AT&T Bell Laboratories Technical Journal
Vol. 63, No. 8, October 1984
Printed in U.S.A.

Contrasting Performance of Faster Binary Signaling With QAM

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(Manuscript received November 15, 1983)

In this paper we determine the performance of Faster Binary Signaling (FBS), an alternative method to Quadrature Amplitude Modulation (QAM) for achieving a high bit rate over an ideal, bandlimited, noisy channel. With this method, signaling is faster than the Nyquist rate. Consequently, there are fewer points in the signal constellation, resulting in a greater separation of the points when the average transmitter power is the same as for QAM. Thus, at the expense of introducing Intersymbol Interference (ISI), there is an apparent improvement in noise immunity. The ISI can be mitigated with maximum likelihood sequence detection. We explore the advisability of trading freedom from ISI for added noise immunity for the extreme case where the system with faster signaling uses a four-point constellation. The question of the efficacy of FBS has been difficult to approach, but FBS has loomed as a possibly strong competitor among alternatives to QAM. We show here how to analyze FBS, and we give examples involving FBS operating at up to five times the QAM rate. In the examples, FBS is revealed to be, at best, of marginal value even if one allows for implementation capabilities far beyond those of forthcoming processors.

I. INTRODUCTION

For data communication over channels such as voiceband analog telephone circuits, satellite links, or terrestrial digital radio hops, there is a search for practical techniques that are more efficient (in bits per cycle) than QAM. This search is intensifying because of the growing

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demand for data communication services over bandlimited channels and because of the continuing drop in the cost of high-speed processing required by advanced communication methods.

The bandlimited channel with additive white Gaussian noise (power $\mathscr{N}$), where the average transmitter power $\mathscr{P}(\mathscr{P} \gg \mathscr{N})$ is constrained, serves as a proving ground for theoretical explorations of the relative efficacy of proposed techniques. Specific methods have been discussed as candidates for improving efficiency. Three candidates are Higher-Dimensional Constellations (HDCs),[1] Ungerbock's Trellis Coding (UTC),[2] and Faster Binary Signaling (FBS).[3] Both HDC and UTC have lent themselves to analysis and their significant value over the QAM method has been established. Moreover, we are beginning to understand the relative value of UTC over HDC.[4] On the other hand, the effectiveness of FBS has hitherto remained a mystery. In Ref. 5* some theoretical results on FBS for some special pulses are presented but the relative effectiveness issue is not settled.

To understand the FBS method, consider the elementary QAM method, 4-PSK (phase-shift keying), in a situation comfortably meeting a stringent probability of error (P_e) constraint. If the bit rate requirement increases it can be met without expanding bandwidth by increasing the number of points in the QAM constellation. FBS is a natural alternative means of transmitting at higher bit rates. In FBS, one fixes the constellation at four points and increases the symbol rate as much as necessary. Maximum Likelihood Sequence Detection (MLSD) is employed to overcome the consequent Intersymbol Interference (ISI) in the best way possible (see Refs. 6 through 8 for treatments of MLSD). The minimum separation between distinct points in the planar FBS constellation is greater than for the QAM constellation. One might say FBS trades freedom from ISI for added noise immunity and then MLSD is used to mitigate the ISI.

With the efficacy of FBS unknown, it looms as a possibly competitive technique. Here we help the process of evaluating the field of candidate methods for moving beyond the capabilities of QAM by showing how to analyze FBS. We show by examples that FBS is, at best, of marginal value relative to QAM, even if one allows for implementation capabilities far beyond those of forthcoming processors. Specifically, we allow for complexity of up to 10^8 states. Given the strides in processor technology in the last few decades and the imminent hardware advances, a number like 10^8 is chosen to avoid outdating of this paper for a long time. Such prudence is needed. Indeed, the analysis that follows leaves open the possibility that FBS

* The terminology "faster binary signaling" is not used in [A] and [M]. In [M] the term "faster than Nyquist signaling" is used.

would offer substantial improvement over QAM if complexity were not a consideration. Moreover, one must also consider that fast detectors could go far beyond conventional MLSD in processing efficiency.[9]

II. SYSTEM DESCRIPTION

2.1 Transmission medium and its use

The data transmission medium is represented here in the simplest idealized form as a lossless characteristic with additive white Gaussian noise. The channelization is shown in Fig. 1. On each channel the transmitted signal is subject to an average power constraint and it is assumed that, for all the systems that we discuss, the average transmitted power is much greater than the noise power. [The ramification of this high signal-to-noise ratio (s/n) assumption is discussed in Section III.] In the analysis that follows, we assume that the channels are isolated from each other in that it is not permitted to mitigate Adjacent Channel Interference (ACI) through some elaborate scheme requiring coordination among channels. It is required that R bits per second be transmitted over the channel. Whatever the form of modulation used, soft maximum likelihood sequence detection is employed at the receiver.[10]

2.2 Comments about the benchmark QAM method

It is because of the current prominence of QAM in applications that the work here is presented with QAM as the benchmark method. Since a flat channel transfer characteristic is assumed, it is trivial to relate results to the equivalent baseband channel representing a QAM rail. We use $M = m^2$ to denote the number of points in the QAM constellation. So, constellations with 16, 64, 256, and 1024 points correspond to FBS operating at 2, 3, 4, and 5 times the conventional rate. We elected to work with square QAM constellations even though certain departures from such constellations yield superior performance.[11] The reason for our choice is that we want to analyze FBS in isolation and the aforementioned departures can be viewed as the first step in using the HDC method. Finally, we employ the harmless expediency of dealing with M as if it is a continuous variable in our calculations and in some of our graphs. When M is a positive integer that is not a

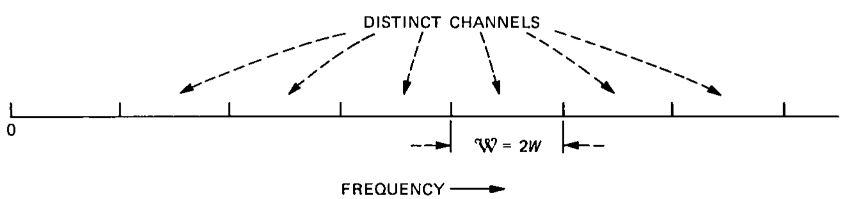

Fig. 1—Adjacent passband channels of bandwidth $\mathcal{W} = 2W$.

perfect square, one can find a two-dimensional constellation that realizes the situation covered by the analysis.

Throughout we will assume that the standard QAM method is meeting a P_e requirement (10^{-3} or less). When we compare alternatives to the QAM method, we will associate primed variables with parameters of the non-QAM system where needed to avoid ambiguity.

2.3 FBS

The generic system structure depicted in Fig. 2 is interpreted here for the special case of FBS. For FBS the binary data are blocked into successive 2-bit words. A pair of independent, synchronous, delta function streams are formed using the two bits to randomly sign the pair of delta functions that are input to the pair of baseband filters. The baseband filters nominally cut off at W cycles/second. On each rail the pulse rate is $r = R/2$. For convenience we use $T' = 1/r$ to denote the time interval between impulses. The filter outputs combine to form the in-phase and quadrature rails of a passband signal. Thus, it may seem that what we have is 4 PSK; but FBS is unusual in that its symbol rate, $1/T'$, is higher than the conventional $1/T \cong 2W$.

The higher rate does not increase bandwidth but it does cause ISI, which will be combatted with MLSD. The ISI is assumed to involve a

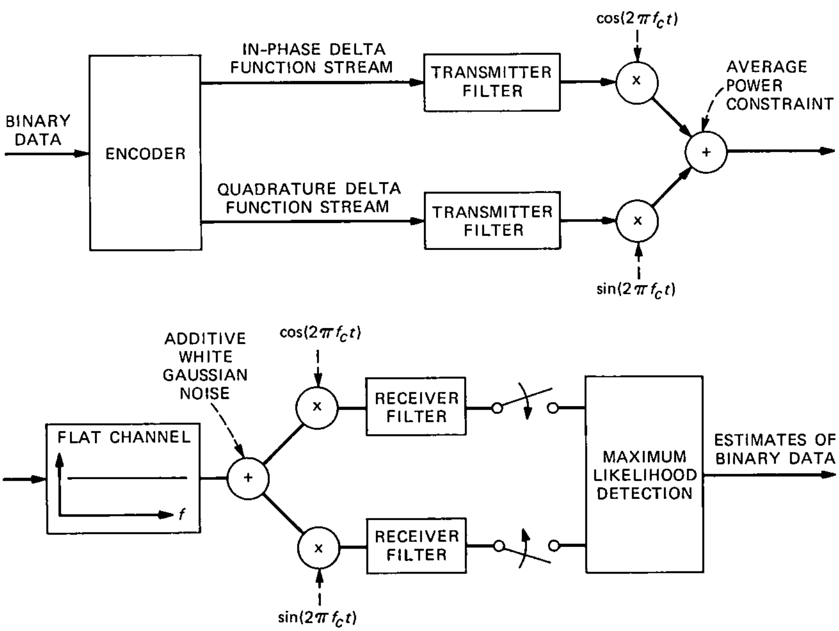

Fig. 2—Generic system structure. For QAM the transmit and receive filters are low-pass filters. For the FBS case they are a matched pair with finite memory in the sampled data domain. The nominal cutoff frequency of the low-pass filters is $W = \mathscr{W}/2$.

memory of ν, i.e., the bandlimited baseband filters are such that the impulse response sampled at times $\{n/T'\}_{n=-\infty}^{\infty}$ have the form $h = \bar{0}, h_0, h_1, \cdots h_\nu, \bar{0}$. Thus MLSD requires $2^{\nu+1}$ states (2^ν per rail).*

We would like to, if we could, make the following idealizations concerning $H(f)$, the Fourier transform of the impulse response of the baseband filter.

A. Each member of a bank of FBS passband systems is spectrally disjoint [in baseband, $H(f)$ vanishes outside $(-W, W)$].

B. $H(f)$ is a trigonometric polynomial of degree ν on $-W' \leq f \leq W'$ $[W' = 1/(2T') > W]$.

The first of the above is needed for spectral efficiency. Statement B is needed to be consistent with the assumption that MLSD involves a memory of ν. Mathematically it would seem impossible to meet A and B. After all, if $H(f)$ is a trigonometric polynomial vanishing on $[W, W']$, it vanishes everywhere. There is no real difficulty here. We will adhere to B with ν the degree of $H(f)$. While A will not strictly hold, one can get as close to ideal as desired as long as ν is large enough to meet out-of-band energy constraints. In MLSD a matched filter is used to initiate the detection process. The matched filter receiver serves to select the desired band $[-W, W]$. We have a dual view of what the frequency band is. From the point of view of where the signal power is concentrated, $[-W, W]$ is the band. From the point of view of MLSD, we are dealing with a sampled temporal response, which can only correspond to a transform that is a polynomial on $[-W', W']$. So long as the degree of the polynomial is large enough the two views of bandwidth can be reconciled.

There are several questions before us. Can we make the energy outside the $[-W, W]$ band so small that the interference between neighboring systems is negligible, yet the number of states involved in MLSD is reasonable? If we can accomplish this, does the FBS system perform better than the comparable QAM system? How much better does it perform and at what complexity?

We investigate these questions in the context of three kinds of discrete impulse responses. The first of these is the Nyquist responses ("brickwall spectra" on $[-W, W]$) truncated to memory ν. For these we shall see that the interference from neighboring systems is prohibitive for reasonable ν. The second set of impulse responses are the discrete prolate spheroidal wave functions, which, for fixed total energy and fixed ν, have the least interference from neighboring systems. We demonstrate that their performance is not good. Finally we explore optimally designed responses and find that for reasonable ν, even for the most favorable cases, the advantage over QAM is very modest.

* Notice that if the constellation were not the product of two one-dimensional constellations, as we have assumed, the complexity would grow as 4^ν instead of $2^{\nu+1}$.

III. PERFORMANCE CRITERION

3.1 Definition of gain

The probability of bit error, P_e, is an important performance measure for data communication systems. For QAM as well as the systems employing HDC, UTC, or FBS, if MLSD is used the probability of bit error decays exponentially as the noise spectral density is decreased (except for an algebraic multiplier). That is, an exponentially tight bound on P_e has the form

$$P_e \leqslant \kappa\sigma e^{-\frac{\xi}{2\sigma^2}} \qquad (\sigma \rightarrow 0),$$

where κ and ξ are independent of σ^2, the noise power spectral density on a single dimension. For FBS viewed in the MLSD context, the exponentially tight bound has the form

$$P_e \leqslant \kappa\sigma e^{\frac{-d^2}{2\sigma^2}} \qquad (\sigma \rightarrow 0).$$

The minimum distance is defined by

$$d_{\min}^2 \triangleq \min_e \,||h{*}e||^2, \tag{1}$$

where $*$ denotes convolution and the minimum is over all doubly infinite sequences e of the form

$$\overline{0}1\epsilon_1\epsilon_2 \cdots \epsilon_K\overline{0},$$

where ϵ_k belongs to $\{0, 1, -1\}$ and K can be any nonnegative integer.

Clearly, $d_{\min}^2 \leqslant ||h||^2$. In cases where $d_{\min}^2 = ||h||^2$ it is common to say that the matched filter bound is attained. What is meant is that the exponent of P_e is the same as if there were only a single data pulse to be detected (no ISI). The terminology stems from the fact that, when there is no ISI, MLSD employs simply a matched filter (along with a threshold comparator).[12]

We take the quantity ξ as a convenient indicator of performance in the high s/n realm. (We stress that, for models of specific systems, more refined computations estimating the actual error probability are often needed.) The "gain" of one system over another is expressed as

$$G = 10 \log_{10} \frac{\xi'}{\xi}.$$

We shall be concerned in this paper with estimating the gain that FBS exhibits over QAM. Both UTC and HDC exhibit substantial gain over QAM. For UTC, gains in the range of 3 to 6 dB have been reported and, for a 3-dB gain, the required complexity is extremely reasonable.[2]

For the conventional QAM system, $||h||^2$ denotes the energy per pulse prior to multiplication by a_i belonging to $[\pm 1, \pm 3, \cdots \pm$

$(L-1)]$, so the modulated pulse has average energy $(L^2-1)/3\,||h||^2$. If there is one symbol every T seconds, the average signal power is $[(L^2-1)/3]\,[||h||^2)/T]$. Since the information rate per rail is $(\log_2 L)/T$ b/s, FBS must operate at a rate of $(\log_2 L)/T$ pulses/s. Let h' be the impulse response for FBS. For the two systems to have identical signal power we must have

$$||h'||^2 = (L^2-1)||h||^2/(3\log_2 L).$$

For FBS, accounting for the wider bandwidth, the noise variance per sample is $\sigma^2\,(\log_2 L)/T$. Not necessarily all of $||h'||^2$ is realized in the error exponent.

The gain over the corresponding QAM system is expressed as

$$G = 10\log_{10}\frac{d_{\min}^2}{||h||^2}\frac{L^2-1}{3\log_2 L} = 10\log_{10}\frac{d_{\min}^2}{||h||^2}\frac{4^{\rho}-1}{3\rho}, \qquad (2)$$

where $\rho = \log_2 L = W'/W = T/T'$. One could interpret $10\log_{10}[(4^{\rho}-1)/3\rho]$ as a noise immunity gain and $10\log_{10}(d_{\min}^2/||h||^2)$ as the penalty for ISI.

Shortly it will prove useful to allow for replacing the noise power spectral density on the FBS system by a level greater than that on the slower system, say $(1+\beta)\sigma^2$ with $\beta > 0$ in place of σ^2. This will enable us to compensate for interference from adjacent channels. When, and if, the matched filter bound is attained the gain is expressed by $10\log_{10}(4^{\rho}-1)/[3\rho(1+\beta)]$. Figure 3 depicts this function with β as a parameter. It is evident from the $\beta = 0$ curve that, depending on ρ, if the matched filter bound is attained, the gain can be considerable.

We consider now the interference in the band $(-W', W')$ that stems from those channels (other than the primary channel centered at zero) whose power spectral density is nonzero in $(-W', W')$. The determination of the additional power due to these interfering channels is straightforward. When measured for a single rail, at the output of the matched filter, $H^*(\omega)$, the power is the same as if σ^2 were replaced by

$$\sigma^2 + \int_{-W'}^{W'} |H'(f)|^2 \frac{\left[\frac{1}{T'}\sum_i |H'(f-i/T)|^2\right]}{\int_{-W'}^{W'} |H'(g)|^2 dg}\, df. \qquad (3)$$

The sum is over all neighboring systems overlapping the $(-W', W')$ band. Because of its genesis, the term that adds to σ^2 in (3) is called the Adjacent-Channel Interference term or ACI. For $H'(f)$ with nearly all the energy in the $(-W, W)$ band, $|H'(f)|^2/\int_{-W'}^{W'}|H'(g)|^2 dg$ has a mean value of approximately T on $(-W, W)$. Since each channel is symmetrically disposed relative to its neighbors, the integral in the

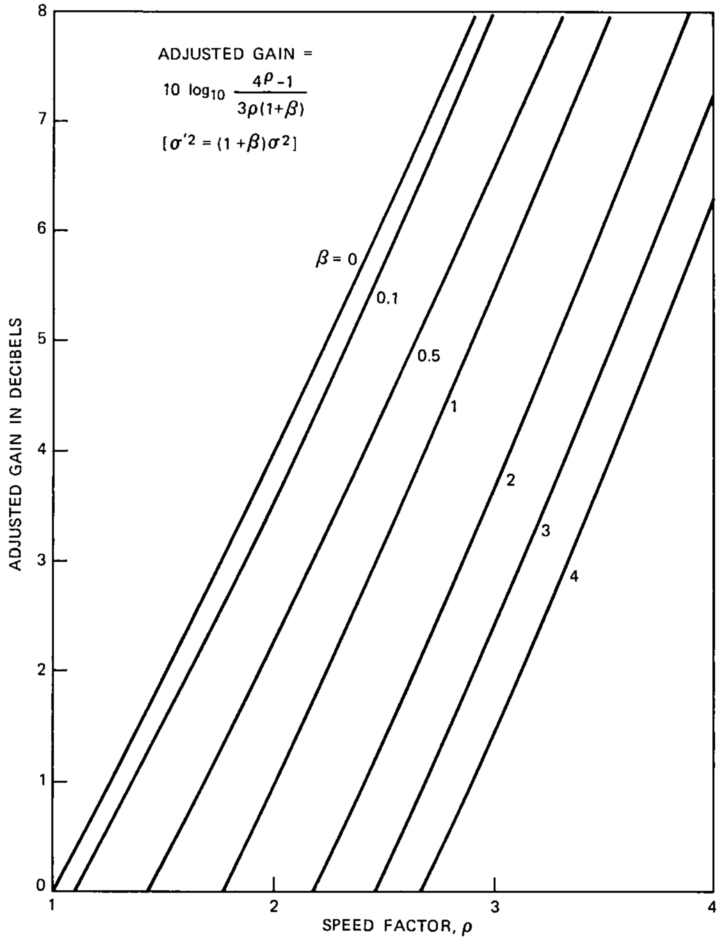

Fig. 3—Adjusted gain versus speed factor when matched filter bound is attained. Adjusted gain $= 10 \log_{10} \dfrac{4^{\rho} - 1}{3\rho(1 + \beta)}$.

numerator of (3) can be replaced by $\int_{-W}^{W}$. It follows that the strength of the ACI term in (3) is roughly indicated by that energy in a pulse with transform $H'(f)$ that is out of the band $(-W, W)$. (We use OBE to denote out-of-band energy.) This approximation becomes more precise if $H'(f)$ is approximately flat in $(-W, W)$ (as is the case in Section IV).

If ACI is not negligible, it is reasonable to modify the gain by subtracting $10 \log_{10}(1 + OBE/\sigma^2)$ or, the more precise but more complex, $10 \log_{10}(1 + ACI/\sigma^2)$. No matter which gain expression is most appropriate, if we insist on some degree of spectral isolation it is unclear how much gain can be attained at specific levels of complexity

($2^{\nu+1}$ with $\nu \leq 26$). In the sequel we will find that the answer is not much gain. For the analysis in Sections IV and V we consider OBE in the range $[0, \sigma^2]$. As we note in Section VI there is no point in considering OBE outside this range.

3.2 Error events

We can see from the formula for minimum distance (1) that, if we slide a window of size ν along an error sequence e, a repeated state is forced to occur by 3^ν shifts. It follows that, to attain the minimum, one need not search over more than $3^{(3^\nu)}$ events. Since $\{3^{(3^\nu)} \mid \nu = 1, 2, 3, 4, \cdots\} = \{27, 19683, 7.63 \times 10^{12}, 4.4 \times 10^{38}, \cdots\}$. We see that, even for $\nu = 3$, a brute force search is extremely ambitious and for $\nu = 4$ it is completely out of the question. (Reference 13 discusses three other state symmetries as well as a repeat.)

For future reference, we borrow from Ref. 13 and list four useful representations and notations for error events in Appendix A.

3.3 Searching the tree of error events for $d^2_{\min}$

For each ν, it is useful to view a set of error events, one of which is guaranteed to achieve $d^2_{\min}$ as a tree. Construct a tree of sequences with three branches emanating out of each node and with the labeling illustrated in Fig. 4. The labels along each upward path represent the beginning string for the nonzero portion of an error event. Once a string of ν consecutive zeros is encountered, the growth out of such a node is pruned from the tree, since continuing the event with nonzero elements will correspond to creating labelings for beginnings of events with a greater $\| h*e \|^2$ than the all-zero continuation.

To envision a computer search for $d^2_{\min}$ for a specific h, one can think of climbing up the tree and to the left and at each node computing the accumulated

$$\sum \langle h^b, \Delta_j^+ \rangle^2 \quad \text{(notation in Appendix A)}$$

on the upward path to the node. There is one summand for each node in the upward path. Climb higher if the record low for a completed

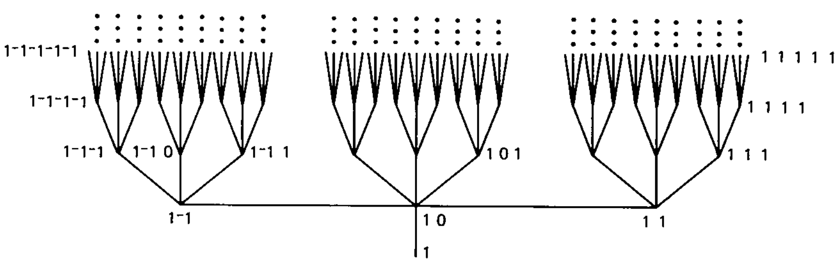

Fig. 4—Error event tree.

error event (a number $\leqslant ||h||^2$) has not been exceeded. Otherwise, climb down to the first node that offers an unclimbed branch and then climb that branch. Whenever a node with ν consecutive zeros is reached, and the old record has not been reached, record the new candidate event for achieving $d^2_{\min}$ and the new record before climbing down.

Some additional special search tools prove useful. Specifically, one can terminate an upward climb whenever any of the four symmetries $\Delta_i = \pm\Delta_j (i < j)$ or $\Delta_i = \pm\Delta_j^b (i \leqslant j)$ is detected (see Ref. 3). Two other search tools are very powerful, one for symmetrical h (see Appendix B) and one for nonsymmetrical h, which is discussed in Section 6.3.

Searching for $d^2_{\min}$ in the manner described will enable us to investigate the efficacy of FBS. In Sections IV, V, and VI, we consider three classes of FBS pulses.

IV. NYQUIST PULSES

4.1 Performance

One of the most elementary results in data communication theory is Nyquist's result that, for $WT = 1/2$, signals of the form $\sum_{-\infty}^{\infty} a_n h(t - nT)$ with $h(t) = \sin t/t$ are ISI-free. In FBS, we replace T by T/ρ with $\rho > 1$ and, as we have already mentioned, the signal bandwidth is invariant to ρ but ISI arises. A hypothetical FBS system based on such a pulse incorporates a level of idealization beyond the standard one associated with the abrupt cutoff. Namely, the system represents the limits of infinite decoding complexity as well as zero energy in the bands, $W < |f| \leqslant W'$.

For each system, assuming the pulse energy is normalized to 1 and W' is normalized to 1/2, we have

$$d^2_{\min} = \inf \frac{\rho}{2\pi} \int_{-\pi/\rho}^{\pi/\rho} \left| 1 - \sum_{1}^{K} \epsilon_k e^{jk\omega} \right|^2 d\omega. \tag{4}$$

The infimum is over all error events with ϵ_k belonging to $\{0, 1, -1\}$ and K ranging over the positive integers. Expression (4) is considered in Ref. 5 where it is demonstrated that $d^2_{\min} > 0$ for all $\rho \geqslant 1$. The intriguing question of whether, for such systems, a positive "gain" is available remains open.

Let

$$H_\rho^{NY}(\omega) \underline{\Delta} \begin{cases} \rho^{1/2} & |\omega| \leqslant \dfrac{\pi}{\rho} \\ 0 & \text{otherwise} \end{cases}$$

denote the FBS pulse transform normalized to unit energy. While expression (4) allows for infinite complexity, for any implementation, approximations of H_ρ^{NY} must be considered. The optimum least-mean-square approximations, the Fourier series $\{H_{\nu,\rho}^{NY}(\omega) \underline{\Delta} \sum_0^{\nu} q_n e^{jn\omega}\}$, are natural responses to use to inquire whether, as ν increases, FBS performs better than QAM. Of course, if ν becomes too large the required detector becomes forbiddingly complex.

The tree search discussed in Section 3.3 was used to determine the "gain" for the least-mean-square approximations. The symmetry of the impulse responses allowed the addition of the test of Appendix B to significantly reduce the running time of the algorithm.

The results are shown in Fig. 5. The "apparent gains" are only meaningful if the Out-of-Band Energies (OBEs) are sufficiently small. Indeed, they are not sufficiently small as we now discuss. Figure 6 shows the out-of-band energy for a unit energy response for $\nu = 26$. Also shown are the noise levels for the benchmark QAM system providing the same information rate at a P_e of 10^{-3} and 10^{-6}. Of the four points $\rho = 2$, 3, 4, and 5, only $\rho = 2$ shows the out-of-band energy below the noise level. The margin for $P_e = 10^{-3}$ is slight ($\approx$4 dB) but, from Fig. 5, we see that for $\nu = 26$ the "gain" is negligible ($\approx$0.1 dB). For $\rho = 2$, if we reduce ν to increase the "gain", the attempt is undermined by the increase in out-of-band energy. The out-of-band energies for $\nu = 20$ and $\nu = 14$ are also shown in Fig. 6, for $\rho = 2$.

We conclude that, for the least-mean-square approximation of a Nyquist pulse, FBS signaling under the mild requirement $P_e = 10^{-3}$ does not offer any significant gain over QAM. In making the comparison, we have allowed FBS the extraordinary complexity of $2^{26} \approx 6 \times 10^7$ states per rail ($>10^8$ states total). If P_e were decreased, FBS would fare even worse.

Figure 7 illustrates approximate Nyquist spectra for $\nu = 26$ and minimizing error events. We note that for $\nu = 26$ the number of candidate error events exceed $10^{(10^{12})} < 3^{(3^{26})}$.

4.2 Infinite complexity, complete spectral confinement asymptote

Allowing for complexity not exceeding 10^8 states, FBS is not attractive relative to QAM for the examples considered thus far. As we mentioned in the last section, for the asymptote of infinite complexity ($\nu \longrightarrow \infty$) and stringent out-of-band energy constraint, the limiting squared distance is expressed in (4). Although we cannot compute the gain G_ρ, we can find an upper bound using candidate error events. We used events revealed to be useful in the tree search for $\nu \leqslant 26$. The list of trigonometric polynomials below $\{E_\rho(\omega)\}_{\rho=2}^5$ was used to bound $d_{\min}^2$:

$$E_2(\omega) = (1 - e^{j\omega} + e^{3j\omega} - e^{4j\omega} + e^{6j\omega} - e^{7j\omega})$$
$$E_3(\omega) = (1 - e^{j\omega} + e^{4j\omega} - e^{5j\omega})$$
$$E_4(\omega) = (1 - e^{j\omega} - e^{2j\omega} + e^{3j\omega} + e^{4j\omega} - e^{5j\omega})(1 + e^{12j\omega})$$
$$E_5(\omega) = (1 - e^{j\omega} - e^{2j\omega} + e^{3j\omega} + e^{4j\omega} - e^{5j\omega}).$$

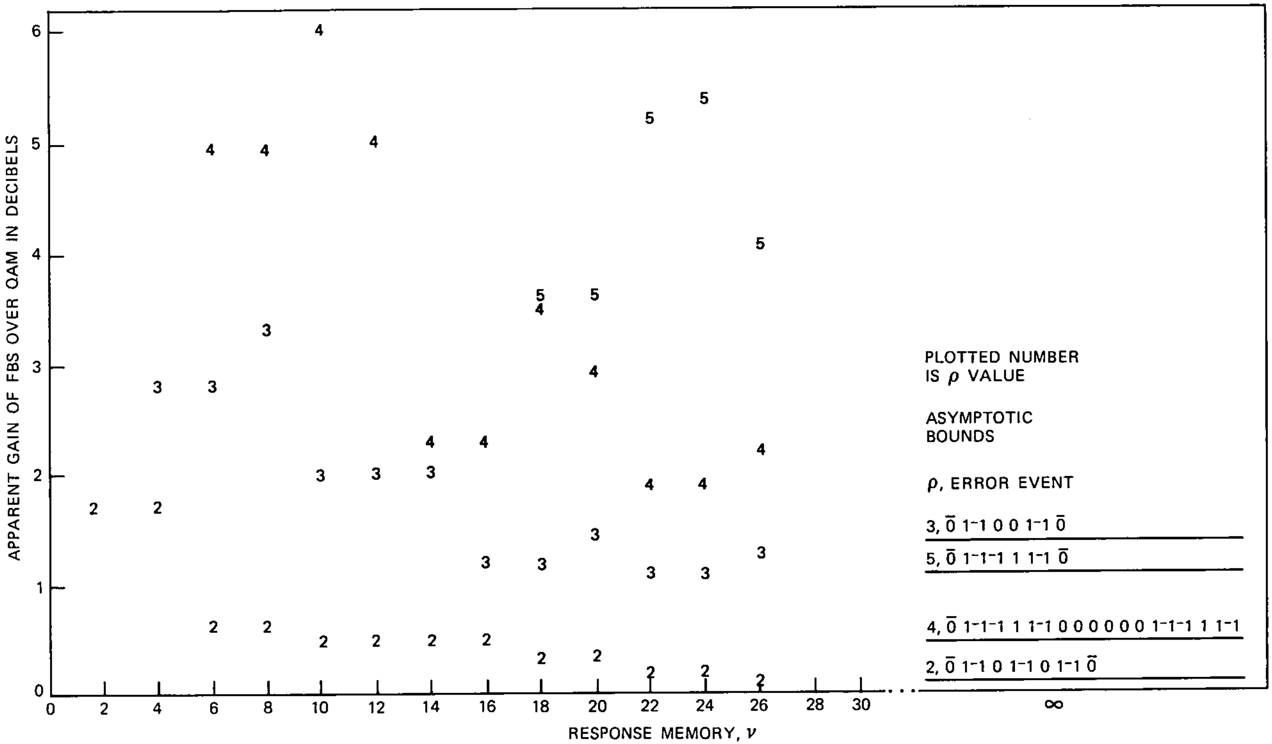

Fig. 5—Apparent gain of FBS over QAM (gains unachievable because of interference from adjacent bands).

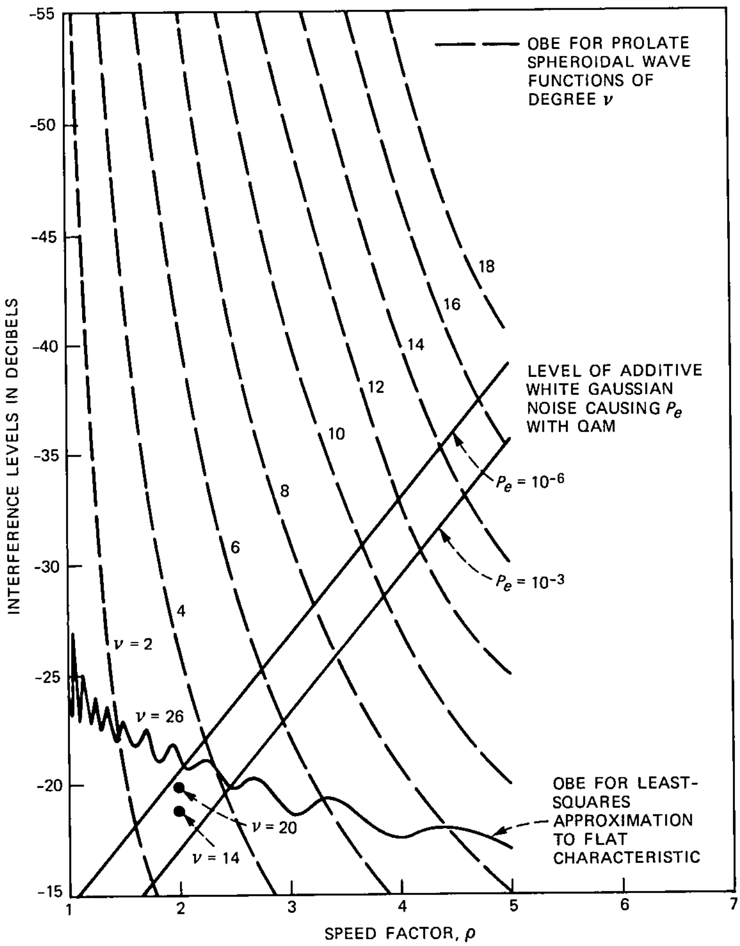

Fig. 6—Comparison of interference levels.

They yield $G_2 < 0.107$ dB, $G_3 < 1.4$ dB, $G_4 < 0.477$ dB, and $G_5 < 1.1$ dB. This shows that, even allowing for an arbitrarily large number of states, in the limit of stringent out-of-band energy requirements, the gains available using a Nyquist pulse are at best very modest. The $E_\rho(\omega)$ characteristics are illustrated in Fig. 8.

V. DISCRETE PROLATE SPHEROIDAL WAVE FUNCTIONS

In Section IV we investigated whether the approximations $H_{\nu,\rho}^{NY}$ have good distance properties for FBS with $\rho = 2, 3, 4$, and 5. We found that, for reasonable complexity and out-of-band energy constraints,

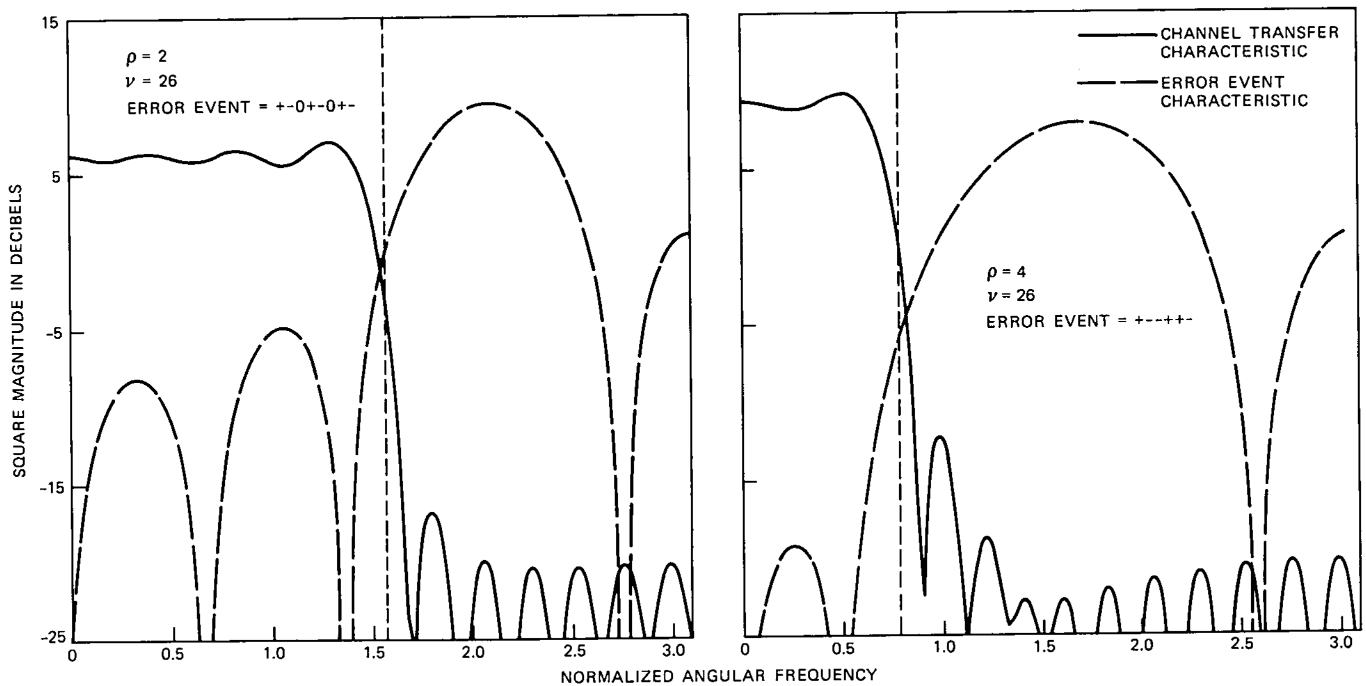

Fig. 7—Examples of $|H^{NY}_{\nu,\rho}|^2$ and extremal error event (+ and − mean +1 and −1). Vertical dotted line marks transition from in-band to out-of-band.

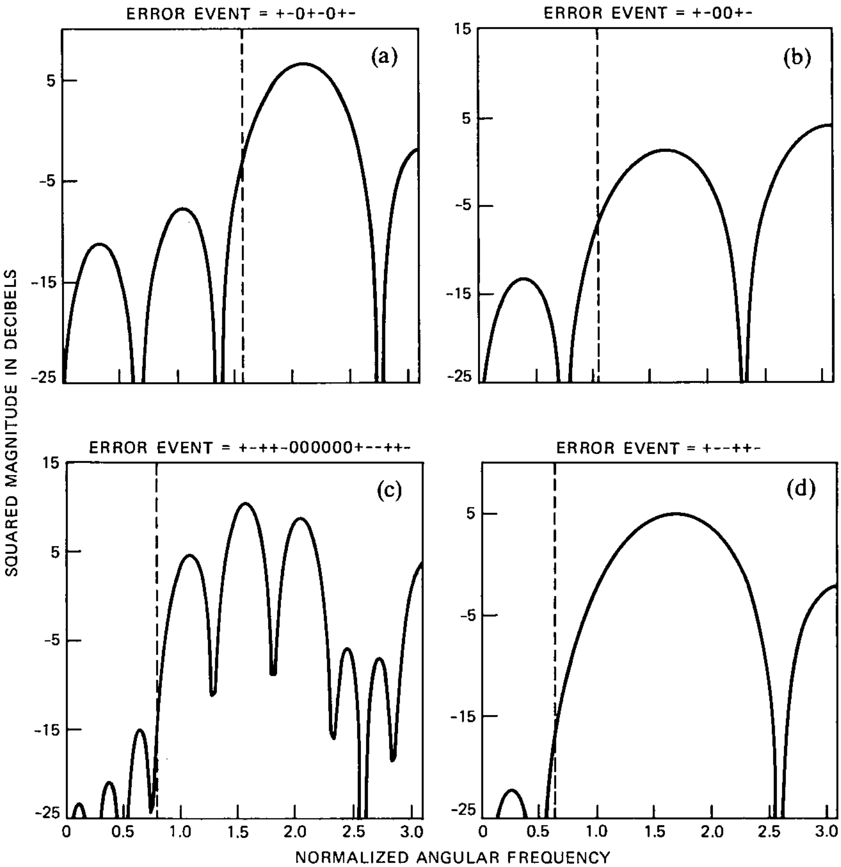

Fig. 8—Extremal error event for (a) $\rho = 2$, (b) $\rho = 3$, (c) $\rho = 4$, and (d) $\rho = 5$.

they do not. The $H^{NY}_{\nu,\rho}$ minimize $\int_{-\pi/\rho}^{\pi/\rho} | H^{NY}_{\rho} - \sum_0^{\nu} q_n e^{jn\omega} |^2 w(\omega) d\omega$ in the special case when the weight function $w(\omega) \equiv 1$. In light of the results of Section IV, we can reformulate the least-mean-square approximation using a $w(\omega)$ that is 0 on $(-\pi/\rho, \pi/\rho)$ and 1 otherwise. The weighting reflects the fact that it is essential to keep the out-of-band energy small but, having seen that the flat transform has no special distance properties, we have no motivation for keeping the transfer characteristic flat within $(-\pi/\rho, \pi/\rho)$.

The extremal responses so obtained are called the Discrete Prolate Spheroidal Wave Functions (DPSWF).[14] Their theory has been developed by Slepian.[15] Wyner has suggested their consideration for use in data communication systems for reasons other than those we are considering here.[16] Let $H^{PS}_{\nu,\rho}$ denote the transform of the discrete spheroidal wave function of memory ν corresponding to an FBS system with parameter ρ. Since minimizing the out-of-band energy corresponds to maximizing the in-band energy, the coefficients of $H_{\nu,\rho}^{PS}$ are then components $(q_o, q_1, \cdots q_\nu)$ of the eigenvector of the matrix of the symmetric quadratic form

$$\frac{\rho}{2\pi}\int_{-\pi/\rho}^{\pi/\rho} |q_o + q_1 e^{j\omega} + \cdots q_\nu e^{j\nu\omega}|^2 d\omega$$

corresponding to the largest eigenvalue, $\lambda_{\nu,\rho}$. Since we normalize by constraining $H_{\nu,\rho}^{PS}$ to have unit energy, the quantity $1 - \lambda_{\nu,\rho}$ is the out-of-band energy.

In Fig. 6, $10 \log_{10}(1 - \lambda_{\nu,\rho})$ is plotted against ρ for various ν (see dashed curves). Unlike $H_{\nu,\rho}^{NY}$ we see that a significant portion of the loci for $H_{\nu,\rho}^{PS}$ are disposed well above curves for the noise levels for $P_e = 10^{-3}$ and $P_e = 10^{-6}$. Thus, there are spaces of systems of moderate complexity with small out-of-band energy, whose distance properties are of interest. What are the distance properties of $H_{\nu,\rho}^{PS}$ in the range $\rho = 2, 3, 4, 5$? They are not good. Use of the search algorithm of Section 3.2 demonstrated no gain for any $H_{\nu,\rho}^{PS}$ whose (ν, ρ) coordinate corresponded to an out-of-band energy below the level of the Gaussian noise for P_e of 10^{-3}. For example, for $\rho = 2$ at $\nu = 4$, the out-of-band energy is -26.3 dB, which is below the level of the additive noise. However, the minimum distance of $H_{2,2}^{PS}$ is poor, specifically $G = -1.33$ dB. For larger ν, the out-of-band energy drops precipitously but distance decreases as well. As ρ increases in the range 3, 4, and 5, the situation worsens: G values significantly below 0 dB occur with out-of-band energy prohibitively above the noise level. As ν is increased, the distance drops markedly.

At this point we have an interesting situation. The least-mean-square approximations to the Nyquist pulse are shown to have attractive "apparent gains" relative to QAM but the gains cannot be realized because the signal spectrum is not adequately confined. On the other hand, the results for DPSWF's show that great spectral confinement is possible, but these pulse shapes do not exhibit any gain over QAM. The question remains as to how much gain we can achieve under a spectral confinement constraint for a specific complexity. This is addressed in Section VI.

VI. OPTIMUM PULSE DESIGN

In this section we investigate the performance of optimally designed FBS responses of prescribed complexity (i.e., prescribed memory ν). By optimum we mean that the minimum distance is maximized. The transmitted power, which is proportional to $1/(2\pi) \int_{-\pi}^{\pi} |H(\omega)|^2 d\omega$, and the out-of-band energy, which is proportional to $1/\pi \int_{\pi\rho^{-1}}^{\pi} |H(\omega)|^2 d\omega$ are both constrained. Once the optimum $\mathbf{h} = h * h^b$ (equivalently $\mathbf{H} = |H(\omega)|^2$) is found, we factor $\mathbf{H}$ to determine an optimal h. (Since $\mathbf{H}$ can have 2ν zeros disposed in inverse conjugate pairs there can be as many as 2^{ν} possible factors of $\mathbf{H}$ that have real coefficients.) The problem of finding optimal $\mathbf{h}$ is essentially a linear programming (LP) problem. The suggestion of viewing optimum MLSE system design as an LP problem appears in my paper with R. R. Anderson.[13] It turns out that, in most cases of interest, the number of constraints corresponding to the various error events is too large for the LP to be useful by itself. The LP is combined with the tree search algorithm that serves to eliminate most error events from consideration. The LP-tree search algorithm solves the design problem. We proceed now to describe the LP and show how it is integrated with a tree search algorithm. Then we present the performance results for optimally designed pulses.

6.1 Linear program

Recall that an LP problem is one of the following type: *Given a vector* $\mathbf{c}$, *find a vector* $\mathbf{y}$ *that maximizes* $\langle \mathbf{y}, \mathbf{c} \rangle$ *subject to a set of linear constraints of the form* $\langle \mathbf{y}, \mathbf{a}_i \rangle \geqslant b_i$, *i belonging to* $\mathscr{I}$, *a finite index set.* The $\mathbf{a}_i$ and b_i are given vectors and scalars, respectively. It is very useful that $\leqslant$ constraints can be converted into $\geqslant$ constraints by changing sign and so equality constraints can be represented by a pair of $\geqslant$ constraints.

In our application, $\mathscr{I}$ is infinite. Since we shall see that the feasible $\mathbf{y}$ exists in a bounded set we can, in principle, obtain a solution as close to optimum as desired by solving an LP with sufficiently many constraints.

6.2 Embedding systems in a $2\nu + 2$ space

Now $\mathbf{h} = \overline{0}\, \mathbf{h}_{-\nu}, \cdots, \mathbf{h}_o, \cdots, \mathbf{h}_\nu\, \overline{0}$. We will represent $\mathbf{h}$ in a $2\nu + 2$ dimensional space where the first $2\nu + 1$ coordinates are $(\mathbf{h}_{-\nu}, \cdots, \mathbf{h}_\nu)$. An additional coordinate augments the projection of $\mathbf{h}$ so that we have $(\mathbf{h}_{-\nu}, \cdots, \mathbf{h}_\nu, \mathbf{s})$. The augmented vector of $2\nu + 2$ components is denoted $\mathbf{y}$. The additional coordinate, $\mathbf{s}$, is a mathematical convenience that will facilitate maximization of the minimum squared distance, as we shall see.

To describe the linear constraints defining the set of admissible $\mathbf{h}$, we need to employ $\mathbf{1_k}$ to denote a vector that has all-zero coordinates except the kth coordinate, which is a one.

In $2\nu + 2$ space, we describe linear constraints defining the set of admissible $\mathbf{h}$.

1. As a convenient normalization, we assume that the energy in h cannot exceed 1, so $\mathbf{h}_o \leqslant 1$; therefore,

$$\langle \mathbf{y}, -\mathbf{1}_{\nu+1} \rangle \geqslant -1.$$

2. $h*h^b = \mathbf{h}$, so for $i \leqslant \nu$, $\mathbf{h}_{-i} = \mathbf{h}_i$; therefore,

$$\langle \mathbf{y}, -\mathbf{1}_i + \mathbf{1}_{2\nu+2-i} \rangle = 0.$$

3. $\mathbf{H}(\omega)$ is nonnegative. $\mathbf{H}(\omega)$ is the function that has $\mathbf{h}$ for Fourier coefficients and the operation of Fourier series is a linear one. So the constraints $\mathbf{H}(\omega) \geqslant 0$, $0 \leqslant \omega \leqslant \pi$, can be put in the form $\langle \mathbf{y}, \mathbf{a}_\omega \rangle \geqslant 0$ by defining $\mathbf{a}_\omega$ appropriately. There is one constraint for each ω on $0 \leqslant \omega \leqslant \pi$. In our application we can use a discrete set of the form $\{\omega_n = (n\pi)/N\}_{n=0}^{N}$, with N sufficiently large to give adequate accuracy.

4. The out-of-band energy cannot exceed a prescribed amount θ, so $1/\pi \int_{\pi/\rho}^{\pi} \mathbf{H}(\omega)\, d\omega \leqslant \theta$. Let $\Theta(\omega)$ be defined to be the function that vanishes for $|\omega| < \pi/\rho$ and is 1 otherwise. Therefore, $1/(2\pi) \int_{-\pi}^{\pi} \mathbf{H}(\omega) \Theta(\omega)\, d\omega \leqslant \theta$. By the Parseval theorem, we can express this constraint as $\langle \mathbf{y}, \mathbf{a} \rangle \leqslant \theta$, where $\mathbf{a}$ is a $2\nu + 2$ vector with a zero in the last position and the first $2\nu + 1$ coordinates are Fourier coefficients of $\Theta(\omega)$ with index of absolute value $\leqslant \nu$.

5. The $2\nu + 2$ component, seemingly extraneous so far, now comes into play. Let $\{\mathbf{E}_j(\omega)\}_{j\in\mathcal{J}}$ be the error polynomials. Project them into a $2\nu + 2$ dimensional space using the successive Fourier coefficients with index of absolute value $\leqslant \nu$ to get the first $2\nu + 1$ components and use -1 for the last component. Call the resulting vectors $\{\mathbf{e}_j\}_{j\in\mathcal{J}}$. It will not bother us if some $\mathbf{E}_j(\omega)$ have nonzero Fourier coefficients with index exceeding ν. Taken together, the constraints $\{\langle \mathbf{y}, \mathbf{e}_j \rangle \geqslant 0\}_{j\in\mathcal{J}}$ amount to a statement that, for each admissible $\mathbf{h}$, the squared minimum distance is never larger than a candidate distance.

The optimal $\mathbf{h}$ is the one maximizing the minimum distance. So the constrained $\mathbf{y}$ attaining max $\langle \mathbf{y}, \mathbf{1}_{2\nu+2} \rangle$ has the optimum design for its first $2\nu + 1$ coordinates and the optimal exponent for the last coordinate.

In $2\nu + 2$ space, the set of all $\mathbf{y}$ meeting constraints is denoted $\mathbf{Y}$. $\mathbf{Y}$ is not empty. For example, it contains $\delta \mathbf{1}_{\nu+1}$, where δ is a number small enough that energy constraints are met. The optimization will not degenerate as $\mathbf{Y}$ is closed and bounded. $\mathbf{Y}$ is closed since it is expressible as the intersection of closed half-spaces. $\mathbf{Y}$ is bounded since each component of $\mathbf{y}$ is bounded by the pulse energy constraint. To see why, note that $\mathbf{e} = \mathbf{1}_{\nu+1}$ shows $y_{2\nu+2} \leqslant 1$. For the remaining bounds on the components of $\mathbf{y}$ we note $\mathbf{H}(\omega) = \sum_{-\nu}^{\nu} x_{m+1+\nu} e^{jm\omega} \geqslant 0$ and so factorization is possible, $\mathbf{H}(\omega) = H(\omega)\, H^*(\omega)$. Fourier coefficients $(x_1, x_2, \cdots, x_{2\nu+1})$ are sums of products and so, by the Schwarz inequality, $y_{\nu+1} = x_{\nu+1}$ bounds all the components of each $\mathbf{y}$ vector.

6.3 The optimization algorithm

The linear constraints include the error events and, in most examples of interest, there are too many error events. For example, for ν as small as 4, the estimate in Section 3.1 indicated that there are over 10^{38} error events about which we should be concerned. The difficulty of too many constraints may sometimes be handled by solving a problem with a manageable number of the constraints. If it can be verified that the optimum meets all constraints (not just the manageable ones), then the solution to the simplified problem is the same as the solution to the difficult problem. We design, via an LP, a response maximizing the minimum distance over some error events and then seek to verify, using a tree search, that the minimum distance is not reduced if one minimizes over all error events.

If the above procedure is unsuccessful, one can repeat it, enlarging $\mathcal{J}$ to include the minimizing error event revealed by the tree search. Eventually, the iteration process will converge. Prior to convergence, LP gives an upper bound while the tree search gives a lower bound to the $d^2_{\min}$ achievable by the optimum design.

The LP provides an **h**, while the tree search requires an h. The minimum-phase deconvolution, $\hat{h}$, is suggested, since, among all h satisfying $h^b * h = \mathbf{h}$, $\hat{h}$ has the greatest $\sum_{i=0}^{k} h_i^2$ for each k.[17,18] This maximal frontal energy concentration expedites the tree search, which operates first on the leading coordinates of **h.** (Orders of magnitude of difference in running time have been observed between minimum- and maximum-phase deconvolutions in the tree search algorithm.)

6.4 Performance

In estimating the performance of the optimum system employing an **h** of memory ν, power levels were set as follows: For the FBS system, as we mentioned in Section 6.2, the pulse energy was bounded above by one. The noise level was set to meet the required P_e in the benchmark system operating at maximum power. Finally, the out-of-band energy constraint was set to a fraction of the noise power.

The resulting gain versus ν curves are shown in Fig. 9 with ν as a parameter. The OBE is constrained to $\sigma^2/10$ so a penalty of $0.414 \approx 10 \log 11/10$ is included in the gain calculation. It is apparent from the curves that, even at extraordinary complexity (exceeding 10^8 states) and a P_e requirement of 10^{-3}, the resulting gains are very modest. This conclusion is not sensitive to the exact premises underlying the computation. Calculation shows that, for $\nu = 26$, if we allow θ to be larger than 1/10, the gains generally decrease because of the OBE penalty. There is little to achieve by making OBE smaller than $\sigma^2/10$ since the design is merely more constrained, and omitting the OBE penalty cannot add more to the gain than 0.414 dB. When the gain is positive, the ACI levels are generally within 0.25 dB of the OBE level so the gains based on ACI are not different in any important

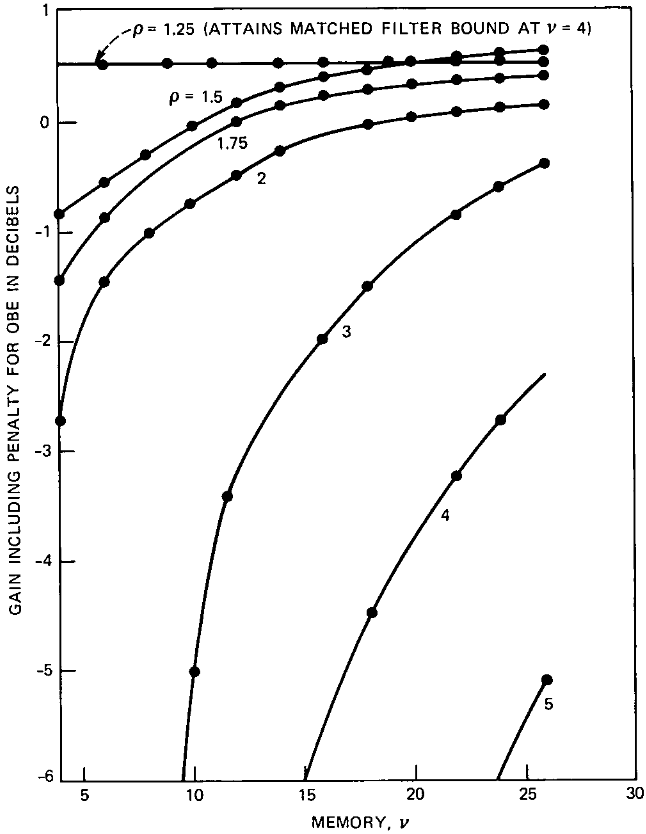

Fig. 9—Gain limit versus memory under spectral confinement constraint for $P_e = 10^{-3}$.

way from those based on OBE. There is no point in showing curves for $P_e < 10^{-3}$ in the benchmark system, as the gains can only decrease if the design is further constrained.

Figure 9 has enabled us to determine the relative merit of FBS for reasonable complexity. The possibility remains that, for extremely large ν, FBS could exhibit substantial gains and that these asymptotic gains could improve as ρ increases.

Figure 9 was derived using a list of 50 error events obtained by running the LP-tree search iteration for successive ν values. To conclude that FBS offers at best a very modest improvement over QAM, it is only necessary to present upper bounds in Fig. 9 rather than exact maximum gains. However, in preparing Fig. 9, we established that it is reasonable to exercise the LP-tree search algorithm to guarantee the precise optimum gain that can be attained for up to 10^5 states. It

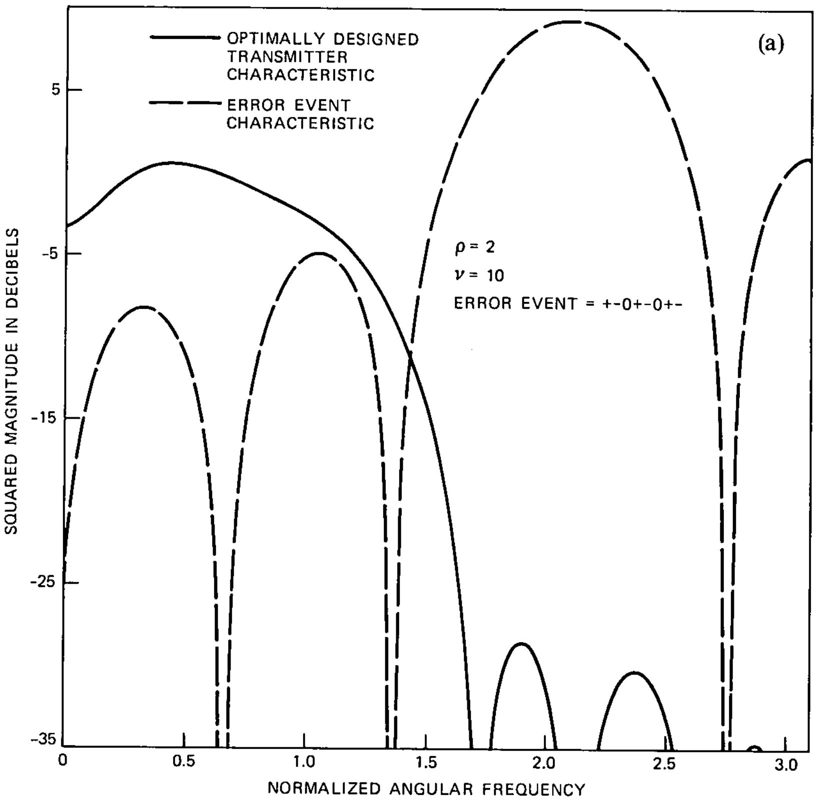

Fig. 10a—Example of an optimally designed transmitter characteristic and an error event characteristic. Vertical dotted line delineates the band edge.

is interesting to note that optimum system design can be accomplished for systems with such an enormous number of states.

Figure 10a illustrates an optimally designed spectrum and a corresponding minimizing error event. Figure 10b illustrates an interesting constrast between a pulse spectrum and an extremal error event.

For $\rho = 5/4$, the gain is only about 0.5 dB but a very interesting behavior is observed. Namely, with little complexity, the maximum distance possible is attained in the sense that the matched filter bound is obtained. From Section III, eq. (2), we can write the following expression for the gain (neglecting the OBE penalty):

$$G(\rho, \nu) = 10 \log_{10} \frac{(4^{\rho} - 1)}{3\rho} + 10 \log_{10} c(\rho, \nu),$$

where the function $c(\rho, \nu)$ gives the fraction of the matched filter energy attained. For fixed ρ, $c(\rho, \nu)$ is a nondecreasing function of ν. With ρ fixed, is $\lim_{\nu \to \infty} c(\rho, \nu) = 1$? In the case $\rho = 5/4$, we have seen

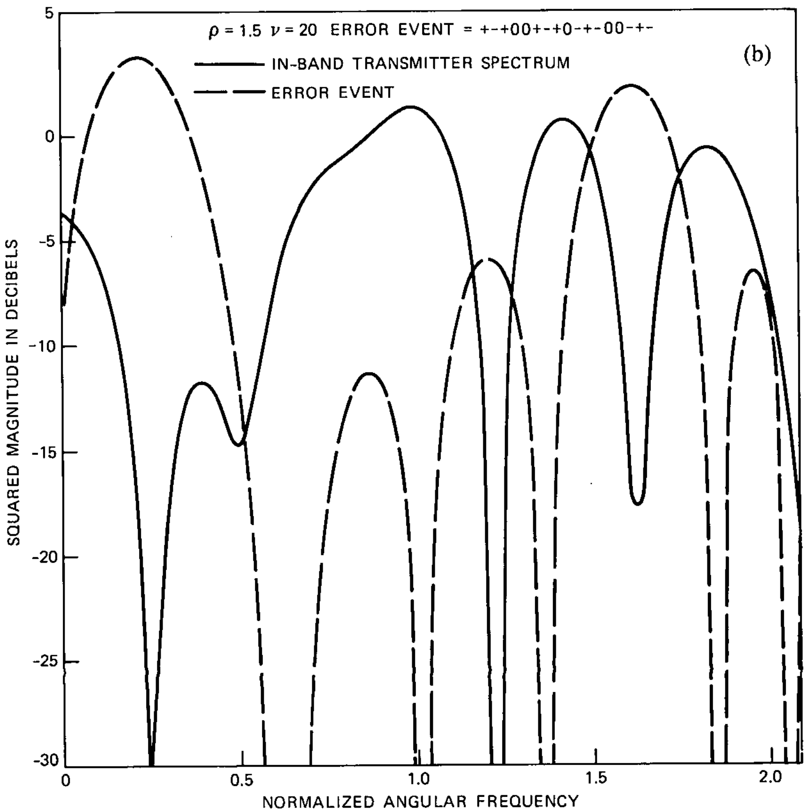

Fig. 10b—The in-band transmitter spectrum is optimized for a limited set of error events, one of which is shown. The extraordinary flexibility afforded by over one million states allows the optimum spectrum to have some peaks and valleys in opposition to those of the minimizing error event.

that the answer is yes. For $\rho > 3$ consideration of the error transform $|1 - e^{j\omega}|^2$ shows that the answer is no, as

$$\lim_{\nu \to \infty} c(\rho, \nu) \leq 4 \sin^2 \left(\frac{\pi}{2\rho}\right) < 1$$

in the limit of stringent out-of-band energy constraints. In light of the limited gains available with optimal FBS, it would be of only academic interest to pinpoint the largest ρ value for which the matched filter bound is attained as complexity is increased. Consequently, we shall not pursue this question further.

VII. DISCUSSION

At this point it is natural to question whether it is worthwhile to generalize and consider Faster Multilevel Signaling (FMS). Motivation for considering FBS comes from Ref. 5, where the first theoretical results on FBS were reported; from Ref. 3, where highly significant benefits of FBS were suggested but not established; and from discussions with J. Salz, who related that the idea of FBS has been around for many years and that it is important to settle the question of its merit. Since FBS proves to be unattractive, why should one consider FMS, especially when we know that increasing the number of levels toward that of the competing QAM system would seem to blur the distinction? Can one generally discount the competitiveness of FMS? We discuss why we cannot dismiss FMS and why, despite the findings on FBS, FMS systems may have some value.

7.1 Relieving the OBE constraint

FBS fared poorly. If we look back on our analysis of FBS it is obvious that it was the OBE constraint that drove the performance level of FBS. We noticed in Section IV suboptimal pulses exhibiting substantial gains that could not be realized because of prohibitive OBE. The stringency of the OBE constraint was necessitated by the substantial overlap of spectra between neighboring systems. As we move away from binary toward more levels, in the class of FMS systems, to compete with a fixed QAM system, the ratio $\rho = W'/W > 1$ decreases. The OBE constraint we need to impose is seen to be more relaxed.

Moreover, as we decrease ρ, systems are represented for which the ACI constraint is not of any direct importance. There is the interesting class of questions pertaining to transmitter filter smoothness considerations. For example, which performs better—a QAM system employing a square root raised cosine pulse with roll-off $\alpha = \rho - 1$, or an optimized FMS system with band-edge nulls of specified order and with system memory ν? The two systems are required to have the same power and information rate. The answer, of course, depends on M, α, ν, and the degree of the band-edge null. The band-edge null is useful for spectral confinement as well as for easing synthesizability. The imposition of nulls of specific order at specific frequencies lead to additional linear constraints and is easily handled by the LP-tree search program.

The simple partial response $\overline{0}$, 1, 1, $\overline{0}$ can be used to illustrate that there are situations where FMS can be very beneficial relative to QAM. Among all systems required to have a band-edge null, the system $\overline{0}$, 1, 1, $\overline{0}$ requires the least number of states, m per rail. It is easily shown that the response attains the matched filter bound independent of m. Figure 11 shows a gain versus roll-off plot, which speaks for itself concerning the substantial gains that are available in certain cases.

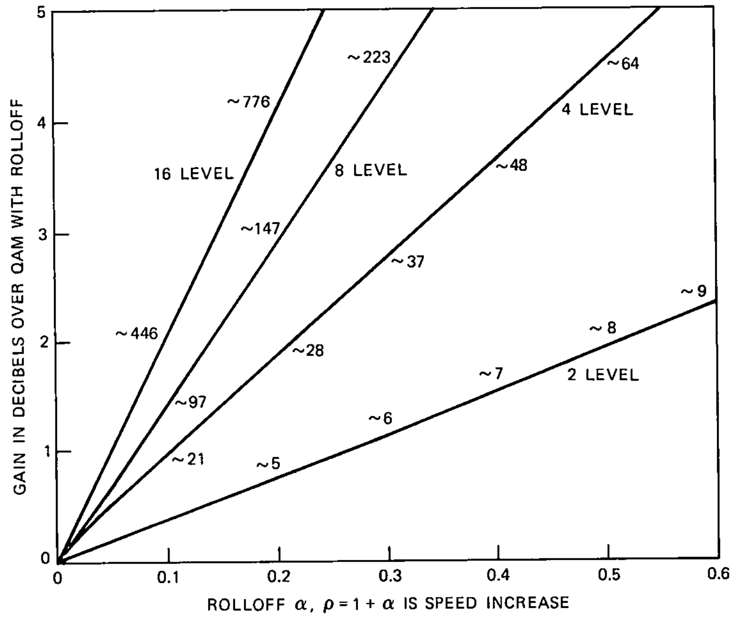

Fig. 11—Gain versus rolloff characteristics for partial response $2^{-1/2}$ $(\bar{0}, 1, 1, \bar{0})$, where the approximate number of constellation points are shown for competing QAM system, and the number of levels equals the number of states per rail.

A class of examples where ACI is not of direct concern occurs with the voiceband channel, which has severe band-edge attenuation. The channel shapes are irregular mounds and there is no obvious spectral support to assume. For a specific information rate, what is the best baud to use if the transmitter spectrum has a null of given order and zero rolloff? This is paraphrasal of the FMS issue coupled with a simpler question of where to center the signal spectrum. (The transmitter design must also account for the effect of nonlinearities and the fact that the exact modulus of the channel transfer characteristic is not known at the transmitter.)

7.2 The LP-tree search algorithm

Aside from the new information on FBS, a major finding of this report is that, for the class of MLSD systems considered, optimum designs can be accomplished involving numbers of states corresponding to the capabilities of forthcoming MLSD implementation technology (and far beyond). We have concentrated here on binary systems and a very special channel. However, the algorithm extends to apply to designing optimum m-ary systems of prescribed complexity operating over arbitrary linear dispersive channels. The astronomical number of error events is not an obstacle.

The extended algorithm, now being programmed in the course of joint work with G. Vannucci, will provide a basic tool for probing the fundamental relationship between attainable rates and system complexity for very general systems. Suppose one wants to achieve a certain information rate, under spectral confinement requirements and with a specific level of complexity. By exercising the LP-tree search algorithm for a sufficient number of ρ values, one can locate ρ_{opt}, the optimum $\rho \geqslant 1$, and the associated optimum gain over a corresponding QAM system with cosine rolloff spectral shaping.

VIII. ACKNOWLEDGMENT

It is a pleasure to acknowledge numerous valuable, stimulating discussions with G. Vannucci. These discussions involved both the theoretical and software aspects of this work. L. J. Domenico's assistance with the programming is greatly appreciated.

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APPENDIX A

Error Event Representations

A.1 Sequence representation

$$\cdots 0 \cdots \overset{\nu}{0}0\epsilon_0\epsilon_1 \cdots \epsilon_K 00 \cdots \overset{\nu}{0} \cdots, \qquad \epsilon_0\epsilon_K \neq 0.$$

A.2 State representation

$\Delta_1, \Delta_2, \Delta_3, \cdots, \Delta_{K+\nu+1}$, where the states Δ_j are defined as the successive ν-tuples of the sequence representation, where the all-zero ν-tuples are omitted, except for the ν-tuple abutting ϵ_K.

$$(0, 0, \cdots, 0, \epsilon_0), (0, 0, \cdots, \epsilon_0, \epsilon_1), \cdots, (\epsilon_K, 0, 0, \cdots 0), (0, 0, 0, \cdots 0).$$

A.3 Augmented state representation

$\Delta_1^+, \Delta_2^+, \Delta_3^+, \cdots \Delta_{K+\nu+1}^+$, where the augmented states Δ_j^+ are defined as the successive $(\nu + 1)$-tuples of the sequence representation

$$\underbrace{(00 \cdots 0\epsilon_0)}_{\nu+1}, (00 \cdots \epsilon_0\epsilon_1) \cdots \underbrace{(\epsilon_K 00 \cdots 0)}_{\nu+1}.$$

This representation derives its usefulness from the equality

$$\sum_{1}^{K+\nu+1} \langle h^b, \Delta_j^+ \rangle^2 = || h*e ||^2,$$

where $h^b \triangleq (h_\nu, h_{\nu-1}, \cdots, h_0)$ and the inner product is defined in the usual way. The b superscript is read "backward" and the b operation is also applied to error events in the memorandum.

A.4 Functional representation

The error sequence maps to the nonnegative cosine polynomial

$$\mathbf{E}(\omega) = |\epsilon_0 + \epsilon_1 e^{i\omega} + \cdots + \epsilon_K e^{iK\omega}|^2$$

on the interval $-\pi \leqslant \omega \leqslant \pi$.

We shall refer to each e as an error sequence, error event, or error pattern. Let $\mathbf{H}(\omega) \triangleq |h_0 + h_1 e^{i\omega} + \cdots + h_\nu e^{i\nu\omega}|^2$ and define $\langle \mathbf{E}, \mathbf{H} \rangle \triangleq 1/(2\pi) \int_{-\pi}^{p} \mathbf{E}(\omega)\mathbf{H}(\omega)\, d\omega$. Then by Parseval's theorem, $\langle \mathbf{E}, \mathbf{H} \rangle = ||e*h||^2$, so we have

$$d_{\min}^2(h) = \min_{\mathbf{E}} \langle \mathbf{E}, \mathbf{H} \rangle = \min_{\mathbf{E}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{E}(\omega)\mathbf{H}(\omega) d\omega.$$

APPENDIX B

Expediting Tree Search When Response Is Symmetrical

We present some observations other than the four symmetry conditions that are useful for efficient calculation of $d^2_{\min}$ and a minimizing error event e for a symmetric transmitter impulse response.

Let $\{S_k\}_{-\infty}^{\infty}$ denote the resulting sequence of scalar products in $h*e$. Let K be the smallest integer satisfying $s_1 s_K \neq 0$ with $s_k = 0$ for $k \notin \{1, 2, \cdots K\}$. If, in the course of searching the tree, an error event breaking the current record, A, is encountered, then, for it to be a minimizing event, we must have

$$s_1^2 + s_2^2 + \cdots s_K^2 < A.$$

Let $[K/2]$ denote the largest integer less than or equal to $K/2$. For a record breaking error event or that error event in reverse (or both) we must have

$$s_1^2 + s_2^2 + \cdots s_{[K/2]}^2 < A/2. \tag{5}$$

Since $||h*e||^2 = ||h^*(\pm e^b)||_1^2$ error events for which it is established that the inequality (5) is reversed need not be explored further in the tree search for $d^2_{\min}$.

To expedite the process of seeking $d^2_{\min}$ among error sequences for which (5) holds, we discuss the calculation of the height at which the exploration of the growth of nodes terminates. Let L be the first integer for which the accumulated sum of $s_1^2 + s_2^2 + \cdots s_{L+1}^2 > A/2$ so that $s_1^2 + s_2 + \cdots s_L^2 \leqslant A/2$. Clearly, $L \geqslant [K/2]$ so $2L + 1 > K$. Once L is detected there is no need to explore any events involving $2L + 2$ scalar products. To put it another way, if $L' = L + 1$ is the first index for which $A/2$ is exceeded, then $s_{2L'} = 0$ and $2L' > K$.

It is not always necessary to search to height $2L'$ to terminate growth exploration. To see this note that from the height, ℓ', of occurrence of the last nonzero element in the event under exploration we have that $K \geqslant \ell' + \nu$. So once L' is determined exploration of growth is terminated if $\ell' + \nu \geqslant 2L'$.

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