ties. In this case each such event corresponds to the arrival of a type 1 circuit (X_1 is incremented by 1). If there is no capacity to the destination ($X_1 + X_3 + X_4 = k$), these must be tied out of the frame. In this case each such event corresponds to the arrival of a type 2 circuit (X_2 is incremented by 1).

Now we consider events that cause type 3 circuits to connect. Every time a circuit is tied out of any frame in the office, it must be tied into another frame with available capacity to the destination. (It would

never be tied into a frame without capacity.) If the frame to be tied into is selected without regard to remaining capacity and without preference for a given frame, then the rate of such tied-in circuit connects should be approximately equal for all frames. Furthermore, this rate should be approximately equal for all states on the frame for which there is available capacity, at least if there are many frames in the office.

Accordingly, we assume that the connects of type 3 circuits correspond to a Poisson process with rate λ_2 (still to be determined) independent of the state, provided that capacity exists to the destination. Later in the section, λ_2 will be determined by a self-consistency requirement. For the present, we assume it is known.

In a similar fashion, it can be reasoned that requests for type 4 circuit connects occur at rate λ_3 (still to be determined) roughly independent of frame and state, provided that capacity to the destination is available at the frame.

To sum up, we assume that the evolution of the state vector (X_1, X_2, X_3, X_4) is given by a continuous-time Markov chain with the following transition rates [all from (x_1, x_2, x_3, x_4)]:

<u>Into State</u>	<u>Rate</u>	
$(x_1 - 1, x_2, x_3, x_4)$	x_1	
$(x_1, x_2 - 1, x_3, x_4)$	x_2	
$(x_1, x_2, x_3 - 1, x_4)$	x_3	
$(x_1, x_2, x_3, x_4 - 1)$	x_4	
$(x_1 + 1, x_2, x_3, x_4)$	λ_1	(if $x_1 + x_3 + x_4 < k$)
$(x_1, x_2 + 1, x_3, x_4)$	λ_1	(if $x_1 + x_3 + x_4 = k$)
$(x_1, x_2, x_3 + 1, x_4)$	λ_2	(if $x_1 + x_3 + x_4 < k$)
$(x_1, x_2, x_3, x_4 + 1)$	λ_3	(if $x_1 + x_3 + x_4 < k$).

Our goal is to find the mean and variance of the number of ties $(X_2 + X_3)$ for the ergodic state distribution.

If we introduce another component to the state space, having no effect on the first four components, the resulting ergodic probabilities can be related to the ergodic probabilities of the classical trunk-overflow process. Since the means and variances of the state variables are readily available for the trunk-overflow process,⁹ we are easily able to solve for the mean and variance of the required number of ties for a destination on a frame.

We thus add a dummy component, X_5 , and also add the following transition rates from $(x_1, x_2, x_3, x_4, x_5)$:

<u>Into State</u>	<u>Rate</u>	
$(x_1, x_2, x_3, x_4, x_5 - 1)$	x_5	
$(x_1, x_2, x_3, x_4, x_5 + 1)$	$\lambda_2 + \lambda_3$	(if $x_1 + x_3 + x_4 = k$).

The other transitions are as before (ignoring the value of X_5), with X_5 unchanged by the transitions.

Define $Y_1 = X_1 + X_3 + X_4$, $Y_2 = X_2 + X_5$, and $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. It is readily seen that the vector (Y_1, Y_2) is a continuous-time Markov chain with transition rate from (y_1, y_2) :

Into State	Rate	
$(y_1 - 1, y_2)$	y_1	
$(y_1, y_2 - 1)$	y_2	
$(y_1 + 1, y_2)$	λ	(if $y_1 < k$)
$(y_1, y_2 + 1)$	λ	(if $y_1 = k$).

These are the transition rates for the classical trunk-overflow process. Physically, one may think of offering λ Erlangs of traffic to k primary trunks, with any overflow going to an infinite group of trunks. The first component represents the number of primary trunks occupied, and the second component represents the number of overflow trunks occupied. The moments of the ergodic distribution of these variables are available:⁹

$$E(Y_1 + Y_2) = \lambda, \tag{1}$$

$$\text{var}(Y_1 + Y_2) = \lambda, \tag{2}$$

$$E(Y_1) = \lambda(1 - B(k, \lambda)), \tag{3}$$

$$E(Y_1^2) = (1 + \lambda)E(Y_1) - k\lambda B(k, \lambda), \tag{4}$$

$$E(Y_2) = \lambda B(k, \lambda), \tag{5}$$

and

$$\text{var}(Y_2) = E(Y_2) \left[1 - E(Y_2) + \frac{\lambda}{k + 1 + E(Y_2) - \lambda} \right], \tag{6}$$

where $B(k, \lambda)$ is the classical Erlang blocking formula

$$B(k, \lambda) = \frac{(\lambda^k/k!)}{\sum_{j=0}^k (\lambda^j/j!)} . \tag{7}$$

We now relate the ergodic distribution of $(X_1, X_2, X_3, X_4, X_5)$ to the ergodic distribution of (Y_1, Y_2) . Let $q_i = \lambda_i/\lambda$. Whenever Y_1 is incremented, X_1 is incremented with probability q_1 , X_3 is incremented with probability q_2 , and X_4 is incremented with probability q_3 . Furthermore, departures from X_1 , X_3 , and X_4 are proportional to the values of these state variables. Therefore, given Y_1 , (X_1, X_3, X_4) is distributed in a multinomial distribution corresponding to Y_1 trials with probabilities of each type q_1 , q_2 , and q_3 , respectively.

Similar reasoning tells us that given Y_2 , (X_2, X_5) has a binomial distribution corresponding to Y_2 trials with probabilities of each type q_1 and $1 - q_1$, respectively. Also, given Y_1 and Y_2 , (X_1, X_3, X_4) and (X_2, X_5) are independent.

Thus, the ergodic probability of $(x_1, x_2, x_3, x_4, x_5)$, denoted $\pi(x_1, x_2, x_3, x_4, x_5)$, is related to the ergodic probability of (y_1, y_2) , denoted $\pi^*(y_1, y_2)$, in the following fashion:

$$\pi(x_1, x_2, x_3, x_4, x_5) = \pi^*(x_1 + x_3 + x_4, x_2 + x_5) \cdot \frac{(x_1 + x_3 + x_4)!(x_2 + x_5)!}{x_1!x_2!x_3!x_4!x_5!} q_1^{x_1} q_2^{x_3} q_3^{x_4} q_1^{x_2} (1 - q_1)^{x_5}. \quad (8)$$

Equation (8) may also be checked by direct substitution into the balance equations, but we omit the details here. The facts stated previously, which allowed us to write equation (8), are sufficient to extract moments of the number of circuits tied. We proceed by conditioning on the values of Y_1 and Y_2 :

$$E(X_2 + X_3 | Y_1, Y_2) = q_1 Y_2 + q_2 Y_1 \quad (9)$$

and

$$E((X_2 + X_3)^2 | Y_1, Y_2) = q_1^2 Y_2^2 + q_1(1 - q_1) Y_2 + 2q_1 q_2 Y_1 Y_2 + q_2^2 Y_1^2 + q_2(1 - q_2) Y_1. \quad (10)$$

Taking expectations with respect to Y_1 and Y_2 in (9) and (10), we obtain (after some algebra)

$$E(X_2 + X_3) = q_1 E(Y_2) + q_2 E(Y_1) \quad (11)$$

and

$$\begin{aligned} \text{var}(X_2 + X_3) &= q_1 q_2 \text{var}(Y_1 + Y_2) + q_2(q_2 - q_1) \text{var}(Y_1) \\ &+ q_1(q_1 - q_2) \text{var}(Y_2) + q_2(1 - q_2) E(Y_1) + q_1(1 - q_1) E(Y_2). \end{aligned} \quad (12)$$

Equations (11) and (12), when combined with eqs. (1) through (6), give the mean and variance of the number of tied circuits to a given control destination on a given frame. These expressions depend implicitly on q_1 and q_2 , which depend on λ_2 and λ_3 . We will determine q_1 and q_2 in Section 2.1.

2.1 Determination of λ , q_1 , and q_2

We have so far analyzed a single frame with available capacity of k circuits to the particular destination considered. It is possible that not all frames in an office have the same number of available circuits to the destination (since the total number of facilities may not be a multiple of the number of frames), and these considerations affect the

determination of λ_2 and λ_3 . We will actually find the derived quantities λ , q_1 , and q_2 . Suppose that there are m types of frames (in terms of circuits to the destination) represented in the office, with available number of circuits to the destination of k_1, k_2, \dots, k_m (m will usually be two). Suppose that the proportions of the frames in the office that are of these types are p_1, p_2, \dots, p_m , respectively. We have argued previously that the values of λ_2 and λ_3 are identical for all the frames, since these represent the rates of arrival of circuits that are selected to be connected to any frame in the office (having capacity) with equal probability.

Denote the expected number of circuit legs to the control destination per frame (including two-point and multipoint circuits) by T . (This is known.) The (random) number of outgoing circuit legs from a frame to the destination is given by Y_1 . Consistency in the average number of legs to the destination per frame demands that the average value of the expectation of Y_1 equals T , or

$$\sum_{i=1}^m p_i[\lambda(1 - B(k_i, \lambda))] = T, \quad (13)$$

where use has been made of (3). The left-hand side of (13) is increasing in λ , is continuous, and takes all values in the range $[0, c]$ for $\lambda \geq 0$, where $c = \sum_{i=1}^m p_i k_i$ is the average capacity to the destination on a frame basis. Therefore, (13) has a unique solution for λ as a function of T in the above quoted range.

We also require, for consistency, that the total number of tied-in circuits in the entire office is equal to the number of tied-out circuits. Recall that X_2 represents tied-out circuits and X_3 represents tied-in circuits, and their expectations are $q_1 E(Y_2)$ and $q_2 E(Y_1)$, respectively, where $E(Y_1)$ and $E(Y_2)$ are given in (3) and (5). This consistency requires that the averages of these values over all frames be equal, or

$$q_1 \sum_{i=1}^m p_i B(k_i, \lambda) = q_2 \sum_{i=1}^m p_i (1 - B(k_i, \lambda)), \quad (14)$$

where a common factor of λ has been eliminated. Use of (13) in (14) with the identity

$$q_1 = \frac{\lambda_1}{\lambda} \quad (15)$$

yields

$$q_2 = \frac{\lambda_1}{T} - \frac{\lambda_1}{\lambda}. \quad (16)$$

2.2 Computation of the mean and variance of total ties per frame

Equations (14), (15), and (16) allow one to compute the parameters q_1 and q_2 used in eqs. (11) and (12), which give the mean and variance of the total number of tied circuits on a particular frame. Equations (11) and (12) depend on the value of k , the number of circuits to the destination on the frame. When the value of k to a particular destination is not constant over all frames, averaging over the values is necessary. Although the rationale for averaging the mean number of ties is clear, some explanation is necessary for the averaging of the variance (as opposed to taking the average of the second moment minus the square of the average first moment).

The ultimate goal of this process is to obtain a number representing the reserved capacity for tied circuits on a frame, where reserved capacity is sufficient to meet tie requirements with a given (user-specified) probability. It can be argued that if the number of control destinations is large, circuits to each of the destinations behave roughly independently. Now choose a particular frame, which has ℓ_i circuits to control destination i , $i = 1, 2, \dots, n$, where n is the total number of control destinations. If n is sufficiently large, the total tie requirements for this frame will be approximately normally distributed. The mean is equal to the sum of the means calculated for each destination (given the number of circuits to that destination on the frame). The variance is equal to the sum of the variances calculated for each destination (given the number of circuits to that destination on the frame).

Although the vector $(\ell_1, \ell_2, \dots, \ell_n)$ will vary from frame to frame, the components are definitely not independent, since the total outgoing capacity from a frame will be more or less fixed. In other words, a frame having a greater than average number of facilities to one destination will tend to have lower than average numbers of facilities to other destinations.

If all destinations have identical characteristics (in terms of the number of frames with given number of circuits to the destination), then the proportion of ℓ_i on a given frame that are equal to a particular value will be very close to the overall proportion (over all frames) of ℓ_i that are equal to that value. In this case, the total variance on the frame will be nearly equal to the sum of the average variances (conditioning on the number of circuits) to each destination.

If all destinations are not identical, there will still be a tendency for the variance of tied circuits to be close to the sum of the average variances over the control destinations.

For this reason it is reasonable to compute the variance of the total number of ties per frame as the sum of the average variance per destination. The latter is equal to the sum of the product of p_i with the value of (12) when k_i replaces k .

2.3 The mean ties for a destination per frame

We obtain a simple expression for the average number of ties per frame.

Equations (11), (3), and (5) give

$$E(X_2 + X_3) = q_1\lambda B(k, \lambda) + q_2\lambda(1 - B(k, \lambda)). \quad (17)$$

The average number of ties per frame, which we denote v , is thus

$$v = \sum_{i=1}^m p_i [q_1\lambda B(k_i, \lambda) + q_2\lambda(1 - B(k_i, \lambda))]. \quad (18)$$

Use of (13), (15), and (16) gives

$$v = 2\lambda_1(1 - T/\lambda). \quad (19)$$

Note that λ_1 is the expected number of two-point circuits per frame, so that $2\lambda_1$ is the total ties if all circuits are tied. (Each tied circuit results in one tie in and one tie out.) The factor $(1 - T/\lambda)$ is the proportion of two-point circuits that are tied, where λ is determined by the implicit solution to (13). As T increases, it can be shown that $1 - T/\lambda$ increases. As T approaches c [facility fill goes to one, see definition following (13)], $1 - T/\lambda$ goes to 1.

As stated earlier, the mean total number of ties required per frame can be found by summing expected ties per destination as given by (19) over all destinations.

2.4 Graphical representation of mean tie requirements

It is possible to summarize the mean number of ties needed for a specific destination per frame in several families of graphs, each of which corresponds to a type of facility. These graphs give a value of v from (19) when λ has been determined from (13) based on four inputs. These are

- j = number of circuits per facility (typically $j = 12$ or 24),
- f = average fill on these facilities,
- a = average number of facilities (to the destination) terminated per frame, and
- r = proportion of circuit legs to that destination that are from two-point circuits.

Figure 4 illustrates one set of these graphs, corresponding to analog group facilities ($j = 12$). The average number of tied circuits per frame is plotted versus the average number of facilities per frame (a) for fills (f) of 0.70, 0.75, 0.80, 0.85, 0.90, and 0.95. These curves assume that all circuits are two-point ($r = 1$). If this is not the case, expected ties obtained from the graph should be multiplied by the value of r . The

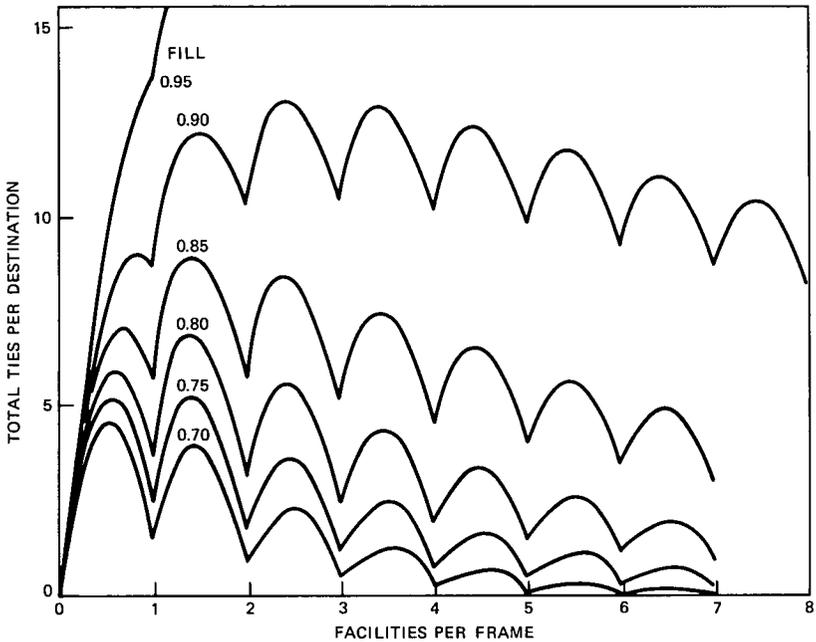


Fig. 4—Ties versus facilities for analog groups; churn model (12 circuits per facility).

“scaloped” nature of the curves is due to the relatively higher mean tie requirement when the mean number of facilities per frame is not integer, forcing uneven spreading of these facilities on the frames.

We now describe the method used to obtain the curves. If facilities are spread as evenly as possible, then $m = 2$, and

$$\left. \begin{aligned} T &= afj, \\ p_1 &= [a] - a, \\ p_2 &= 1 - p_1, \\ k_1 &= ([a] - 1)j, \\ \text{and } k_2 &= [a]j \end{aligned} \right\} \quad (20)$$

Here $[x]$ represents the smallest integer greater than or equal to x . We can now determine λ by numerical solution of (13) and v , the average number of ties, can be obtained from (19).

III. SIMULTANEOUS ASSIGNMENT OF CIRCUITS

In this section we consider the second subcase of the semicontrol model in which circuits (and ties) are assigned to control facilities after assignment of all facilities to frames and after observation of the

assignment of the no-control legs of the circuits. This scenario excludes the possibility that circuits will be both tied in and tied out to the same destination on the same frame. Here again (as in the last section), there are two types of ties on a frame—those due to circuits tied out and those due to circuits tied in.

We first obtain the distribution of the number of circuits tied out of a frame. By assumption, we know that the distribution of the number of no-control legs of two-point circuits on a frame that require cross-connection to a particular control destination is a Poisson distribution. Ties out are required if this number exceeds the available number of circuits to the destination on the frame. In this case the number required is exactly equal to the excess.

The distribution of the number of tied-in circuits on a frame depends on the methodology used to assign tied-in circuits. The total number of tied-in circuits in an office is equal to the number of tied-out circuits in the same office (this can be determined by the method described in the last paragraph). The specific way that individual frames are assigned these tied-in circuits is up to the office administrator. Of course, the number assigned to any particular frame cannot exceed the remaining capacity (after satisfying through-connects on the frame to the destination). Thus, circuits will never be tied into and tied out of the same frame (unlike the churn model of Section II).

Here we compute distributions of tied-in circuits under each of two assignment policies. In the first, tied-in circuits are assigned to minimize the variance of ties on a frame. In the second, tied-in circuits are assigned to maximize the variance of tied circuits on a frame. The resulting variances provide lower and upper bounds for the variance of any assignment policy.

We obtain first and second moments for the number of circuits tied in and tied out (for a given control destination) per frame. The second (first) moment of the total number of ties per destination per frame equals the sum of the second (first) moments of the number of circuits tied in and the number of circuits tied out. The second moment is additive since circuits are never tied into and tied out of the same frame in the simultaneous assignment case. (This can, however, occur in the churn model of the previous section.)

3.1 Moments of circuits tied out of a frame

Let X be the number of legs of two-point circuits incident on a frame that must be cross-connected to a given control destination, let k be the capacity in circuits to the controlled destination on a frame, let Y be the number of circuits tied out to this destination on the frame, and let Z be the remaining capacity available for tied-in circuits (or multipoints) on the frame. Then,

$$Y = (X - k)^+, \quad (21)$$

and

$$Z = (k - X)^+, \quad (22)$$

where $a^+ = \max[a, 0]$.

Under the assumption that X is a Poisson random variable with mean λ , i.e.,

$$P\{X = i\} = (\lambda^i/i!)e^{-\lambda}, \quad (23)$$

we can compute the first two moments of Y :

$$E(Y) = \lambda S_k - k S_{k+1}, \quad (24)$$

and

$$E(Y^2) = \lambda^2 S_{k-1} + (1 - 2k)\lambda S_k + k^2 S_{k+1}, \quad (25)$$

where

$$S_k = \sum_{j=k}^{\infty} (\lambda^j/j!)e^{-\lambda}. \quad (26)$$

That is, S_k is the tail sum of the Poisson probability distribution.

A good approximation for S_k for large values of k is given by the normal approximation

$$S_k \approx \bar{\Phi} \left(\frac{k - 1/2 - \lambda}{\sqrt{\lambda}} \right), \quad (27)$$

where $\bar{\Phi}$ represents the complement of the standard normal c.d.f.; that is,

$$\bar{\Phi}(y) = \int_y^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx. \quad (28)$$

When the value of k varies, i.e., a proportion p_i of the frames has k_i circuits to the destination for $i = 1, 2, \dots, m$, then the appropriate average moments are found by taking the expectations of (24) and (25) with respect to the distribution p .

3.2 Moments of the number of tied-in circuits

We first find the distribution of the available capacity to the destination after the assignment of circuits cross-connected through the frame, but before circuits are tied in (the random variable Z). We obtain

$$P\{Z = i\} = (\lambda^{k-i}/(k-i)!)e^{-\lambda}, \quad 0 < i \leq k; \quad (29)$$

$$P\{Z = i\} = 0, \quad i > k; \quad (30)$$

and

$$P\{Z = 0\} = 1 - \sum_{i=1}^k P\{Z = i\}. \quad (31)$$

Equations (29) through (31) are based on a given frame with k available circuits to the destination. Let α_i be the overall probability (considering all frames with possibly different values of k) that there is a remaining capacity of i circuits to the destination after the assignment of circuits cross-connected through the frame, but before circuits are tied in. We can compute α_i , $i > 0$, by conditioning on the value of k

$$\alpha_i = e^{-\lambda} \sum_{j:k_j \geq i} p_j \lambda^{(k_j-i)} / (k_j - i)!, \quad (32)$$

where p_j is the proportion of frames in the office with capacity k_j circuits to the control destination under consideration. The maximum subscript i for which $\alpha_i > 0$ is $\max \{k_j\}$.

We next compute the average number of circuits that must be tied in per frame, which we denote by B . This must equal the average number of circuits tied out per frame. Using (24) for various k values yields

$$B = \sum_{j=1}^m p_j (\lambda S_{k_j} - k_j S_{k_j+1}). \quad (33)$$

Now assume that there is a large number of frames. It is possible to have a proportion θ_i of frames with i tied-in circuits to the destination under consideration $\forall i$ if and only if

$$\sum_{i=1}^{\infty} i \theta_i = B \quad (34)$$

and

$$\sum_{i=j}^{\infty} \theta_i \leq \sum_{i=j}^{\infty} \alpha_i, \quad \forall j \geq 0. \quad (35)$$

Equation (34) follows from the fact that the expected number of circuits tied in equals the expected number tied out. Equation (35) states that the proportion of frames with i or more circuits tied in has to be no greater than the proportion of frames with capacity for i or more tied-in circuits.

We now determine the values of θ_i that give minimum and maximum values for the second moment (denoted by V) subject to the constraints (34) and (35), where

$$V = \sum_{i=1}^{\infty} i^2 \Theta_i. \quad (36)$$

3.3 Minimum second moment

Choose ℓ such that

$$\sum_{i=1}^{\ell} i\alpha_i + \sum_{i=\ell+1}^{\infty} \ell\alpha_i < B \leq \sum_{i=1}^{\ell+1} i\alpha_i + \sum_{i=\ell+2}^{\infty} (\ell+1)\alpha_i. \quad (37)$$

If $B \leq \sum_{i=1}^{\infty} i\alpha_i$, i.e., if there is average capacity to handle tied-in requirements, then this is always possible. This follows because the left-hand side of (37) is a nondecreasing function of ℓ , with a value of 0 for $\ell = 0$, and a value of $\sum_{i=1}^{\infty} i\alpha_i$ for $\ell = \infty$. Note that ℓ is uniquely determined. Define

$$\rho = B - \sum_{i=1}^{\ell} i\alpha_i - \sum_{i=\ell+1}^{\infty} \ell\alpha_i. \quad (38)$$

Now, the minimal value of V is produced by setting

$$\Theta_i = \alpha_i, \quad i < \ell; \quad (39)$$

$$\Theta_{\ell} = \sum_{i=\ell}^{\infty} \alpha_i - \rho; \quad (40)$$

$$\Theta_{\ell+1} = \rho; \quad (41)$$

and

$$\Theta_i = 0, \quad i > \ell + 1. \quad (42)$$

Physically, this distribution is obtained when one assigns tied-in circuits sequentially in the following fashion: Assign the next tied-in circuit to a frame with available capacity having the fewest tied-in circuits already assigned.

Clearly, Θ_i forms a valid probability distribution, since (37) implies that Θ_{ℓ} and $\Theta_{\ell+1}$ are nonnegative, and also since the sum of the Θ_i 's is one. This distribution is also easily seen to satisfy (34) and (35). As a matter of fact, (35) is tight for all $j \leq \ell$.

This last property uniquely specifies Θ , i.e., a probability distribution $\hat{\Theta}$ satisfying (34) and (35) with the property that, for every m , if

$$\hat{\Theta}_{m+1} > 0, \quad \text{then} \quad \sum_{i=j}^{\infty} \hat{\Theta}_i = \sum_{i=j}^{\infty} \alpha_i, \quad \forall j \leq m \quad (43)$$

must be the distribution Θ given in (39) through (42).

Proof: Choose $n = \sup\{m: \hat{\Theta}_{m+1} > 0\}$. The condition of the last paragraph tells us that $\hat{\Theta}_j = \alpha_j$, $\forall j < n$. The normalization of $\hat{\Theta}$ and its satisfying (34) uniquely determine $\hat{\Theta}_n$ and $\hat{\Theta}_{n+1}$. Also,

$$B = \sum_{i=1}^{n+1} i\hat{\theta}_i = \sum_{i=1}^{n-1} i\alpha_i + n\hat{\theta}_n + (n+1)\hat{\theta}_{n+1}.$$

However,

$$n\hat{\theta}_n + (n+1)\hat{\theta}_{n+1} > n \sum_{i=n}^{\infty} \alpha_i,$$

and

$$n\hat{\theta}_n + (n+1)\hat{\theta}_{n+1} \leq (n+1) \sum_{i=n}^{\infty} \alpha_i \leq n\alpha_n + \sum_{i=n+1}^{\infty} (n+1) \alpha_i.$$

Comparison with (37) shows that $n = \ell$ and we are done. \square

We use this fact to show that the distribution Θ has minimum second moment over all distributions satisfying (34) and (35). Choose $\Theta' \neq \Theta$, which satisfies the constraints; Θ' must violate (43). Thus, we can find m and $j \leq m$ with $\Theta'_{m+1} > 0$ and $\sum_{i=j}^{\infty} \Theta'_i < \sum_{i=j}^{\infty} \alpha_i$. Choose the largest such m . Since both Θ' and α sum to one, we can find $k < j$ with $\Theta'_k > 0$. Choose the largest such k . The new distribution Θ'' with

$$\Theta''_k = \Theta'_k - \frac{\epsilon}{j-k},$$

$$\Theta''_j = \Theta'_j + \frac{\epsilon}{j-k},$$

$$\Theta''_m = \Theta'_m + \epsilon,$$

and

$$\Theta''_{m+1} = \Theta'_{m+1} - \epsilon,$$

with $\epsilon = \min[\Theta'_{m+1}, (j-k)\Theta'_k, (j-k)(\sum_{i=j}^{\infty} (\alpha_i - \Theta'_i))] > 0$, and other components identical to Θ' , is still a probability distribution satisfying (34) and (35) and

$$\sum_{i=1}^{\infty} i^2 \Theta''_i = \sum_{i=1}^{\infty} i^2 \Theta'_i + \epsilon(j+k-2m-1),$$

or

$$\sum_{i=1}^{\infty} i^2 \Theta''_i < \sum_{i=1}^{\infty} i^2 \Theta'_i.$$

With a series of such transformations, Θ' can be transformed into Θ . \square

3.4 Maximum second moment

We proceed in a similar fashion. In this case choose ℓ such that

$$\sum_{i=\ell+1}^{\infty} i\alpha_i < B \leq \sum_{i=\ell}^{\infty} i\alpha_i. \quad (44)$$

Define

$$\rho^* = B - \sum_{i=\ell+1}^{\infty} i\alpha_i. \quad (45)$$

The maximum value of V is produced by setting

$$\theta_i = 0, \quad 0 < i < \ell; \quad (46)$$

$$\theta_{\ell} = \rho^*/\ell; \quad (47)$$

$$\theta_i = \alpha_i, \quad i > \ell; \quad (48)$$

and

$$\theta_0 = 1 - \sum_{i=1}^{\infty} \theta_i. \quad (49)$$

Physically, this distribution is obtained when one assigns tied-in circuits in the following sequential fashion: Completely fill the frame with maximum available capacity before assigning to another frame. Here we omit the proof, which is similar to that of the last subsection.

IV. CONCLUDING REMARKS

This paper describes the assumptions of a semicontrol model for facility and circuit assignment in a multiframe office. We compute the mean and variance of the number of ties per frame for each of two subcases. The first scenario assumes that we treat a mature frame in a churn environment without planned rearrangements. The second one assumes a simultaneous assignment of circuits, and is appropriate following the cutover of a new office or a wholesale rearrangement of existing circuits within an office. These computations are incorporated as subroutines in the computer program DACSPET (DACS Planning and Engineering Tool), which computes frame requirements and resulting port usages for a multiframe office.

4.1 Preliminary insights

The previously described computer program has been run on an assortment of sample data. These runs provide preliminary insights into the question of tie requirements for multiframe offices. We summarize these insights below.

1. The effect of fill is clear in all cases. Higher facility fills imply more ties per frame. The reason for this is simply that higher fills imply less spare capacity on fewer facilities, and therefore less flexibility in assigning circuits.

2. Tie requirements under the churn scenario are larger than under the simultaneous circuit assignment scenario. The effect is relatively minor at low facility fills, although it increases rapidly as the fill does.

3. The size of an office, in terms of number of frames or circuits, seems to have less of an effect on tie requirements per frame than other variables, such as the number of control destinations served and the proportion of traffic to each destination. An office with many destinations, each with a small proportion of traffic, will clearly incur more tied circuits than an office with a few large destinations under the semicontrol model.

4. The ratio of two-point and multipoint circuits also affects the number of tied circuits per frame under the assumption of an external multipoint bridge. For the same overall facility fill, the number of tied circuits is roughly proportional to the percentage of two-point circuits, all other factors being equal.

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REFERENCES

1. J. R. Colton, "Cross-Connections—DACs Makes Them Digital," *Bell Lab. Rec.*, 58, No. 8 (September 1980), pp. 248–55.
2. R. C. Drechsler, "DACs Cross-Connects—And That's Just the Beginning," *Bell Lab. Rec.*, 58, No. 9 (October 1980), pp. 305–11.
3. J. R. Colton and A. J. Osofsky, "DACs Features and Applications," *Nat. Telecommun. Conf. Rec.*, 1 (November 1981).
4. R. P. Abbot and D. C. Koehler, "Digital Access and Cross-Connect System Architecture," *Nat. Telecommun. Conf. Rec.*, 1 (November 1981).
5. A. J. Cirillo, L. F. Horney, and J. D. Moore, "DACs Microprocessor System," *Nat. Telecommun. Conf. Rec.*, 1 (November 1981).
6. L. C. Sweeney, "DACs in an Associated Company," *Nat. Telecommun. Conf. Rec.*, 1 (November 1981).
7. G. P. Ashkar, G. A. Ford, and T. Pecsavaradi, "Reshaping the Network for Special Services," *Bell Lab. Rec.*, 61 (September 1983), pp. 4–10.
8. P. Soni, private communication.
9. R. B. Cooper, *Introduction to Queuing Theory*, New York: MacMillan, 1972.

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